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The Theta Ideal, Dense Submodules and the Forcing Linearity Number for a Multiplication Module

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Abstract. In this article we study the theta ideal of a multiplication module and obtain some results involving them. We further look at the dense submodules of a multiplication module and show that they are themselves multiplication. We also show that the forcing linearity number of a multiplication module is zero.

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1. Introduction

Let R be a commutative ring with unity and let M be a (unitary) multiplication R-module. Let I be an ideal of R.

The theta ideal $\theta(M)$ and the trace ideal $T(M)$ of M are defined to be $\sum_{x\in M}(Rx : M)$ and $\sum_{f\in M^*} f(M)$ respectively. Emphasis in this article is on the properties and the applications of the ideal $\theta(M)$.

It is known (see, [1, Theorem 2.6]) that if M is faithful then $\theta(M) = T(M)$. We prove the converse of this result, that is, if $\theta(M) = T(M)$ then M is faithful. In [1, Lemma 2.1], it has been shown that if I is finitely generated and $I \subseteq \theta(M)$ then IM is finitely generated. We show that if M is faithful and IM is finitely

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generated then I is also finitely generated (see, Lemma 2.6). We further show that if M is faithful then IM is multiplication if and only if $I\theta(M)$ is a multiplication ideal of R (see, Lemma 2.15). As a consequence of Lemma 2.15, we show that if J and J' are ideals of R such that JM and $J'M$ are multiplication then so is $JJ'M$. We also obtain some known results regarding M as corollaries using the properties of the ideal $\theta(M)$.

Let N be a submodule of M. Then N is said to be dense in M if $\sum f(N) = M$, where the summation is taken over all $f \in \mathrm{Hom}_R(N, M)$. We first show that if N is dense in M then N is multiplication. We further show that N is dense in M if and only if $\theta(N) = \theta(M)$ and $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$. As a consequence of these results, we show that if N is dense in M then N is finitely generated if and only if M is finitely generated. These results give complete characterization of dense submodules of multiplication modules and improve upon several results in this direction, for example, Corollary 8 and Theorem 9 of [11].

In [10], some partial result have been obtained about the forcing linearity number of M. We show that this number is always zero for multiplication modules.

2. The trace and theta ideals

Throughout this article, rings are assumed to be commutative with unity and modules are assumed to be unitary.

In this section we study the trace ideal and the theta ideal of a multiplication module and obtain some results involving them.

Henceforth, let R denote a ring. We recall the following definition:

Definition 2.1. *An R-module M is said to be a multiplication R-module if for each submodule N of M there exists some ideal I of R such that $N = IM$.*

An ideal J of R is said to be a multiplication ideal of R if J is a multiplication R-module.

Remark 2.2. Let M be an R-module.

(a) Let $I \subseteq \mathrm{ann}_R(M)$ be an ideal. Then M is a multiplication R-module if and only if M is a multiplication R/I-module. Therefore, in many applications, one may assume that M is faithful by treating it as an $R/\mathrm{ann}_R(M)$-module.

(b) Now assume that M is a multiplication R-module. Then all quotients of M are multiplication R-modules. Furthermore, if $S \subset R$ is a multiplicative set then $S^{-1}M$ is a multiplication $S^{-1}R$-module. If N is a submodule of M then $N = (N : M)M$.

Notation. For an R-module M let $M^* = \mathrm{Hom}_R(M, R)$.

Definition 2.3. *Let M be an R-module. Then we have*

$$\begin{aligned} T(M) &= \textstyle\sum_{f\in M^*} f(M) \\ \tau(M) &= \cap\{I \mid I \subseteq R \textit{ is an ideal, } \mathrm{ann}_R(M) \subseteq I \textit{ and } M = IM\} \\ \theta(M) &= \textstyle\sum_{x\in M} (Rx : M) \\ D_0(M) &= \textstyle\sum_{x\in M} \mathrm{ann}_R(\mathrm{ann}_R(x)) \end{aligned}$$

The ideal $T(M)$ is called the trace ideal of M. The ideals $\tau(M)$, $\theta(M)$ and $D_0(M)$ are respectively known as τ-ideal, θ-ideal and D_0-ideal of M.

The following lemma is a collection of some elementary observations.

Lemma 2.4. *Let M be a multiplication R-module and let N be a submodule of M. Then we have the following:*

(a) *$M = \theta(M)M$ and $N = \theta(M)N$.*

(b) *If N is finitely generated then there exists some $a \in \theta(M)$ such that $(1-a)N = 0$.*

(c) *$\mathrm{ann}_R(\theta(M)) \subseteq \mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(T(M))$.*

(d) *If M is faithful then so is $\theta(M)$.*

(e) *If I is an ideal of R such that $M = IM$ then $T(M) = IT(M)$.*

(f) *Let $I \subseteq \mathrm{ann}_R(M)$ be an ideal. Denote M by M' while treating M as an R/I-module. Then $\theta(M') = \theta(M)/I$.*

In [1, Theorem 2.6], it is shown that if M is a faithful multiplication R-module then $T(M) = \theta(M)$. However, if $\mathrm{ann}_R(M)$ contains a nonzero divisor of R then $\theta(M) \neq 0 = T(M)$. Hence, in general, $\theta(M)$ does not equal $T(M)$. Naturally, one faces the following question: For a multiplication module M, what are the necessary and sufficient conditions for $T(M) = \theta(M)$? We answer this question in Lemma 2.5.

Lemma 2.5. *Let M be a multiplication R-module. Then $T(M) = \theta(M)$ if and only if M is faithful.*

Proof. We only need to prove the necessity. Note that by Lemma 2.4, $\mathrm{ann}_R(\theta(M)) = \mathrm{ann}_R(M)$. Let $a \in \mathrm{ann}_R(M)$. Then there exist $f_1, \dots, f_n \in M^*$ and $x_1, \dots, x_n \in M$ such that $a = \sum_{i=1}^n f_i(x_i)$. Let $N = \sum_{i=1}^n Rx_i$. Then, by Lemma 2.4, there exists some $b \in \theta(M)$ such that $(1-b)N = 0$. In particular, $x_i = bx_i$ for all $i = 1, \dots, n$. Thus $a = \sum_{i=1}^n f_i(x_i) = \sum_{i=1}^n f_i(bx_i) = ba$. As $a\theta(M) = 0$, we have $a = 0$. □

In [1, Lemma 2.1], it has been shown that if M is a multiplication R-module and $I \subseteq \theta(M)$ is a finitely generated ideal then IM is finitely generated. In the following result we prove the converse of the same.

Lemma 2.6. *Let M be a faithful multiplication R-module and let I be an ideal of R such that IM is finitely generated. Then I is finitely generated.*

Proof. There exist $a_1, \dots, a_n \in I$ and $x_1, \dots, x_n \in M$ such that $IM = \sum_{i=1}^n Ra_ix_i$. By [1, Lemma 2.1], $I \subseteq \theta(M)$. As M is faithful, by [1, Theorem 2.3], $I = I\theta(M)$. Furthermore, by [1, Theorem 2.6], $T(M) = \theta(M)$. Hence $I = I\theta(M) = IT(M) = I\sum_{f\in M^*} f(M) = \sum_{f\in M^*} f(IM) \subseteq (a_1, \dots, a_n)$. Thus $I = (a_1, \dots, a_n)$. □

Therefore, Lemma 2.6, together with [1, Lemma 2.1], leads to the following result:

Theorem 2.7. *Let M be a faithful multiplication R-module and let I be an ideal of R. Then IM is finitely generated if and only if $I \subseteq \theta(M)$ and I is finitely generated.*

In the next result we list a property of the ideal $\theta(M)$.

Lemma 2.8. *Let M be a multiplication R-module and let P be a prime ideal of R. Then $\theta(M) \subseteq P$ if and only if $M_P = 0$ and $\mathrm{ann}_R(M) \subseteq P$.*

Proof. First suppose that $M_P = 0$ and $\mathrm{ann}_R(M) \subseteq P$. Let $x \in M$. Then $(Rx)_P = 0$ and therefore, there exists some $s \in R \setminus P$ such that $sRx = 0 = s(Rx\colon M)M$. Thus $s(Rx\colon M) \subseteq \mathrm{ann}_R(M) \subseteq P$. Clearly, $(Rx\colon M) \subseteq P$. As x is arbitrary, we have $\theta(M) \subseteq P$.
Conversely, suppose that $\theta(M) \subseteq P$. As $M = \theta(M)M$, we get $M_P = \theta(M)_P M_P$. By [3, Proposition 4], M_P is cyclic and therefore, by Nakayama Lemma, $M_P = 0$. Furthermore, $\mathrm{ann}_R(M) \subseteq \theta(M) \subseteq P$. □

We now recall a definition.

Definition 2.9. *A proper submodule N of an R-module M is said to be a prime submodule of M if $(N\colon M)$ is a prime ideal and M/N is a torsion-free module over the integral domain $R/(N\colon M)$.*

If N is a prime submodule of M then N is also referred as a P-prime submodule of M, where $P = (N\colon M)$.

In, [4, Corollary 2.11], a characterization of prime submodules of a multiplication module is given. We obtain the same result as a consequence of Lemma 2.8. Compare the proof given below with that of [4, Corollary 2.11].

Corollary 2.10. *Let M be a multiplication R-module and let P be a prime ideal of R such that $\mathrm{ann}_R(M) \subseteq P$. Then PM is a prime submodule of M if and only if $M \neq PM$.*

Proof. First suppose that $M \neq PM$. Clearly, $P \subseteq (PM\colon M)$. We claim that $(PM\colon M) = P$. Assume the contrary. Choose some $a \in (PM\colon M) \setminus P$. Then $aM \subseteq PM$. Therefore, $M_P = (aM)_P = PM_P$. By [3, Proposition 4], M_P is cyclic and therefore, by Nakayama Lemma, $M_P = 0$. Now, by Lemma 2.8, $\theta(M) \subseteq P$ and hence $M = PM$, a contradiction. Thus, M/PM is a faithful multiplication R/P-module and therefore, by lemma [4, Lemma 4.1], a torsion-free R/P-module. Hence PM is a P-prime submodule of M. Converse is trivial as every prime submodule is proper. □

As a further application of Lemma 2.8, we give an equivalent condition for an R-module to be multiplication. Note that it is essentially a reformulation of [4, Theorem 1.2].

Theorem 2.11. *Let M be an R-module. Then the following statements are equivalent:*

(a) *M is a multiplication R-module.*

(b) *For every maximal ideal $\mathfrak{m}$ of R either $M_{\mathfrak{m}} = 0$ or $\theta(M) \nsubseteq \mathfrak{m}$.*

Proof. The implication (a) to (b) follows by Lemma 2.8. We now prove (b) implies (a). Let $\mathfrak{m}$ be a maximal ideal of R. Suppose that $M_{\mathfrak{m}} = 0$. Then for every $x \in M$ there exists some $a \in \mathfrak{m}$ such that $(1-a)x = 0$. Now suppose that $\theta(M) \nsubseteq \mathfrak{m}$. Then there exists some $x \in M$ such that $(Rx\colon M) \nsubseteq \mathfrak{m}$, that is, $\mathfrak{m} + (Rx\colon M) = R$. Therefore, there exists some $a \in \mathfrak{m}$ such that $1-a \in (Rx\colon M)$. Thus $(1-a)M \subseteq Rx$. Now (a) follows by [4, Theorem 1.2]. □

In [1, Corollary 2.2], it has been shown that if M is a multiplication R-module then M is finitely generated if and only if $\theta(M) = R$. The next result is a refinement of the same.

Corollary 2.12. *Let M be an R-module. Then $\theta(M) = R$ if and only if M is a finitely generated multiplication R-module.*

Proof. The necessity follows by Theorem 2.11 and by [1, Corollary 2.2], whereas the sufficiency follows by [1, Corollary 2.2]. □

We have another corollary to Theorem 2.11.

Corollary 2.13. *Let M be a multiplication R-module and let N be a submodule of M. If $\theta(N) = \theta(M)$ then N is also multiplication.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Assume that $N_{\mathfrak{m}} \neq 0$. Then $M_{\mathfrak{m}} \neq 0$. Therefore, by Theorem 2.11, $\theta(N) = \theta(M) \nsubseteq \mathfrak{m}$. Now, again by Theorem 2.11, N is multiplication. □

By [1, Theorem 2.3], if M is a faithful multiplication R-module then $\theta(M)$ is an idempotent multiplication ideal of R. In addition, by [1, Theorem 2.6] and by Lemma 2.4, $\theta(M)$ is faithful. Naturally, one asks the following question: Given an idempotent, faithful, multiplication ideal I of R, does there exist a multiplication R-module M such that $\theta(M) = I$? We answer this question in the next result.

Lemma 2.14. *Let I be an idempotent, faithful, multiplication ideal of R. Then there exists a faithful multiplication R-module M such that $I = \theta(M)$.*

Proof. Take $M = I$. As I is faithful and $I = I^2$, by [1, Theorem 2.6], we have $\theta(I) \subseteq I$. On the other hand, $I = \theta(I)I \subseteq \theta(I)$ and hence $I = \theta(I)$. □

By [4, Corollary 1.4], if M is a multiplication R-module and I is a multiplication ideal of R then IM is a multiplication R-module. In the next result we prove a variant and a converse of the same.

Lemma 2.15. *Let M be a faithful multiplication module over R and let I be an ideal of R. Then IM is a multiplication R-module if and only if $I\theta(M)$ is a multiplication ideal of R.*

Proof. First assume that IM is a multiplication R-module. Let $J \subseteq I\theta(M)$ be an ideal. As $JM \subseteq I\theta(M)M = IM$, there exists some ideal J_0 of R such that $JM = J_0IM$. As M is faithful, by [1, Theorem 2.6], we have $\theta(M) = T(M)$ and by [1, Theorem 2.3], $J = J\theta(M)$. Therefore, $J = J\theta(M) = JT(M) = J\sum_{f\in M^*} f(M) = \sum_{f\in M^*} f(JM) = \sum_{f\in M^*} f(J_0IM) = J_0I\theta(M)$. Thus $I\theta(M)$ is a multiplication ideal of R.

Conversely, assume that $I\theta(M)$ is a multiplication ideal of R. Then $IM = I\theta(M)M$ is a multiplication R-module, by [4, Corollary 1.4]. Note that for the converse we do not need faithfulness of M. □

As a consequence of Lemma 2.15, we get the next result.

Corollary 2.16. *Let M be a multiplication R-module and let I and J be ideals of R. If IM and JM are multiplication R-modules then so is IJM.*

Proof. Put $I_0 = \mathrm{ann}_R(M)$, $R' = R/I_0$, $I' = (I + I_0)/I_0$ and $J' = (J + I_0)/I_0$. Then M, IM and JM are multiplication R'-modules. We shall write M' for M, when we treat M as an R'-module. Note that $I'M' = IM$, $J'M' = JM$ and that M' is a faithful R'-module.

Now, by Lemma 2.15, $I'\theta(M')$ is a multiplication ideal of R'. Therefore, by [4, Corollary 1.4], $I'\theta(M')J'M'$ is a multiplication R'-module and hence a multiplication R-module. Clearly, $I'\theta(M')J'M' = I'J'M' = IJM$. □

In [5, Lemma 1.5], it is shown that if M is a multiplication R-module and N is a finitely generated submodule of M then M/IM is a finitely generated R/I-module, where $I = \mathrm{ann}_R(N)$. In the next result we show that M/IM is also faithful over R/I. In fact, we do not even assume that N is finitely generated.

Lemma 2.17. *Let M be a multiplication R-module and let N be a submodule of M. Let $I = \mathrm{ann}_R(N)$. Then M/IM is a faithful R/I-module.*

Proof. We may assume that $N \neq 0$. As $\mathrm{ann}_R(M) \subseteq I$, we may also assume that M is faithful. Let $a \in (IM : M)$. Then $aM \subseteq IM$. Let $f \in T(M)$. Then $af(M) \subseteq If(M)$. Thus, $af(M)N = 0$. As f is arbitrary, we get $aT(M)N = 0$. Now, by [1, Theorem 2.6], $T(M) = \theta(M)$ and by Lemma 2.4, $N = \theta(M)N$. Therefore, $aN = a\theta(M)N = aT(M)N = 0$. Thus $a \in I$. Hence the assertion. □

Continuing in the same vein we investigate the conditions for a quotient of a multiplication module to be finitely generated.

Lemma 2.18. *Let M be a multiplication R-module and let N be a submodule of M. Then M/N is finitely generated if and only if $(N :_R M) + \theta(M) = R$.*

Proof. First suppose that M/N is finitely generated. As $M/N = \theta(M)(M/N)$, there exists some $a \in \theta(M)$ such that $(1-a)(M/N) = 0$, that is, $1-a \in (N :_R M)$. Thus $(N :_R M) + \theta(M) = R$.

Conversely, suppose that $(N :_R M)+\theta(M) = R$. Then there exists some $a \in \theta(M)$ such that $1 - a \in (N :_R M)$. Let $\phi\colon aM \longrightarrow M/N$ be the natural map, that is,

$\phi(ax) = ax + N$. As $(1 - a)M \subseteq N$, we have $\phi(ax) = x + N$. Thus ϕ is onto. As $a \in \theta(M)$, by [1, Lemma 2.1], aM is finitely generated and therefore, so is M/N. $\square$

The following corollary improves on [5, Lemma 1.5]:

Corollary 2.19. *Let M be a multiplication R-module and let N be a submodule of M. Let $I = \operatorname{ann}_R(N)$. Then M/IM is finitely generated if and only if N is contained in some finitely generated submodule of M.*

Proof. By Lemma 2.17, $(IM :_R M) = I$. First assume that M/IM is finitely generated. Then by Lemma 2.18, $I + \theta(M) = R$. Therefore, there exists some $a \in \theta(M)$ such that $1 - a \in I$, that is, $(1 - a)N = 0$. As a consequence, we have $N = aN \subseteq aM$. Note that aM is finitely generated by [1, Lemma 2.1].

Conversely, assume that N is contained in a finitely generated submodule K of M. Then by Lemma 2.4, there exists some $a \in \theta(M)$ such that $(1 - a)K = 0$. Therefore, $(1 - a)N = 0$, that is, $1 - a \in I$. Thus $I + \theta(M) = R$. Now, by Lemma 2.18, M/IM is finitely generated. $\square$

In Corollary 2 of [5, Theorem 1.3], it is shown that if M is a finitely generated faithful multiplication R-module then M^* is a (finitely generated faithful) multiplication R-module. We now give an example to show that if M is not finitely generated, then this result is not true, in general.

Example 2.20. Let k be a field and let $x_1, x_2, x_3, \ldots$ be indeterminates. Put $A = k[x_1, x_2, x_3, \ldots]$. Let I_0 denote the ideal of A generated by the set $\{x_i^2 - x_i \mid i \in \mathbb{N}\} \cup \{x_i x_j \mid i, j \in \mathbb{N}, i \neq j\}$ and let $R = A/I_0$. Put $e_i = x_i + I_0$. Let $I = (e_1, e_2, e_3, \ldots)$. As I is generated by idempotents, I is multiplication. Note that I is faithful and idempotent. Further note that I is a vector-space over k with basis $\{e_1, e_2, e_3, \ldots\}$.

One easily checks that

$$I^* = \{f : I \longrightarrow R \mid f(e_i) = a_i e_i \text{ for some unique } a_i\text{s in } k, i \geq 1\}.$$

Therefore, I^* is isomorphic to $k^\infty = k \times k \times k \times \cdots$ as a vector-space over k.

Note that if I^* is generated by a set S as an R-module then I^* is generated by $S \cup \{e_i f \mid f \in S, i \geq 1\}$ as a vector-space over k. As the cardinality of any basis of k^∞ over k is uncountable, I^* is not finitely generated as an R-module.

We claim that I^* is not multiplication. Assume the contrary. Let $\phi \in I^*$ denote the inclusion of I into R. Note that $\operatorname{ann}_R(\phi) = 0$. Therefore, by Corollary 1 of [5, Lemma 1.5], I^* is finitely generated, a contradiction.

Note that $I^* = \operatorname{End}_R(I)$. Thus, in general, $\operatorname{End}_R(M)$ is not multiplication for a multiplication R-module M. Note further that if M is finitely generated multiplication then $\operatorname{End}_R(M) \cong R/\operatorname{ann}_R(M)$ by [8, Corollary 3.3] and therefore, a multiplication R-module.

In the next result we look at the trace ideal of the tensor product of multiplication modules. Note that the tensor product of two multiplication modules is also multiplication, by [2, Theorem 2.1].

Lemma 2.21. *Let M and N be R-modules. Then we have the following:*
(a) $T(M)T(N) \subseteq T(M \otimes_R N) \subseteq T(M) \cap T(N)$.
(b) *If M (or N) is faithful and multiplication then $T(M \otimes_R N) = T(M)T(N)$.*
(c) *If both M and N are faithful and multiplication then*

$$T(M \otimes_R N) = T(M)T(N) = T(M)T(N)T(M \otimes_R N).$$

Proof. Let $f \in M^*$ and $g \in N^*$. Define $h\colon M \otimes_R N \longrightarrow R$ by setting $h(x \otimes y) = f(x)g(y)$. Clearly, h is R-linear. Therefore, $f(M)g(N) \subseteq T(M \otimes_R N)$. Thus $T(M)T(N) \subseteq T(M \otimes_R N)$.

Now, let $\phi \in (M \otimes_R N)^*$. Fix $y \in N$. Define $\psi_y\colon M \longrightarrow R$ by setting $\psi_y(x) = \phi(x \otimes y)$. Then ψ_y is R-linear and therefore, $\phi(x \otimes y) \in T(M)$ for all $(x, y) \in M \times N$. It follows that $T(M \otimes_R N) \subseteq T(M)$. Similarly, $T(M \otimes_R N) \subseteq T(N)$. This proves (a).

We now prove (b). By [1, Theorem 2.6], $T(M) = \theta(M)$. Therefore, by [1, Theorem 2.3] and part (a), $T(M \otimes_R N) = T(M)T(M \otimes_R N)$. Thus, again by part (a), $T(M)T(N) \subseteq T(M \otimes_R N) = T(M)T(M \otimes_R N) \subseteq T(M)T(N)$.

Part (c) is immediate from the fact that $T(M)T(N) \subseteq T(M \otimes_R N) \subseteq T(M) \cap T(N)$ and [1, Theorem 2.3]. □

Remark 2.22. If M and N are multiplication R-modules then, in general, $T(M \otimes_R N) \neq T(M)T(N)$ as can be seen from the following example: Let k be a field and $k[x]$ be the polynomial ring in indeterminate x. Let $R = k[x]/(x^2)$ and let y denote the natural image of x in R. Let $M = R/(y)$. Then $T(M) = (y)$ and $T(M \otimes_R M) = (y) \neq 0 = T(M)T(M)$.

However, the following result is immediate from Lemma 2.21.

Corollary 2.23. *If M and N are faithful multiplication R-modules then so is $M \otimes_R N$.*

Proof. Put $L = M \otimes_R N$ and $I = \operatorname{ann}_R(L)$. By [2, Theorem 2.1], L is multiplication. We now prove that $I = 0$. By Lemma 2.4, $IT(L) = 0$. Therefore, by Lemma 2.21, $IT(M)T(N) = 0$. As M and N are faithful, by [1, Theorem 2.6], we have $T(M) = \theta(M)$ and $T(N) = \theta(N)$. Again, by Lemma 2.4, $\theta(M)$ and $\theta(N)$ are faithful. Therefore, $I = 0$. □

3. A neat proof

Let M be a multiplication R-module. In [9, Theorem 2.4], the equality of $T(M)$ and $D_0(M)$ is proved. Incorporating this result as a part of the proof, in [1, Theorem 2.6], it is shown that if M is faithful then $T(M) = \tau(M) = \theta(M) = D_0(M)$. However, the proof in [9, Theorem 2.4] is bit too complicated and long. We present here a simple and neat proof of $T(M) = D_0(M)$ along with the relations which hold among $T(M), \tau(M), \theta(M)$ and $D_0(M)$ in general. Obviously, not much is new in the remaining proofs.

Theorem 3.1. *Let M be a multiplication R-module. Then*

(a) $T(M) = T(M)\tau(M) \subseteq \tau(M)$.

(b) $\tau(M) = \theta(M)$.

(c) $T(M) = D_0(M)$.

(d) *If M is faithful then* $T(M) = \tau(M) = \theta(M) = D_0(M)$.

Proof. The proof of (a) and (b) is given in the proof of [1, Theorem 2.6]. For the sake of completeness, we reproduce them here.

By [4, Corollary 1.7], $M = \tau(M)M$. Therefore, we have $T(M) = \sum_{f\in M^*} f(M) = \sum_{f\in M^*} f(\tau(M)M) = \tau(M)T(M) \subseteq \tau(M)$. Hence (a) is proved.

We now prove (b). As $\theta(M)M = \sum_{x\in M}(Rx : M)M = \sum_{x\in M} Rx = M$ and $\operatorname{ann}_R(M) \subseteq \theta(M)$, we get $\tau(M) \subseteq \theta(M)$. We now show that $\theta(M) \subseteq \tau(M)$. Let $x \in M$. Then $Rx = (Rx : M)M = (Rx : M)\tau(M)M = \tau(M)(Rx : M)M = \tau(M)Rx$. Therefore, there exists some $a \in \tau(M)$ such that $(1 - a)Rx = 0 = (1-a)(Rx : M)M$. Hence $(1-a)(Rx : M) \subseteq \operatorname{ann}_R(M) \subseteq \tau(M)$. Thus $(Rx : M) \subseteq \tau(M)$. As $x \in M$ is arbitrary, we get $\theta(M) \subseteq \tau(M)$. Thus, (b) is proved.

We now prove (c). We first show that $D_0(M) \subseteq T(M)$. Let $x \in M$ and let $a \in \operatorname{ann}_R(\operatorname{ann}_R(x))$. It suffices to show that $a \in T(M)$.

Let $\phi\colon Rx \longrightarrow R$ be the map given by $\phi(bx) = ab$. Note that as $a \in \operatorname{ann}_R(\operatorname{ann}_R(x))$, ϕ is well defined.

As $Rx = (Rx : M)M$, there exist $a_1, \dots, a_n \in (Rx : M)$, $x_1, \dots, x_n \in M$ and $b_1, \dots, b_n \in R$ such that $x = \sum_{i=1}^n a_i x_i$ and $a_i x_i = b_i x$ for all $i = 1, \dots, n$. Therefore, $(1 - \sum_{i=1}^n b_i)x = 0$ and hence $a(1 - \sum_{i=1}^n b_i) = 0$.

For $i = 1, \dots, n$ let $f_i : M \longrightarrow R$ denote the map given by $f_i(y) = \phi(a_i y)$. Then $f_i(x_i) = ab_i$ and therefore, $\sum_{i=1}^n f_i(x_i) = a$. Thus $a \in T(M)$.

Conversely, let $f \in M^*$ and let $y \in M$. Then we have $\operatorname{ann}_R(y)f(y) = f(\operatorname{ann}_R(y)y) = 0$. Therefore, $f(y) \in \operatorname{ann}_R(\operatorname{ann}_R(y)) \subseteq D_0(M)$. Clearly, $T(M) \subseteq D_0(M)$. Thus (c) is proved.

Now assume that M is faithful. To show $T(M) = \tau(M) = \theta(M) = D_0(M)$, it suffices to show that $\theta(M) \subseteq D_0(M)$. Let $z \in M$. Then $Rz = (Rz : M)M$ and therefore, $\operatorname{ann}_R(z)Rz = 0 = \operatorname{ann}_R(z)(Rz : M)M$. As M is faithful, $\operatorname{ann}_R(z)(Rz : M) = 0$. Hence $(Rz : M) \subseteq \operatorname{ann}_R(\operatorname{ann}_R(z))$. Thus $\theta(M) \subseteq D_0(M)$. □

4. Dense submodules of a multiplication module

Recall that a submodule N of an R-module M is said to be dense in M if $\sum f(N) = M$, where the summation is taken over all $f \in \operatorname{Hom}_R(N, M)$.

Remark 4.1. Let M be an R-module and let N be a submodule of M. Let $I \subseteq \operatorname{ann}_R(M)$ be an ideal. Put $R' = R/I$. Then M is an R'-module and N is an R' submodule of M. It is immediate that if N is dense in M as an R-submodule if and only if N is dense in M as an R'-submodule.

We now prove two lemmas.

Lemma 4.2. *Let M be an R-module and let N be a dense submodule of M. Then we have*

(a) $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$.

(b) $T(M) \subseteq T(N)$.

(c) *If M is faithful and multiplication then $T(N) = T(M)$.*

Proof. Proof of part (a) is trivial. We now prove (b). Let $x \in M$ and let $f \in M^*$. As N is dense in M, there exist $g_1, \dots, g_n \in \mathrm{Hom}_R(N, M)$ and $x_1, \dots, x_n \in N$ such that $x = \sum_{i=1}^n g_i(x_i)$. Therefore, $f(x) = \sum_{i=1}^n f o g_i(x_i) \in T(N)$. This proves (b).

Now assume that M is a faithful multiplication R-module. Then, by [1, Theorem 2.6], $T(M) = \theta(M)$ and by Lemma 2.4, $N = \theta(M)N$. Therefore, $T(N) = \theta(M)T(N) \subseteq \theta(M) = T(M)$. Thus (c) is proved. □

Lemma 4.3. *Let M be an R-module and let N be a submodule of M such that $T(N) = R$. Then N is dense in M.*

Proof. As $T(N) = R$, there exist $f_1, \dots, f_n \in N^*$ and $y_1, \dots, y_n \in N$ such that $\sum_{i=1}^n f_i(y_i) = 1$. Let $x \in M$. For $i = 1, \dots, n$, define $\phi_i : N \longrightarrow M$ by setting $\phi_i(y) = f_i(y)x$. Clearly, $\sum_{i=1}^n \phi_i(y_i) = x$ and therefore, N is dense in M. □

As a corollary to Lemma 4.3, we obtain [11, Theorem 11].

Corollary 4.4. *Let M be an R-module and let N be a finitely generated submodule of M such that N is multiplication. Then N is dense in M if and only if $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$.*

Proof. The necessity follows by Lemma 4.2. We now prove the sufficiency. In view of Remark 4.1, we may assume that M is faithful. Then, N is also faithful. As N is finitely generated and faithful, by [1, Theorem 2.6] and [1, Corollary 2.2], $T(N) = \theta(N) = R$. Now apply Lemma 4.3. □

In the next result we show that every dense submodule of a multiplication module is itself multiplication. This generalises several results which have been proved with some additional condition either on M or on N (see, for example, Corollary 8 and Theorem 9 of [11]).

Theorem 4.5. *Let M be a multiplication R-module and let N be a dense submodule of M. Then N is multiplication.*

Proof. In view of Remark 4.1, we may assume that M is faithful. Then by Lemma 4.2, N is faithful. Let $x \in N$. Then, by Lemma 2.4, there exists some $a \in \theta(M)$ such that $(1-a)x = 0$. Now, by [1, Lemma 2.1], aM is finitely generated. Furthermore, as aM is a homomorphic image of M, it is a multiplication R-module. Note that aN is a dense submodule of aM. Therefore, by [11, Theorem 9], aN is a finitely generated multiplication R-module. As $x = ax \in aN$, there exists some ideal I of R such that $Rx = aIN$. Thus, by [4, Proposition 1.1], N is multiplication. □

Theorem 4.6. *Let M be a multiplication R-module and let N be a submodule of M. Then N is dense in M if and only if* $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ *and* $\theta(N) = \theta(M)$.

Proof. In view of Remark 4.1 and Lemma 2.4(f), we may assume that M is faithful.

First assume that N is dense in M. Then by Theorem 4.5, N is multiplication. Now, by Lemma 4.2, N is faithful and $T(N) = T(M)$. Therefore, by [1, Theorem 2.6], $\theta(N) = T(N) = T(M) = \theta(M)$.

Conversely, assume that N is faithful and $\theta(N) = \theta(M)$. Then, by Corollary 2.13, N is multiplication. Furthermore, by [1, Theorem 2.6], $T(N) = \theta(N) = \theta(M)$. Let $x \in M$. Then, by Lemma 2.4, there exists some $a \in \theta(M) = T(N)$ such that $(1-a)x = 0$. Hence there exist $f_1, \dots, f_n \in N^*$ and $y_1, \dots, y_n \in N$ such that $\sum_{i=1}^n f_i(y_i) = a$. For $i = 1, \dots, n$, define $\phi_i : N \longrightarrow M$ by setting $\phi_i(y) = f_i(y)x$. Clearly, $\sum_{i=1}^n \phi_i(y_i) = ax = x$ and therefore, N is dense in M. □

Remark 4.7. Let M be a multiplication R-module and let N be a submodule of M. If $\theta(M) = \theta(N)$ then N need not be dense in M as can be seen from the following example: Let $(R, \mathfrak{m})$ be a Noetherian local ring of depth zero. Assume further that R is not a field. Choose $a \in \mathfrak{m} \setminus \{0\}$ such that $a\mathfrak{m} = 0$. Put $I = Ra$. Then I is a multiplication ideal of R, $\theta(I) = R = \theta(R)$ and I is not dense in R as $\mathrm{ann}_R(I) = \mathfrak{m} \neq 0 = \mathrm{ann}_R(R)$.

Similarly, if $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ then N is not necessarily dense in M even if N is multiplication. For example see, [11, Example 13].

Thus the sufficiency conditions in Theorem 4.6 are strict.

The following result improves on [11, Corollary 8] and [11, Theorem 9].

Corollary 4.8. *Let M be a multiplication R-module and let N be a dense submodule of M. Then N is a multiplication R-module. Furthermore, N is finitely generated if and only if M is finitely generated.*

Proof. By Theorem 4.5 and Theorem 4.6, N is multiplication and $\theta(N) = \theta(M)$. Now apply [1, Corollary 2.2]. □

The next result is an easy consequence of Theorem 4.6 and Corollary 4.8.

Theorem 4.9. *Let M be a finitely generated multiplication R-module and let N be a submodule of M. Then N is dense in M if and only if N is finitely generated multiplication and* $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$.

Proof. The necessity follows by Corollary 4.8 and Lemma 4.2. We now prove the sufficiency. As both M and N are finitely generated and multiplication, by [1, Corollary 2.2], we have $\theta(N) = R = \theta(M)$. Now, apply Theorem 4.6. □

In the next result, we show that the denseness of a submodule N of a multiplication module M can be described in terms of denseness of certain ideal of $\theta(M)$ in $\theta(M)$.

Theorem 4.10. *Let M be a faithful multiplication R module, N be a submodule of M and let $I = (N :_R M)\theta(M)$. Then N is dense in M if and only if I is dense in $\theta(M)$.*

Proof. As M is faithful, by Lemma 2.4, $\theta(M)$ is faithful. Furthermore, by [1, Theorem 2.3 and Theorem 2.6], $\theta(M)$ is an idempotent multiplication ideal of R, $\theta(\theta(M)) = \theta(M)$, $I = I\theta(M)$ and $\theta(M) = T(M)$. Now, assume that N is a dense submodule of M. By Theorem 4.5 and Theorem 4.6, N is faithful, multiplication and $\theta(N) = \theta(M)$. Moreover, by Lemma 2.15, I is a multiplication ideal of R. As N is faithful and $N = IM$ it follows that I is faithful. Thus it suffices to prove that $\theta(I) = \theta(\theta(M)) = \theta(M)$. As $I = I\theta(M)$, we have $\theta(I) \subseteq \theta(M)$. Furthermore, $\theta(I)N = \theta(I)IM = IM = N$ and therefore, $\theta(N) \subseteq \theta(I)$. As $\theta(N) = \theta(M)$, we have $\theta(I) = \theta(M)$.

Conversely, assume that I is dense in $\theta(M)$. Then, again by Theorem 4.5 and Theorem 4.6, I is faithful, multiplication and $\theta(I) = \theta(M)$. It follows that $N = IM$ is faithful. Furthermore, by Lemma 2.15, N is multiplication. As $N = \theta(M)N$, we have $\theta(N) \subseteq \theta(M)$. Note that $N = \theta(N)N = I\theta(N)M$. Thus $I = I\theta(M) = IT(M) = I\sum_{f\in M^*} f(M) = \sum_{f\in M^*} f(IM) = \sum_{f\in M^*} f(I\theta(N)M) = I\theta(N)\sum_{f\in M^*} f(M) = I\theta(N)T(M) = I\theta(N)\theta(M) = I\theta(N)$. As I is faithful, by [1, Theorem 2.6], we have $\theta(I) = T(I)$. Therefore, by Lemma 2.4, we have, $\theta(M) = \theta(I) = T(I) = \theta(N)T(I) \subseteq \theta(N)$. Now, by Theorem 4.6, N is dense in M. □

5. Forcing linearity number for a multiplication module

Throughout this section let M denote an R-module. We now recall the following definition:

Definition 5.1. *A map $f : M \longrightarrow M$ is said to be R-homogeneous if $f(ax) = af(x)$ for all $a \in R$ and for all $x \in M$.*

Let $\mathfrak{M}_R(M) = \{f : M \to M \mid f \text{ is } R\text{-homogeneous}\}$. Then $\mathrm{End}_R(M) \subseteq \mathfrak{M}_R(M)$.

Following [6], we say that a collection $\mathfrak{F}$ of proper submodules of M forces linearity on M if the following happens: Let $f \in \mathfrak{M}_R(M)$. Then $f \in \mathrm{End}_R(M)$ if and only if f is linear on each $N \in \mathfrak{F}$.

We now define the forcing linearity number $fln(M)$ for M as follows:

(a) If $\mathrm{End}_R(M) = \mathfrak{M}_R(M)$, then $fln(M)$ is defined to be 0.

(b) If $\mathrm{End}_R(M) \neq \mathfrak{M}_R(M)$ then $fln(M)$ is defined to be a positive integer n if there exists a collection $\mathfrak{F}$ of proper submodules of M such that $|\mathfrak{F}| = n$ and $\mathfrak{F}$ forces linearity on M but no collection $\mathfrak{F}'$ of proper submodules of M with $|\mathfrak{F}'| < n$ forces linearity on M.

(c) If neither of the above conditions holds then $fln(M)$ is defined to be ∞.

In this section we show that the forcing linearity number for a multiplication module is 0.

We now prove two lemmas.

Lemma 5.2. *If M is cyclic then* $\mathrm{End}_R(M) = \mathfrak{M}_R(M)$.

Proof. Trivial. □

Lemma 5.3. *Let $S \subset R$ be a multiplicative set and let $f \in \mathfrak{M}_R(M)$. Define $S^{-1}f : S^{-1}M \longrightarrow S^{-1}M$ by setting $S^{-1}f(\frac{x}{s}) = \frac{f(x)}{s}$. Then $S^{-1}f$ is well defined and $S^{-1}f \in \mathfrak{M}_{S^{-1}R}(S^{-1}M)$.*

Proof. Let $\frac{x}{s}, \frac{y}{t} \in S^{-1}M$ such that $\frac{x}{s} = \frac{y}{t}$. Then there exists $u \in S$ such that $utx = usy$. As $f \in \mathfrak{M}_R(M)$, we have $utf(x) = f(utx) = f(usy) = usf(y)$. Clearly, $S^{-1}f(\frac{x}{s}) = S^{-1}f(\frac{y}{t})$. Thus $S^{-1}f$ is well defined.

Now let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}M$. Then

$$S^{-1}f(\frac{a}{s}\frac{x}{t}) = S^{-1}f(\frac{ax}{st}) = \frac{f(ax)}{st} = \frac{af(x)}{st} = \frac{a}{s}\frac{f(x)}{t} = \frac{a}{s}S^{-1}f(\frac{x}{t}).$$

Therefore, $S^{-1}f \in \mathfrak{M}_{S^{-1}R}(S^{-1}M)$. □

We now prove the following theorem:

Theorem 5.4. *If M is a multiplication R-module then $fln(M) = 0$.*

Proof. Let $f \in \mathfrak{M}_R(M)$ and let P be a prime ideal of R. Let $S = R \setminus P$ and let $f_P = S^{-1}f$. Then by Lemma 5.3, $f_P \in \mathfrak{M}_{R_P}(M_P)$. As M_P is a multiplication module over the local ring R_P, by [3, Proposition 4], M_P is a cyclic R_P-module and hence by Lemma 5.2, $f_P \in \mathrm{End}_{R_P}(M_P)$. Let $x, y \in M$. Put $z = f(x+y) - f(x) - f(y)$. Clearly, $\frac{z}{1} = 0$ in M_P. As P is arbitrary, we have $z = 0$. Therefore, $f \in \mathrm{End}_R(M)$. □

If M is an R-module with $fln(M) = 0$ then M need not be multiplication even if M is finitely generated and R is Noetherian (or Noetherian local). This we show by the following example: Let k be a field and let x, y be indeterminates. Let $R = k[x, y]$ (or $k[[x, y]]$) and $I = (x, y)$. One easily checks that if $f \in \mathfrak{M}_R(M)$ then there exists some $\alpha \in R$ such that $f(a) = \alpha a$ for all $a \in I$. Thus $\mathfrak{M}_R(M) = \mathrm{End}_R(M)$. As I is not locally principal, by [3, Proposition 4], I can not be multiplication.

However, if R is Noetherian and M is finitely generated then the following, a reformulation of [7, Theorem 2.3], does hold:

Theorem 5.5. *Let R be Noetherian and let M be finitely generated. Let S be the set of nonzero divisors of M. Then $flm(M) = 0$ if and only if $S^{-1}M$ is a multiplication $S^{-1}R$-module.*

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