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Three Dimensional Contact Metric Manifolds with Vanishing Jacobi Operator

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Abstract. We study 3-dimensional contact metric manifolds, the Jacobi operator, of which, vanishes identically. The local description and construction as well as some global results of this class of manifolds are given. Our results are followed by several examples.

1. Introduction

In contact geometry, the Jacobi operator $l = R(.,\xi)\xi$ plays a fundamental role. The class of contact metric manifolds with $l = 0$ is particularly large. For example, the normal bundle of an m-dimensional integral submanifold of a $(2m+1)$-dimensional Sasakian manifold admits a contact metric structure with $l = 0$ ([1], [2, p. 153]). Thus, the study of these manifolds is of considerable interest. Some results concerning the 3-dimensional case are given in [5] and the references therein, [7].

In the present paper we continue the study of 3-dimensional contact metric manifolds $M(\eta, \xi, \phi, g)$ with $l = 0$. First, we explicitly describe locally all these manifolds. For their local description we make use of a special coordinate system and we write down the equations that characterize these manifolds. So, we are led to a simple system of 1st order partial differential equations. The solution of this system depends on two arbitrary functions of two variables and on three functions of one variable. Secondly, for any function $G : V \subseteq R^3 \to R$ differentiable on an open subset $V \subseteq R^3$ and such that $\frac{\partial^2 G}{\partial x^2} = 0$, we construct a family of contact metric manifolds $V(\eta, \xi, \phi, g)$ with $l = 0$. Thirdly, we classify those which additionally satisfy $||Q\xi|| =$ constant, where Q is the Ricci operator. Finally, we classify the ones which are closed and have non-negative (or non-positive) scalar curvature.

2. Preliminaries

In this section, we give the definitions, the formulas and some lemmas we need. For more details concerning contact metric manifolds the reader is referred to [2]. Throughout this paper, all manifolds are assumed to be connected, and all functions to be of class C^∞.

A differentiable $(2m+1)$-dimensional manifold M is called a *contact manifold*, if it admits a global differential 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. Given a contact manifold (M, η) there exists a unique global vector field ξ (called the *Reeb vector field* or the *characteristic vector field*), which satisfies $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field $X \in \mathcal{X}(M)$. Polarizing $d\eta$ on the contact subbundle D, defined by $\eta = 0$, one obtains a Riemannian metric g and a $(1,1)$-tensor field ϕ such that:

$$d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi$$

for any $X, Y \in \mathcal{X}(M)$. The metric g is called an *associated metric* of η, and (η, ξ, ϕ, g) is called a *contact metric structure*. A differentiable $(2m+1)$-dimensional manifold equipped with a contact metric structure (η, ξ, ϕ, g) is called a *contact metric (Riemannian) manifold* and it is denoted by $M(\eta, \xi, \phi, g)$. The set $\mathcal{A}(\eta)$ of associated metrics to η is of infinite dimension and each metric $g \in \mathcal{A}(\eta)$ has the same volume element dv. On a contact metric manifold $M(\eta, \xi, \phi, g)$ we define the $(1,1)$-tensor fields l and h by

$$lX = R(X, \xi)\xi, \quad hX = \frac{1}{2}(\mathcal{L}_\xi \phi)X,$$

where $\mathcal{L}_\xi$ and R are the Lie differentiation in the direction of ξ and the curvature tensor respectively, given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \mathcal{X}(M)$. The tensors l and h are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad Trh = Trh\phi = 0, \quad h\phi = -\phi h.$$

Since h anti-commutes with ϕ, if X is a non-zero eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. On a contact metric manifold $M(\eta, \xi, \phi, g)$ the following formulas are valid:

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \nabla_\xi \phi = 0,$$

$$\nabla\xi = -\phi - \phi h, \quad \nabla_\xi \xi = 0, \quad \phi l \phi - l = 2(\phi^2 + h^2), \quad \nabla_\xi h = \phi - \phi l - \phi h^2,$$

where ∇ is the Levi-Civita connection.

Now, let $M(\eta, \xi, \phi, g)$ be a 3-dimensional contact metric manifold with $l = 0$. Then, from $\phi l \phi - l = 2(\phi^2 + h^2)$ we have $h^2 = -\phi^2$ and so the non-zero eigenvalues of h are ± 1 and their eigenvectors are orthogonal to ξ. Thus, the contact subbundle D is decomposed into the orthogonal eigenspaces ± 1, which we denote by $[+1]$ and $[-1]$ respectively.

From now on, by an M_l-manifold we will mean a 3-dimensional contact metric manifold $M(\eta, \xi, \phi, g)$ satisfying $l = 0$. Moreover, by $(\xi, X, \phi X)$ we will denote a local orthonormal frame of eigenvectors of h such that $h\xi = 0$, $hX = X$ and $h\phi X = -\phi X$.

Now, we will give some well known results concerning M_l-manifolds.

Lemma 2.1. *On any M_l-manifold the following formulas are valid:*

$$\nabla_X \xi = -2\phi X, \quad \nabla_{\phi X}\xi = 0, \quad \nabla_\xi X = 0, \quad \nabla_\xi \phi X = 0, \tag{1}$$

$$\nabla_X \phi X = -AX + 2\xi, \quad \nabla_{\phi X} X = -B\phi X, \tag{2}$$

$$\nabla_X X = A\phi X, \;\; \nabla_{\phi X}\phi X = BX, \;\; (A = -div\phi X, B = -divX), \tag{3}$$

$$[\xi, X] = 2\phi X, \;\; [\xi, \phi X] = 0, \;\; [X, \phi X] = -AX + B\phi X + 2\xi, \tag{4}$$

$$\xi A = 0, \quad \xi B = 2A, \tag{5}$$

$$Q\xi = 2AX + 2B\phi X, \, QX = \frac{S}{2}X + 2A\xi, \, Q\phi X = \frac{S}{2}\phi X + 2B\xi, \tag{6}$$

$$S = TrQ = 2(\phi XA + XB - A^2 - B^2), \tag{7}$$

$$\xi S = 4(\phi XB + XA - 2AB), \tag{8}$$

where Q is the Ricci operator ($QZ = \sum_i R(Z, e_i)e_i$, e_i, $i = 1, 2, 3$, is an orthonormal basis), div denotes the divergence ($divZ = \sum_i g(\nabla_{e_i} Z, e_i)$), and S is the scalar curvature.

Lemma 2.2. *If the scalar curvature of an M_l-manifold is constant, then either $S = 4$, or $QY \in [+1]$ for any $Y \in [+1]$.*

For the proofs of Lemmas 2.1 and 2.2 see [5]. Especially, the relations $A = -div\phi X$ and $B = -divX$ are immediate consequences of (1), (2) and the definition of the divergence.

3. Local description and construction of M_l-manifolds

In the next theorem all M_l-manifolds are locally determined.

Theorem 3.1. *Let $M(\eta, \xi, \phi, g)$ be an M_l-manifold. Then, for any point $P \in M$, there exists a chart $\{U, (x, y, z)\}$ with $P \in U \subseteq M$, such that*

$$\eta = dx - \frac{a}{c}dz, \quad \xi = \frac{\partial}{\partial x},$$

$$g = \begin{pmatrix} 1 & 0 & -\frac{a}{c} \\ 0 & 1 & -\frac{b}{c} \\ -\frac{a}{c} & -\frac{b}{c} & \frac{1+a^2+b^2}{c^2} \end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix} 0 & -a & \frac{ab}{c} \\ 0 & -b & \frac{1+b^2}{c} \\ 0 & -c & b \end{pmatrix}$$

with respect to the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, where a, b, c ($c \neq 0$ everywhere) are smooth functions on U given by

$$\left.\begin{aligned} a &= \{f_1(z) - 2\textstyle\int_{y_0}^{y} e^{-\int_{y_0}^{s} C_1(t,z)dt} ds\} e^{\int_{y_0}^{y} C_1(t,z)dt} \\ b &= 2x + \{f_2(z) - \textstyle\int_{y_0}^{y} C_2(s,z) e^{-\int_{y_0}^{s} C_1(t,z)dt} ds\} e^{\int_{y_0}^{y} C_1(t,z)dt} \\ c &= f_3(z) e^{\int_{y_0}^{y} C_1(t,z)dt} \end{aligned}\right\} \tag{9}$$

and $C_1(y, z), C_2(y, z), f_1(z), f_2(z), f_3(z)$, ($f_3(z) \neq 0$ everywhere) are integration smooth functions on U.

Proof. Let $(\xi, X, \phi X)$ be a local orthonormal frame of eigenvectors of h, such that $hX = X$ and $h\phi X = -\phi X$ in an appropriate neighborhood V of an arbitrary point of M. Since from (4): $[\xi, \phi X] = 0$ on V, the distribution obtained by ξ and ϕX is integrable, and so for any point $P \in V$ there exists a chart $\{U, (x, y, z)\}$ such that $P \in U \subseteq V$ and

$$\xi = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y}, \quad X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \tag{10}$$

where a, b, c are smooth functions defined on U. Since $\xi, X, \phi X$ are linearly independent we have $c \neq 0$ at any point of U. Now, we will determine the functions a, b, c. Substituting ξ, X in $[\xi, X] = 2\phi X$ and $X, \phi X$ in $[X, \phi X] = -AX + B\phi X + 2\xi$ we easily get

$$\left.\begin{aligned} &\tfrac{\partial a}{\partial x} = 0, \quad \tfrac{\partial b}{\partial x} = 2, \quad \tfrac{\partial c}{\partial x} = 0 \\ &\tfrac{\partial a}{\partial y} = Aa - 2, \quad \tfrac{\partial b}{\partial y} = Ab - B, \quad \tfrac{\partial c}{\partial y} = Ac. \end{aligned}\right\} \tag{11}$$

Therefore, using (5) we get $\frac{\partial A}{\partial x} = 0$ (so $A = C_1(y, z)$) and $\frac{\partial B}{\partial x} = 2A$ (and so $B = 2xC_1(y, z) + C_2(y, z)$), where $C_1(y, z)$ and $C_2(y, z)$ are integration functions. Solving the system of equations (11) we find (9). In what follows, we will calculate the tensor fields η, ϕ, g with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the components g_{ij} of the Riemannian metric g, using (10) we have:

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -\frac{a}{c},$$
$$g_{23} = g_{32} = -\frac{b}{c}, \quad \text{and} \quad 1 = g(X, X) = c^2 g_{33} - a^2 - b^2,$$

from which we get $g_{33} = \frac{1+a^2+b^2}{c^2}$.

The components of the tensor field ϕ are immediate consequences of

$$\phi(\frac{\partial}{\partial x}) = \phi(\xi) = 0, \quad \phi(\frac{\partial}{\partial y}) = -a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y} - c\frac{\partial}{\partial z}$$

and

$$\phi(\frac{\partial}{\partial z}) = \frac{1}{c}\{ab\frac{\partial}{\partial x} + (1 + b^2)\frac{\partial}{\partial y} + bc\frac{\partial}{\partial z}\}.$$

The expression of the contact form η, immediately follows from

$$\eta(\frac{\partial}{\partial x}) = \eta(\xi) = 1, \quad \eta(\frac{\partial}{\partial y}) = \eta(\phi X) = 0 \quad \text{and} \quad \eta(\frac{\partial}{\partial z}) = -\frac{a}{c}.$$

This completes the proof of the theorem.

In the next theorem all M_l-manifolds are locally constructed in R^3.

Theorem 3.2. *Let $G : V \subseteq R^3 \to R$ be a smooth function on an open subset V of R^3 so that $\frac{\partial^2 G(x,y,z)}{\partial x^2} = 0$, where (x, y, z) are the standard coordinates of R^3. Then there exists a family of contact metric structures (η, ξ, ϕ, g) on V satisfying $l = 0$. This family is determined by G and three smooth functions $f_i(z)$, $i = 1, 2, 3$, ($f_3 \neq 0$ everywhere) on V.*

Proof. The equation $\frac{\partial^2 G}{\partial x^2} = 0$ implies $G(x, y, z) = 2xC_1(y, z) + C_2(y, z)$, where C_1, C_2 are arbitrary smooth functions of y, z on V. Now, we consider the linearly independent vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y},$$

where a, b, c are the smooth functions defined by (9) on V and $f_i(z)$, $i = 1, 2, 3$, ($f_3 \neq 0$ everywhere) are smooth functions on V. Let g be the Riemannian metric on V defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$, and ξ, η, ϕ the tensor fields defined by

$$\xi = \frac{\partial}{\partial x}, \quad \eta(.) = g(., \xi), \quad \phi\xi = 0, \quad \phi e_2 = e_3, \quad \text{and} \quad \phi e_3 = -e_2.$$

We easily find that $\eta \wedge d\eta \neq 0$, $\phi^2 Z = -Z + \eta(Z)\xi$, $d\eta(Z,W) = g(Z,\phi W)$ and $g(\phi Z,\phi W) = g(Z,W) - \eta(Z)\eta(W)$ for any $Z,W \in \mathcal{X}(M)$. Hence, $V(\eta,\xi,\phi,g)$ is a contact metric manifold. Using the definitions of e_1, e_2, e_3 and (9), we calculate

$$[e_1,e_2] = 2e_3, \quad [e_1,e_3] = 0, \quad [e_2,e_3] = 2e_1 - C_1e_2 + (2xC_1 + C_2)e_3.$$

Moreover, if ∇ is the Levi-Civita connection of g, then using the above formulas for $[e_i,e_j]$, $g(e_i,e_j) = \delta_{ij}$ and the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

(which is valid on any Riemannian manifold), we obtain

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = 0, \quad \nabla_{e_2}e_1 = -2e_3$$

and finally

$$le_2 = R(e_2,e_1)e_1 = 0, \quad le_3 = R(e_3,e_1)e_1 = 0.$$

Hence, $V(\eta,\xi,\phi,g)$ defines a family of 3-dimensional contact metric manifolds with $l = 0$.

4. Global results

Lemma 4.1. *On any M_l-manifold $M(\eta,\xi,\phi,g)$ the scalar curvature is given by*

$$S = div\phi hQ\xi. \tag{12}$$

Proof. Using the first of (6) we find

$$\phi hQ\xi = \phi h(2AX + 2B\phi X) = 2(A\phi X + BX)$$

and so, from this, (3) and (7) we obtain

$$\begin{aligned} div\phi hQ\xi &= 2div(A\phi X + BX) \\ &= 2(Adiv\phi X + \phi XA + BdivX + XB) \\ &= 2(-A^2 + \phi XA - B^2 + XB) = S. \end{aligned}$$

An immediate consequence of the Lemma 4.1 and of the divergence theorem is the following proposition concerning closed (compact without boundary) contact manifolds.

Proposition 4.2. *On any 3-dimensional closed contact manifold (M,η) there is no associated metric with $l = 0$ and strictly positive (or strictly negative) scalar curvature.*

In Proposition 4.2 the closeness assumption is of vital importance. In the next example, a contact form η is defined on R^3, such that, there are associated metrics with $l = 0$ and strictly positive (or strictly negative) scalar curvature.

Example 4.1. We consider on R^3 the contact form $\eta = dx + \frac{2y}{1+z^2}dz$ and an arbitrary function $f(y,z)$ of variables y, z. The tensor fields (η, ξ, ϕ, g), where $\xi = \frac{\partial}{\partial x}$, $g = (g_{ij})$:

$$g_{11} = g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = 2y(1+z^2)^{-1}$$
$$g_{23} = g_{32} = \frac{yf - 2x}{1+z^2}, \quad g_{33} = \frac{1 + 4y^2 + (yf-2x)^2}{(1+z^2)^2}$$

and

$$\phi = (\phi_{ij}) : \quad \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2y,$$
$$\phi_{22} = yf - 2x, \quad \phi_{32} = -(1+z^2)$$
$$\phi_{13} = \frac{2y(yf-2x)}{1+z^2}, \quad \phi_{23} = \frac{1+(yf-2x)^2}{1+z^2}, \quad \phi_{33} = 2x - yf$$

define a contact metric structure on R^3 with $l = 0$ and scalar curvature

$$S = 2\{(2x - yf)(2f_y + yf_{yy}) + (1+z^2)(f_z + yf_{yz}) - (yf_y + f)^2\},$$

where $f_y = \frac{\partial f}{\partial y}$. Choosing the function f properly we achieve associated metrics to η with scalar curvature of any sign. For example:

a) If $f = -z$, then $S = -2(1+2z^2) < 0$.

b) If $f = z$, then $S = 2$.

c) If $f = a$ (const.) $\neq 0$, then $S = -2a^2$. (If $a = 0$, then M is flat).

d) If $f = atan(atan^{-1}z)$, $a =$ const. > 0 and if we restrict ourselves to the set $D = \{(x,y,z) \in R^3 | -\frac{\pi}{2a} < tan^{-1}z < \frac{\pi}{2a}\}$, then $S = 2a^2 > 0$.

The next theorem concerns closed M_l-manifolds.

Theorem 4.3. *Let $M(\eta, \xi, \phi, g)$ be a closed M_l-manifold. If $S \geq 0$ or $S \leq 0$, then M is flat.*

Proof. Using (12) and the divergence theorem we obtain $S = 0$. Hence, from Lemma 2.2 and the second of (6) we get $A = 0$. The latter, (5), (7), (8) and the first of (6) yield

$$\xi B = \phi XB = 0, \quad XB = B^2 \quad \text{and} \quad g(Q\xi, Q\xi) = 4B^2.$$

From the last relation, it follows that the function B^2 is defined and differentiable on M. Calculating the Laplacian of B^2 and using the above relations, we easily obtain

$$\Delta B^2 = 4B^4.$$

Using the divergence theorem once more, we get, from the last relation, $B = 0$. The relations $A = B = S = 0$ and (6) imply $Q = 0$ and so M is flat. This completes the proof of the theorem.

Remark 1. An example of a 3-dimensional closed metric manifold with $l = 0$ is the 3-torus T^3 with contact form $\eta = \frac{1}{2}(coszdx + sinzdy)$ and the associated flat metric $g_{ij} = \frac{1}{4}\delta_{ij}$ (see [2, p. 68]). Concerning 3-dimensional flat contact metric manifolds, Rukimbira [9] showed that a closed flat contact metric manifold is isometric to the quotient for a flat 3-torus by a finite cyclic group of isometries of order 1,2,3,4 or 6.

We note that the assumption of closeness in Theorem 4.3 is crucial. In the following example, a non-closed, non-flat M_l-manifold with $S = 0$ is given.

Example 4.2. The tensor fields (η, ξ, ϕ, g), where $\eta = dx + 2ye^{-z}dz, \xi = \frac{\partial}{\partial x}$,

$$g = \begin{pmatrix} 1 & 0 & 2ye^{-z} \\ 0 & 1 & 2xe^{-z} - y \\ 2ye^{-z} & 2xe^{-z} - y & \{1 + 4y^2 + (2x - ye^z)^2\}e^{-2z} \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} 0 & 2y & 2(y^2 - 2xye^{-z}) \\ 0 & ye^z - 2x & (1 + (2x - ye^z)^2)e^{-z} \\ 0 & -e^z & 2x - ye^z \end{pmatrix}$$

define on R^3 a non-flat contact metric manifold with $l = 0$ and $S = 0$.

In order to prove the next theorem we recall the following well known result of Lie group theory (see for instance [8, Lemma 2.5]).

Lemma 4.4. *Let (M, g) be an n-dimensional complete, simply connected Riemannian manifold and let $X_1, X_2, \ldots, X_n$ be orthonormal vector fields, satisfying*

$$[X_i, X_j] = \sum_k c_{ij}^k X_k$$

where the coefficients c_{ij}^k are constant. Then, for any point $P \in M$, the manifold M has a unique Lie group structure, such that P is the identity, the vector fields X_i and the Riemannian metric g are left invariant.

Theorem 4.5. *Let $M(\eta, \xi, \phi, g)$ be a M_l-manifold with $||Q\xi|| = c$ (constant, $c \geq 0$). If $c = 0$, then M is flat. If $c > 0$ and M is complete and simply connected, then for each point $P \in M$, the manifold M has a unique Lie group structure, such that P is the identity, the orthonormal vector fields $\xi, \frac{1}{c}Q\xi, -\frac{1}{c}\phi Q\xi$ and the Riemannian metric g are left invariant. Moreover, M has constant negative scalar curvature $S = -\frac{c^2}{2}$.*

Proof. From the hypothesis $||Q\xi|| = c$ and the first of (6) we obtain

$$4(A^2 + B^2) = c^2. \tag{13}$$

Differentiating (13) with respect to ξ and using (5) we get $AB = 0$. Similarly, differentiating the last equation and using (5) and (13) we find

$$A = 0 \quad \text{and} \quad B^2 = \frac{c^2}{4}. \tag{14}$$

If $c = 0$, then $B = 0$, and from (7) and (14) we have $S = 0$, and so M is flat. Now, we suppose $c \neq 0$. Then, from (7) and (14) we get $S = -\frac{c^2}{2}$. Using the first of (6) we find that the vector field $\phi X = \frac{1}{c}Q\xi$ is globally defined and differentiable, and so is $X = -\frac{1}{c}\phi Q\xi$. The rest of the proof is an immediate consequence of (4) and Lemma 4.4.

Remark 2. Using (13), (14), (7), (12) and the divergence theorem we easily get that the only closed M_l-manifolds with $||Q\xi|| =$ constant, are the flat ones.

In the next example the structure of a Lie group contact metric manifold with $l = 0$ and $||Q\xi|| =$ constant is given on R^3.

Example 4.3. We consider the manifold $M = R^3$ and the vector fields

$$e_1 = \frac{\partial}{\partial x}, e_2 = (f_1(z) - 2y)\frac{\partial}{\partial x} + (2x - \kappa y + f_2(z))\frac{\partial}{\partial y} + f_3(z)\frac{\partial}{\partial z}, e_3 = \frac{\partial}{\partial y},$$

where f_1, f_2, f_3, ($f_3 \neq 0$ everywhere) are arbitrary smooth functions of z and $\kappa =$ const. $\neq 0$. We define the tensor fields ξ, η, ϕ, g by $\xi = e_1$, $g(e_i, e_j) = \delta_{ij}$, $i, j = 1, 2, 3$, $\eta(X) = g(e_1, X)$ for any $X \in \mathcal{X}(\mathcal{M})$, $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. The M_l-manifold $M(\eta, \xi, \phi, g)$ is a Lie group with scalar curvature $S = -2\kappa^2$. If $\kappa = 0$, then M is flat.

In the following example we construct an M_l-manifold $M(\eta, \xi, \phi, g)$ with $||Q\xi|| =$ constant on an open subset U of M. The scalar curvature of this manifold is not constant on $M - U$.

Example 4.4. On $M = R^3$, we consider the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y},$$

where

$$a = \begin{cases} -2e^{\frac{1}{z^2}}(-1 + e^{ye^{-\frac{1}{z^2}}}), & z > o \\ -2y, & z \leq 0 \end{cases}$$

$$b = \begin{cases} -\kappa e^{\frac{1}{z^2}}(-1 + e^{ye^{-\frac{1}{z^2}}}) + 2x, & z > o \\ 2x - \kappa y, \quad (\kappa = \text{const.}) & z \leq 0 \end{cases}$$

$$c = \begin{cases} e^{ye^{-\frac{1}{z^2}}}, & z > o \\ 1, & z \leq 0. \end{cases}$$

Defining ξ, η, ϕ, g as in the Example 4.3, then $M(\eta, \xi, \phi, g)$ is a contact metric manifold with $l = 0$ and scalar curvature

$$S = \begin{cases} 2\{4 - 4e^{ye^{-\frac{1}{z^2}}} - e^{-\frac{2}{z^2}} - (\kappa + 2xe^{-\frac{1}{z^2}})^2 + \frac{4xe^{(ye^{-\frac{1}{z^2}} - \frac{1}{z^2})}}{z^3}\}, & z > o \\ -2\kappa^2, & z \leq 0. \end{cases}$$

We denote that M has an open subset $U = \{(x, y, z) \epsilon R^3 | z < 0\}$, which is the restriction of a Lie group (see Example 4.3 for $f_1 = f_2 = 0, f_3 = 1$).

Remark 3. According to Lemma 2.2, the contact metric manifolds with $l = 0$ and constant scalar curvature S are those for which $S = 4$ or $QY \in [+1]$ for any $Y \in [+1]$. In cases (b) and (c) of Example 4.1 we have $S = c$ (const.)$\neq 4$ and $QY \in [+1]$ for any $Y \in [+1]$. In case (d) for $a = \sqrt{2}$ of the same example, we have the coexistence of $S = 4$ and $QY \in [+1]$ for any $Y \in [+1]$. The following example also shows the existence of case $S = 4$ and $QY \notin [+1]$ for any $Y \in [+1]$.

Example 4.5. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 | y - z > 0$. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = (z - y)\frac{\partial}{\partial x} + (2x + \frac{1}{y - z})\frac{\partial}{\partial y} + \frac{1}{y - z}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y}$$

are linearly independent at each point of M. We define ξ, η, ϕ, g as in the Example 4.3. The manifold $M(\eta, \xi, \phi, g)$ is a contact metric manifold with $l = 0$ and scalar curvature $S = 4$. For the tensor fields h and Q we have $he_2 = e_2$, $he_3 = -e_3$, $Qe_2 = \frac{2}{z-y}e_1 + 2e_2$, and so $Qe_2 \notin [+1]$, while $e_2 \in [+1]$.

Remark 4. i) On a contact metric manifold the operator τ defined by $\tau = \mathcal{L}_\xi g$ plays an interesting role. Using $\nabla_\xi h = \phi - \phi l - \phi h^2$ and $\phi l \phi - l = 2(\phi^2 + h^2)$ it follows that the conditions

$$\nabla_\xi \tau = 0, \quad \nabla_\xi h = 0, \quad \phi l = l\phi$$

are equivalent ([7]). A 3-dimensional contact metric manifold, on which $Q\phi = \phi Q$, satisfies $\phi l = l\phi$, but not conversely. Examples of contact metric manifolds with $\phi l = l\phi$ and $Q\phi \neq \phi Q$ were initially given by Blair [2, p. 183] and later by Calvaruso-Perrone [4]. We note that all M_l-manifolds with $B \neq 0$ satisfy $\phi l = l\phi$ and $Q\phi \neq \phi Q$ as follows from (6). For 3-dimensional contact metric manifolds with $Q\phi = \phi Q$ see [3].

ii) For closed contact metric manifolds, Perrone [6] has proved the following:

"Let (M, η) be a 3-dimensional closed contact manifold (M, η). Then a metric $g \in \mathcal{A}(\eta)$ is a critical point for the functional

$$I(g) = \int_M S dv, \quad g \in \mathcal{A}(\eta)$$

if and only if $\nabla_\xi \tau = 0$."

Using this result we conclude that the Riemannian metric of any closed M_l-manifold is a critical point for $I(g)$.

Remark 5. On an M_l-manifold the eigenfunctions of the Ricci operator are given by

$$\lambda_1 = \frac{S}{2},\ \ \lambda_2 = \frac{1}{2}(\kappa + \sqrt{\kappa^2 + 16(A^2 + B^2)}),\ \ \lambda_3 = \frac{1}{2}(\kappa - \sqrt{\kappa^2 + 16(A^2 + B^2)}),$$

where $\kappa = \frac{S}{2}$. The manifolds of Theorem 4.5 are homogeneous, but in general, an M_l-manifold is not homogeneous or, more generally, non curvature homogeneous ([10]), since the eigenfunctions $\lambda_1, \lambda_2, \lambda_3$ are not constant. The contact metric manifolds of the Examples 4.1(c) and 4.3 are curvature homogeneous. For 3-dimensional homogeneous contact metric manifolds see [8].

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