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Special Directions in Complex Contact Manifolds

Dedicated to Professor Gerald D. Ludden on his 70th birthday

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Abstract. Ideas of holomorphic and real special directions on complex contact manifolds are developed. It is shown that such directions exist under certain curvature conditions. The special case of the Lie group $SL(2,\mathbb{C})$ is studied in detail.

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Introduction

In the geometry of contact metric manifolds, a vector Y orthogonal to the characteristic (Reeb) vector field ξ is called a special direction if the covariant derivative of ξ in the direction Y is collinear with Y. If Y is a vector field with this property, then one can think of ξ as falling forward or backward as one moves along integral curves of Y and when one such direction exists exhibiting one direction of fall, another direction exists exhibiting the opposite direction of fall. If the sectional curvatures of plane sections containing ξ are negative, such directions always exist. If ξ generates an Anosov flow on a 3-dimensional contact manifold, one can investigate the situation in which special directions agree with the stable and unstable directions of the Anosov flow. In particular the Lie group $SL(2,\mathbb{R})$ admits a family of contact metric structures exhibiting these properties. This theory was developed in [1], [2].

In the present paper we develop the ideas of holomorphic and real special directions on a complex contact metric manifold. We first prove some existence results for special directions and then discuss a 2-parameter family of complex contact metric structures on the Lie group $SL(2,\mathbb{C})$. The notion of a holomorphic Anosov flow in general and on $SL(2,\mathbb{C})$ was introduced by E. Ghys in [5]. We show that for our 2-parameter family of structures, the holomorphic special directions determine subbundles that agree with the stable and unstable subbundles of the corresponding holomorphic Anosov flow. Reducing to a 1-parameter family of complex contact metric structures, we also show that $SL(2,\mathbb{C})$ admits a real vector field generating a partially hyperbolic flow whose central bundle has dimension 2.

1. Preliminaries

A *complex contact manifold* is a complex manifold M of odd complex dimension $2n+1$ together with an open covering $\{\mathcal{O}\}$ of coordinate neighborhoods such that:

1) On each $\mathcal{O}$ there is a holomorphic 1-form ω such that $\omega \wedge (d\omega)^n \neq 0$.
2) On $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$ there is a non-vanishing holomorphic function f such that $\omega' = f\omega$.

The complex contact structure determines a non-integrable subbundle $\mathcal{H}$ by the equation $\omega = 0$; $\mathcal{H}$ is called the *complex contact subbundle* or the *horizontal subbundle.*

On the other hand if M is an Hermitian manifold with almost complex structure J, Hermitian metric g and open covering by coordinate neighborhoods $\{\mathcal{O}\}$, it is called a *complex almost contact metric manifold* if it satisfies the following two conditions:

1) In each $\mathcal{O}$ there exist 1-forms u and $v = u \circ J$ with dual vector fields U and $V = -JU$ and (1,1) tensor fields G and $H = GJ$ such that
$$G^2 = H^2 = -I + u \otimes U + v \otimes V,$$
$$GJ = -JG, \quad GU = 0, \quad g(X, GY) = -g(GX, Y).$$
2) On $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$,
$$u' = Au - Bv, \quad v' = Bu + Av,$$
$$G' = AG - BH, \quad H' = BG + AH$$
where A and B are functions with $A^2 + B^2 = 1$.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form ω is $u - iv$ to within a non-vanishing complex-valued function multiple and the local tensor fields G and H are related to du and dv by
$$du(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y),$$

$$dv(X,Y) = g(X, HY) - (\sigma \wedge u)(X,Y)$$

where $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi-Civita connection of g (see [2], [3] and [7]). Moreover on $\mathcal{O} \cap \mathcal{O}'$ it is easy to check that $U' \wedge V' = U \wedge V$ and hence we have a global vertical bundle $\mathcal{V}$ orthogonal to $\mathcal{H}$ which is typically assumed to be integrable. We refer to a complex contact manifold with a complex almost contact metric structure satisfying these conditions as a *complex contact metric manifold.*

The definition of a complex contact manifold is analogous to that of a contact structure in the wider sense. This is natural in view of the result [8] that for a compact complex manifold with $H^1(M, \mathbb{Z}_{n+1}) = 0$, i.e. no $(n+1)$-torsion, a complex contact structure is given by a global 1-form if and only if its first Chern class vanishes. On the other hand there are a number of interesting cases of complex contact manifolds with global contact form including a complex Boothby-Wang fibration as developed by Foreman [4]. When the form is global the structure tensors may be chosen so that $\sigma = 0$. We will retain the generality of σ being non-zero for some of our discussion, but then specialize to $\sigma = 0$ when appropriate.

There are two other local structure tensors that will be important for us, namely $h_U = \frac{1}{2}\mathrm{sym}\mathcal{L}_U G \circ p$ and $h_V = \frac{1}{2}\mathrm{sym}\mathcal{L}_V H \circ p$ where $\mathcal{L}$ denotes Lie differentiation, "sym" denotes the symmetric part and p denotes the projection $TM \longrightarrow \mathcal{H}$. These operators enjoy the following properties [2], [9]:

$$h_U G = -G h_U, \quad h_V H = -H h_V$$

$$h_U U = h_U V = h_V U = h_V V = 0$$

$$\begin{aligned} \nabla_X U &= -GX - Gh_U X + \sigma(X)V \\ \nabla_X V &= -HX - Hh_V X - \sigma(X)U. \end{aligned} \tag{1}$$

Defining an operator l_U in terms of the curvature tensor R by $l_U X = R_{XU}U$ we have the following formula ([9])

$$G l_U G - l_U = 2(G^2 + h_U^2 + d\sigma(U,V) v \otimes V). \tag{2}$$

Next we will review some ideas related to hyperbolicity. A diffeomorphism f of a (usually compact) Riemannian manifold M is said to be *partially hyperbolic in the narrow sense* (see e.g. [10, pp. 13–14]) if there exist numbers $C > 0$ and

$$0 < \lambda_1 \le \mu_1 < \lambda_2 \le \mu_2 < \lambda_3 \le \mu_3, \quad \mu_1 < 1, \quad \lambda_3 > 1,$$

independent of $p \in M$, and an invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that for $n > 0$

$$C^{-1}\lambda_1^n ||v|| \le ||f_*^n v|| \le C\mu_1^n ||v||, \quad v \in E_p^s,$$

$$C^{-1}\lambda_2^n||v|| \leq ||f_*^n v|| \leq C\mu_2^n||v||, \quad v \in E_p^c,$$
$$C^{-1}\lambda_3^n||v|| \leq ||f_*^n v|| \leq C\mu_3^n||v||, \quad v \in E_p^u.$$

When M is compact the notion is independent of the choice of metric, but when M is non-compact, the notion is in general metric dependent. The subbundles E^s, E^c and E^u are called the *stable*, *central* and *unstable subbundles* respectively.

For example let ψ_t be an Anosov flow corresponding to a vector field ξ. For fixed t the diffeomorphism ψ_t is partially hyperbolic with 1-dimensional central direction generated by ξ. Later in this paper we will encounter a real vector field whose corresponding 1-parameter group is a group of partially hyperbolic diffeomorphisms for which the central subbundle has dimension 2.

In [5, p. 586] E. Ghys defines the notion of a holomorphic Anosov flow as a particular $\mathbb{C}^*$-action on a complex manifold which gives rise to an invariant splitting of the real tangent bundle together with natural growth conditions on the subbundles. Lemma 2.1 of [5] shows that the resulting stable and unstable subbundles extend to complex subbundles in the complexified tangent bundle. In remarks on p. 600 of [5] Ghys discusses the possible consideration of starting with a holomorphic vector field and the flow it generates. This is described in terms of a splitting of the complexified tangent bundle, but one could equally well begin with a splitting of the holomorphic tangent bundle and we take this point of view here.

Let ξ be a holomorphic vector field on a Hermitian manifold M; strictly speaking ξ does not determine a flow due to the lack of a natural ordering of the complex numbers. However for a holomorphic vector field, the theory of complex differential equations goes through as in the real case. Let $w_1, \dots, w_n$ be local complex coordinates on M and $f^j(w_1, \dots, w_n)$ the holomorphic component functions of ξ with respect to the basis $\{\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_n}\}$. Then the system of complex differential equations

$$\frac{dw_j}{dz} = f^j(w_1, \dots, w_n)$$

satisfies existence and uniqueness theorems similar to those in the real case (see e.g. [6, Theorem 2.2.2]).Thus for a given point $w_0 \in M$ one can construct a unique local holomorphic curve through the point by the solution $w(z) = (w_1(z), \dots, w_n(z))$, $w(0) = w_0$. Therefore given ξ one can define a "flow" ψ_z mapping a point w_0 to the point $w(z)$. We say that ξ (and the flow) is a *holomorphic Anosov flow* if there exists an invariant splitting of the holomorphic tangent bundle τM as a direct sum of type (1,0) subbundles E^s, E^u and the 2-dimensional bundle tangent to the orbits of ψ_z together with numbers $C > 0$, $\lambda > 0$ such that

$$||\psi_{z*}v|| \leq Ce^{-\lambda\Re(z)}||v||, \quad v \in E_p^s,$$

$$||\psi_{z*}v|| \geq Ce^{\lambda\Re(z)}||v||, \quad v \in E_p^u$$

where $\Re$ denotes the real part.

2. Holomorphic special directions

In this section, we will work with complex contact metric manifolds of complex dimension 3 with the property $h_U \neq h_V$. Let $k = h_U - h_V$.

Lemma. $kJ + Jk = 0$.

Proof. We can compute $(\nabla_X J)V$ directly using (1) to get

$$(\nabla_X J)V = -GkX$$

and applying G to both sides, we obtain

$$kX = G(\nabla_X J)V.$$

Therefore

$$\begin{aligned} kJX &= G(\nabla_{JX} J)V \\ &= -G(\nabla_{JX} J)JU \\ &= -G(\nabla_X J)U \end{aligned}$$

since (M, J, g) is a Hermitian manifold. We can continue the computation as follows:

$$\begin{aligned} kJX &= -JH(-\nabla_X V - J\nabla_X U) \\ &= -JH(HX + Hh_V X + \sigma(X)U + J(GX + Gh_U X - \sigma(X)V)) \\ &= -JH(HX + Hh_V X + \sigma(X)U - HX - Hh_U X - \sigma(X)U) \\ &= -JH(-HkX) \\ &= -JkX. \end{aligned}$$

Therefore $kJX + JkX = 0$. □

As a result, k anti-commutes with the complex structure J, and it vanishes on the vertical subbundle generated by U and V. Let $\kappa > 0$ be an eigenvalue of k with unit eigenvector X. Then, X is a horizontal vector field and $-\kappa$ is also an eigenvector of k with eigenvector JX. Since the real dimension of M is 6, there is another eigenvector W of k with non-negative eigenvalue ν. Then again, W is a horizontal vector field and $-\nu$ is also an eigenvalue of k with eigenvector JW.

Definition. *A vector* $Y - iJY$ *of type* $(1,0)$ *on a complex contact metric manifold* M *where* Y *is horizontal, is called a holomorphic special direction if*

$$\nabla_{Y-iJY}(U + iV) = (\gamma + i\delta)(Y - iJY)$$

for some non-zero complex number $\gamma + i\delta$.

Since

$$\nabla_{Y-iJY}(U+iV) = \nabla_Y U + \nabla_{JY} V + i(\nabla_Y V - \nabla_{JY} U),$$

for $Y - iJY$ to be a holomorphic special direction we must have

$$\nabla_Y U + \nabla_{JY} V = \gamma Y + \delta JY \quad \text{and} \quad \nabla_Y V - \nabla_{JY} U = \delta Y - \gamma JY. \tag{3}$$

We know that

$$\begin{aligned} \nabla_Y U + \nabla_{JY} V &= -GY - Gh_U Y + \sigma(Y)V - HJY - Hh_V JY - \sigma(JY)U \\ &= h_U GY + h_V HJY - \sigma(JY)U + \sigma(Y)V \\ &= (h_U - h_V)GY - \sigma(JY)U + \sigma(Y)V \\ &= kGY - \sigma(JY)U + \sigma(Y)V \end{aligned}$$

and

$$\begin{aligned} \nabla_Y V - \nabla_{JY} U &= -HY - Hh_V Y - \sigma(Y)U + GJY + Gh_U JY - \sigma(JY)V \\ &= h_V HY - h_U GJY - \sigma(Y)U - \sigma(JY)V \\ &= -kHY - \sigma(Y)U - \sigma(JY)V. \end{aligned}$$

(We see that if $h_U = h_V$, and hence $k = 0$, then there are no holomorphic special directions.) Since Y is horizontal and k anti-commutes with J, the conditions for a special direction are

$$kGY = \gamma Y + \delta JY \tag{4}$$

and $\sigma(Y) = \sigma(JY) = 0$. Since the eigenspace of k relative to the eigenvalue ν (resp. κ) is perpendicular to $\{X, JX\}$ (resp. $\{W, JW\}$), $W \in \{GX, HX\}$ and $X \in \{GW, HW\}$ and hence there are constants α and β with $\alpha^2+\beta^2 = 1$ such that $W = \alpha GX + \beta HX$, $GW = -\alpha X - \beta JX$, $HW = \alpha JX - \beta X$, $GX = \alpha W + \beta JW$ and $HX = -\alpha JW + \beta W$.

Again since Y is horizontal, there are constants a, b, c and e with $a^2 + b^2 + c^2 + e^2 = 1$ such that

$$Y = aX + bJX + cW + eJW.$$

Then

$$\begin{aligned} GY &= a(\alpha W + \beta JW) + b(-\alpha JW + \beta W) \\ &\quad + c(-\alpha X - \beta JX) + e(\alpha JX - \beta X) \\ &= -(c\alpha + e\beta)X + (e\alpha - c\beta)JX + (a\alpha + b\beta)W + (a\beta - b\alpha)JW, \\ kGY &= -(c\alpha + e\beta)\kappa X - (e\alpha - c\beta)\kappa JX \\ &\quad + (a\alpha + b\beta)\nu W - (a\beta - b\alpha)\nu JW, \\ JY &= aJX - bX + cJW - eW \end{aligned}$$

and

$$\gamma Y + \delta JY = (a\gamma - b\delta)X + (b\gamma + a\delta)JX + (c\gamma - e\delta)W + (e\gamma + c\delta)JW.$$

Thus, for (4) to be satisfied, the following four equations must hold:

$$\begin{aligned} a\gamma - b\delta &= -(c\alpha + e\beta)\kappa && (5)\\ b\gamma + a\delta &= (-e\alpha + c\beta)\kappa && (6)\\ c\gamma - e\delta &= (a\alpha + b\beta)\nu && (7)\\ e\gamma + c\delta &= (b\alpha - a\beta)\nu. && (8) \end{aligned}$$

If $\nu = 0$, (7) and (8) gives $c = e = 0$ and then (5) and (6) gives $a = b = 0$. This is a contradiction since $a^2 + b^2 + c^2 + e^2 = 1$. Therefore $\nu \neq 0$.

Multiplying (5) by $-e\alpha + c\beta$, (6) by $c\alpha + e\beta$ and adding we get

$$[a(-e\alpha + c\beta) + b(c\alpha + e\beta)]\gamma + [-b(-e\alpha + c\beta) + a(c\alpha + e\beta)]\delta = 0.$$

Similarly, multiplying (7) by $-b\alpha + a\beta$, (8) by $a\alpha + b\beta$, and adding we have

$$[-c(b\alpha - a\beta) + e(a\alpha + b\beta)]\gamma + [e(b\alpha - a\beta) + c(a\alpha + b\beta)]\delta = 0.$$

We can rewrite the above two equations as follows:

$$[(bc - ae)\alpha + (ac + be)\beta]\gamma + [(be + ac)\alpha + (ae - bc)\beta]\delta = 0 \quad (9)$$
$$[-(bc - ae)\alpha + (ac + be)\beta]\gamma + [(be + ac)\alpha - (ae - bc)\beta]\delta = 0. \quad (10)$$

Since not both of γ and δ are 0, the determinant of the above system must be 0. Using the fact that $\alpha^2 + \beta^2 = 1$, we get

$$(bc - ae)(be + ac) = 0,$$

so either $bc - ae = 0$ or $be + ac = 0$ (but not both).

Suppose $bc - ae = 0$. Then, equations (9) and (10) give

$$\beta\gamma + \alpha\delta = 0. \quad (11)$$

Multiply (5) by c, (6) by e, add them together and use our assumption that $bc - ae = 0$ to get

$$(ac + be)\gamma = -(c^2 + e^2)\alpha\kappa. \quad (12)$$

On the other hand, a times (7) plus b times (8) gives

$$(ac + be)\gamma = (a^2 + b^2)\alpha\nu. \quad (13)$$

Subtract (12) from (13) to get

$$\alpha((a^2 + b^2)\nu + (c^2 + e^2)\kappa) = 0. \quad (14)$$

Suppose $(a^2 + b^2)\nu + (c^2 + e^2)\kappa = 0$. Since $a^2 + b^2 + c^2 + e^2 = 1$, we can take $c^2 + e^2 = 1 - a^2 - b^2$ and write the above equation as $(a^2 + b^2)(\nu - \kappa) + \kappa = 0$ which implies $\kappa > \nu$ since $a^2 + b^2 > 0$ and $\kappa > 0$. On the other hand, we can take

$a^2+b^2=1-c^2-e^2$ to get $(c^2+e^2)(\kappa-\nu)+\nu=0$ which implies $\kappa<\nu$ since $c^2+e^2>0$ and $\nu>0$. This is a contradiction. Hence $\alpha=0$.

Similarly, $-e$ times (5) plus c times (6) and b times (7) plus $-a$ times (8) give $\beta=0$. Since $\alpha^2+\beta^2=1$, we get a contradiction. Therefore

$$bc-ae\neq 0 \quad \text{and} \quad be+ac=0. \tag{15}$$

Working now with $be+ac=0$, (9) and (10) give

$$\alpha\gamma-\beta\delta=0. \tag{16}$$

Multiplying (5) by c, (6) by e and adding gives

$$(bc-ae)\delta=(c^2+e^2)\alpha\kappa.$$

Similarly, a times (7) plus b times (8) gives

$$(bc-ae)\delta=(a^2+b^2)\alpha\nu.$$

Comparing the above two equations we get

$$\alpha((a^2+b^2)\nu-(c^2+e^2)\kappa)=0.$$

Similarly, $-e$ times (5) plus c times (6) and $-b$ times (7) plus a times (8) give

$$\beta((a^2+b^2)\nu-(c^2+e^2)\kappa)=0.$$

Since not both of α and β are 0, we get

$$a^2+b^2=\frac{\kappa}{\kappa+\nu} \quad \text{and} \quad c^2+e^2=\frac{\nu}{\kappa+\nu}. \tag{17}$$

Now multiply (5) by $a\gamma$ and use (8) and (16) to get

$$a^2\gamma^2-ab\gamma\delta=(a^2\beta-ab\alpha)\beta\nu\kappa. \tag{18}$$

Similarly, multiply (6) by $b\gamma$ and then use (7) and (16) to get

$$b^2\gamma^2+ab\gamma\delta=(b^2\beta+ab\alpha)\beta\nu\kappa. \tag{19}$$

Now adding (18) and (19) gives

$$\gamma^2=\beta^2\nu\kappa. \tag{20}$$

Then, (16) gives

$$\delta^2=\alpha^2\nu\kappa. \tag{21}$$

We can use (16) in γ times (5) plus δ times (6) to get

$$e=-\frac{\nu}{\beta\gamma+\alpha\delta}a. \tag{22}$$

Similarly, we can use (16) again, this time in γ times (7) plus δ times (8), to get

$$c=\frac{\beta\gamma+\alpha\delta}{\kappa}b. \tag{23}$$

Therefore we have the following result.

Theorem 2.1. *Let M be a complex contact metric manifold of complex dimension 3 with $h_U \neq h_v$. If $\sigma = 0$ and if k is non-singular on the horizontal subbundle, then there exist holomorphic special directions on M.*

Note that, when $\sigma = 0$, we have two choices for γ and δ. We can choose them to have the same signs as β and α, or we can choose them to have opposite signs as β and α.

For $\gamma = \beta\sqrt{\nu\kappa}$ and $\delta = \alpha\sqrt{\nu\kappa}$, let

$$Y = aX + bJX + \sqrt{\frac{\nu}{\kappa}}bW - \sqrt{\frac{\nu}{\kappa}}aJW$$

where a and b are two numbers satisfying $a^2 + b^2 = \frac{\kappa}{\kappa+\nu}$. Then $Y - iJY$ is a holomorphic special direction. If Z is a linear combination of Y and JY, then $Z - iJZ$ is also a holomorphic special direction with the same factor, $\gamma + i\delta$. Therefore Y determines a J-invariant plane section of special directions.

If $\gamma = -\beta\sqrt{\nu\kappa}$ and $\delta = -\alpha\sqrt{\nu\kappa}$, then $Y = aX + bJX - \sqrt{\frac{\nu}{\kappa}}bW + \sqrt{\frac{\nu}{\kappa}}aJW$ gives a special direction independent of the preceding one.

When $\sigma \neq 0$ we may still get special directions. If we take $\gamma = \beta\sqrt{\nu\kappa}$ and $\delta = \alpha\sqrt{\nu\kappa}$, then $\beta\gamma + \alpha\delta = \sqrt{\nu\kappa}$, and $c = \sqrt{\frac{\nu}{\kappa}}b$ and $e = -\sqrt{\frac{\nu}{\kappa}}a$ as before. In order to have $\sigma(Y) = \sigma(JY) = 0$, we must have

$$a\sigma(X) + b\sigma(JX) + \sqrt{\frac{\nu}{\kappa}}b\sigma(W) - \sqrt{\frac{\nu}{\kappa}}a\sigma(JW) = 0$$

and

$$-b\sigma(X) + a\sigma(JX) + \sqrt{\frac{\nu}{\kappa}}a\sigma(W) + \sqrt{\frac{\nu}{\kappa}}b\sigma(JW) = 0.$$

We can rewrite the above two equations as follows:

$$(\sigma(X) - \sqrt{\frac{\nu}{\kappa}}\sigma(JW))a + (\sigma(JX) + \sqrt{\frac{\nu}{\kappa}}\sigma(W))b = 0$$

$$(\sigma(JX) + \sqrt{\frac{\nu}{\kappa}}\sigma(W))a - (\sigma(X) - \sqrt{\frac{\nu}{\kappa}}\sigma(JW))b = 0.$$

Since not both of a and b are not 0, we must have

$$(\sigma(X) - \sqrt{\frac{\nu}{\kappa}}\sigma(JW))^2 + (\sigma(JX) + \sqrt{\frac{\nu}{\kappa}}\sigma(W))^2 = 0$$

which is equivalent to

$$\sigma(X) = \sqrt{\frac{\nu}{\kappa}}\sigma(JW) \quad \text{and} \quad \sigma(JX) = -\sqrt{\frac{\nu}{\kappa}}\sigma(W).$$

The other choice, $\gamma = -\beta\sqrt{\nu\kappa}$ and $\delta = -\alpha\sqrt{\nu\kappa}$, gives the conditions

$$\sigma(X) = -\sqrt{\frac{\nu}{\kappa}}\sigma(JW) \quad \text{and} \quad \sigma(JX) = \sqrt{\frac{\nu}{\kappa}}\sigma(W).$$

Also note that, any choice of the pair of numbers a and b satisfying the condition $a^2+b^2=\frac{\kappa}{\kappa+\nu}$ gives us a pair of vectors Y and JY which satisfy (4) but they all lie in the same plane.

Now, let us go back to the case $\sigma=0$. Everything we have been doing so far is pointwise. Now we will show that the plane section given by the vector fields Y and JY is global. To do this, we have to show that the local sections agree on the overlaps. On the intersection $\mathcal{O}\cap\mathcal{O}'$ of two coordinate neighborhoods with structure tensors (u,v,U,V,G,H) on $\mathcal{O}$ and (u',v',U',V',G',H') on $\mathcal{O}'$, there are two functions A, B with $A^2+B^2=1$ such that

$$U'=AU-BV,\ V'=BU+AV,\ G'=AG-BH,\ H'=BG+AH.$$

We want to evaluate $k'X$ and $k'JX$ for the eigenvector X of k corresponding to the eigenvalue κ. First, we evaluate $h_{U'}X$:

$$\begin{aligned}
h_{U'}X &= G'\nabla_X U'+G'^2X\\
&= (AG-BH)\nabla_X(AU-BV)-X\\
&= A^2G\nabla_XU-ABG\nabla_XV-ABH\nabla_XU\\
&\quad +B^2H\nabla_VV-(A^2+B^2)X\\
&= A^2h_UX+B^2h_VX\\
&\quad -AB(G(-HX-Hh_VX)+H(-GX-Gh_UX))\\
&= A^2h_UX+B^2h_VX-AB(JX+Jh_VX-JX-Jh_UX)\\
&= A(Ah_UX+BJh_UX)+B(Bh_VX-AJh_VX).
\end{aligned}$$

Similarly,

$$h_{V'}X=A(Ah_VX+BJh_VX)+B(Bh_UX-AJh_UX).$$

Therefore

$$\begin{aligned}
k'X &= h_{U'}X-h_{V'}X\\
&= (A^2-B^2)h_UX+2ABJh_UX+(B^2-A^2)h_VX-2ABJh_VX\\
&= (A^2-B^2)kX+2ABJkX\\
&= (A^2-B^2)\kappa X+2AB\kappa JX
\end{aligned}$$

and similarly

$$k'JX=-(A^2-B^2)\kappa JX+2AB\kappa X.$$

Hence we see that

$$k'(AX+BJX)=\kappa(AX+BJX).$$

This says that κ is also an eigenvalue of k' with eigenvector $AX+BJX$. Then, ν is also an eigenvalue of k' with eigenvector $AW+BJW$. The plane section on $\mathcal{O}$ is generated by

$$Y=aX+bJX+\sqrt{\frac{\nu}{\kappa}}bW-\sqrt{\frac{\nu}{\kappa}}aJW$$

for some a, b with $a^2+b^2=\frac{\kappa}{\kappa+\nu}$, and the plane section on $\mathcal{O}'$ is generated by

$$Y'=a'X'+b'JX'+\sqrt{\frac{\nu}{\kappa}}b'W'-\sqrt{\frac{\nu}{\kappa}}a'JW'$$

for some a', b' with $a'^2+b'^2=\frac{\kappa}{\kappa+\nu}$. On the intersection $\mathcal{O}\cap\mathcal{O}'$ we have

$$\begin{aligned}
Y' &= a'X'+b'JX'+\sqrt{\frac{\nu}{\kappa}}b'W'-\sqrt{\frac{\nu}{\kappa}}a'JW'\\
&= a'(AX+BJX)+b'(-BX+AJX)\\
&\quad+\sqrt{\frac{\nu}{\kappa}}b'(AW+BJW)-\sqrt{\frac{\nu}{\kappa}}a'(-BW+AJW)\\
&= (a'A-b'B)X+(a'B+b'A)JX\\
&\quad+\sqrt{\frac{\nu}{\kappa}}(a'B+b'A)W+\sqrt{\frac{\nu}{\kappa}}(b'B-a'A)JW
\end{aligned}$$

and

$$(a'A-b'B)^2+(a'B+b'A)^2=\frac{\kappa}{\kappa+\nu}.$$

Therefore, the plane sections agree on the overlaps.

3. Real special directions

In this section we consider the existence problem of real special directions.

Definition. *A horizontal real vector Y on a complex contact metric manifold M is called a real special direction for U (resp. for V) if*

$$\nabla_Y U=\gamma Y$$

(resp. $\nabla_Y V=\gamma Y$) for a non-zero number γ.

Since

$$\nabla_Y U=-GY-Gh_UY+\sigma(Y)V,$$

to find real special directions of U, we will consider eigenvalues of h_U. If the complex dimension of M is $2n+1$, h_U has $2n$ non-negative eigenvalues λ_i, $i=1,\dots,2n$ with eigenvectors orthogonal to $\mathcal{V}$. Suppose X_i is a unit eigenvector of h_U corresponding to the eigenvalue λ_i. Then, since h_U anti-commutes with G, GX_i is also an eigenvector of h_U with eigenvalue $-\lambda_i$. A unit horizontal vector field Y can be written in the form

$$Y=\sum_{i=1}^{2n}(a_iX_i+b_iGX_i)$$

where $\sum(a_i^2 + b_i^2) = 1$. Then

$$\begin{aligned}\nabla_Y U &= -GY - Gh_U Y + \sigma(Y)V \\ &= -\sum_{i=1}^{2n}(b_i(\lambda_i - 1)X_i + a_i(\lambda_i + 1)GX_i) + \sigma(Y)V.\end{aligned}$$

For $\nabla_Y U = \gamma Y$, we must have $\sigma(Y) = 0$, $\gamma a_i = -b_i(\lambda_i - 1)$ and $\gamma b_i = -a_i(\lambda_i + 1)$. Then we get $\gamma^2 a_i = a_i(\lambda^2 - 1)$ and $\gamma^2 b_i = b_i(\lambda^2 - 1)$. So, if not both of a_i and b_i are 0, then the corresponding λ_i's are equal, say to λ, and $\gamma^2 = \lambda^2 - 1$ implying that $\lambda > 1$.

Theorem 3.1. *Let M be a complex contact metric manifold with $\sigma = 0$. If h_U has an eigenvalue $\lambda > 1$, then there are real special directions of U. In particular, if all plane sections generated by U and a horizontal vector field have negative sectional curvatures, real special directions of U exist.*

Proof. Suppose h_U has an eigenvalue $\lambda > 1$ with unit eigenvector X. Let

$$Y = \sqrt{\frac{\lambda - 1}{2\lambda}}X + \sqrt{\frac{\lambda + 1}{2\lambda}}GX.$$

Then it can be seen by a direct computation that

$$\nabla_Y U = -\sqrt{\lambda^2 - 1}Y.$$

If all plane sections generated by U and a horizontal vector field have negative sectional curvature, then $K(U, X) + K(U, GX) < 0$ where X is a unit eigenvector of h_U with eigenvalue λ. Then $K(U, X) + K(U, GX) = 2(1 - \lambda^2)$. So $\lambda > 1$ and we have real special directions of U as above. $\square$

Note that, when $\lambda > 1$ if we take $Z = -\sqrt{\frac{\lambda-1}{2\lambda}}X + \sqrt{\frac{\lambda+1}{2\lambda}}GX$ then $\nabla_Z U = \sqrt{\lambda^2 - 1}Z$. The special directions Y and Z are independent and since $g(Y, Z) = \frac{1}{\lambda}$ they are not orthogonal.

We can apply the same procedure to $\nabla_Y V$ to get the following result.

Theorem 3.2. *Let M be a complex contact metric manifold with $\sigma = 0$. If h_V has an eigenvalue $\mu > 1$, then there are real special directions of V. In particular, if all plane sections generated by V and a horizontal vector field have negative sectional curvatures, real special directions of V exist.*

Note that the special directions in this case are given by

$$Y = \sqrt{\frac{\mu - 1}{2\mu}}W + \sqrt{\frac{\mu + 1}{2\mu}}HW$$

and

$$Z = -\sqrt{\frac{\mu - 1}{2\mu}}W + \sqrt{\frac{\mu + 1}{2\mu}}HW$$

where W is a unit eigenvector of h_V corresponding to the eigenvalue μ.

4. The Lie group $SL(2,\mathbb{C})$

We now study the Lie group

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} | z_1 z_4 - z_2 z_3 = 1 \right\}.$$

We form a 2-parameter family of complex contact metric structures on $SL(2,\mathbb{C})$ as follows. Take $\lambda > \mu > 0$ and consider the matrices

$$\frac{1}{2}\sqrt{\lambda^2-\mu^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sqrt{\frac{\lambda+\mu}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sqrt{\frac{\lambda-\mu}{2}}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

in the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ which we regard as the tangent space of $SL(2,\mathbb{C})$ at the identity. Applying the differential of left translation to these matrices gives the vector fields

$$\begin{aligned}
\xi_1 &= \frac{1}{2}\sqrt{\lambda^2-\mu^2}\Big(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\Big),\\
\xi_2 &= \sqrt{\frac{\lambda+\mu}{2}}\Big(z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3} - z_3\frac{\partial}{\partial z_4}\Big),\\
\xi_3 &= -\sqrt{\frac{\lambda-\mu}{2}}\Big(z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3} + z_3\frac{\partial}{\partial z_4}\Big).
\end{aligned}$$

The complex contact form on $SL(2,\mathbb{C})$ is

$$\omega = \frac{2}{\sqrt{\lambda^2-\mu^2}}(z_4 dz_1 - z_2 dz_3) = u - iv.$$

Set $\xi_1 = \frac{1}{2}(U+iV)$, $\xi_2 = \frac{1}{2}(E_2 - iJE_2)$, $\xi_3 = \frac{1}{2}(E_3 - iJE_3)$. The metric is determined by left translation of the basis $\{U, V, E_2, JE_2, E_3, JE_3\}$, as an orthonormal basis at the identity, and the structure tensors G and $H = GJ$ are determined by $GE_2 = E_3$ and $GJE_2 = -JE_3$. Computing the Lie brackets we have

$$[U,V] = 0,$$

$$\begin{gathered}
[U,E_2] = (\lambda+\mu)E_3, \quad [U,JE_2] = (\lambda+\mu)JE_3,\\
[U,E_3] = (\lambda-\mu)E_2, \quad [U,JE_3] = (\lambda-\mu)JE_2,\\
[V,E_2] = -(\lambda+\mu)E_3, \quad [V,JE_2] = (\lambda+\mu)JE_3,\\
[V,E_3] = -(\lambda-\mu)E_2, \quad [V,JE_3] = (\lambda-\mu)JE_2,\\
[E_2,JE_2] = 0, \quad [E_3,JE_3] = 0,\\
[E_2,E_3] = 2U, \quad [E_2,JE_3] = -2V,\\
[JE_2,E_3] = -2V, \quad [JE_2,JE_3] = -2U.
\end{gathered}$$

The basis $\{U, V, E_2, JE_2, E_3, JE_3\}$ is an eigenvector basis of the operators h_U and h_V. In particular

$$h_U E_2 = \lambda E_2, \quad h_U JE_2 = -\lambda JE_2, \quad h_V E_2 = \mu E_2, \quad h_V JE_2 = -\mu JE_2.$$

Using the anti-commutivities $h_U G + G h_U = 0$ and $h_V H + H h_V = 0$, we get

$$h_U E_3 = -\lambda E_3, \quad h_U JE_3 = \lambda JE_3, \quad h_V E_3 = \mu E_3, \quad h_V JE_3 = -\mu JE_3.$$

Also from this Lie algebra of vector fields, it is straightforward to compute covariant derivatives and, in particular, to easily show that $\sigma = 0$.

We now turn to the question of holomorphic special directions in $SL(2, \mathbb{C})$. Using the eigenvalues and the eigenvectors of h_U and h_V, we see that E_2 and JE_3 are eigenvectors of $k = h_U - h_V$ corresponding to the eigenvalues $\lambda - \mu$ and $\lambda + \mu$ respectively. From the solutions obtained in Section 2, if a and b are two numbers satisfying $a^2 + b^2 = \frac{\lambda-\mu}{2\lambda}$, then

$$Y = aE_2 + bJE_2 + \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}bJE_3 + \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}aE_3 \tag{24}$$

is a holomorphic special direction with

$$\nabla_{Y-iJY}(U + iV) = -\sqrt{\lambda^2 - \mu^2}(Y - iJY),$$

and

$$Z = aE_2 + bJE_2 - \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}bJE_3 - \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}aE_3 \tag{25}$$

is an independent holomorphic special direction with

$$\nabla_{Z-iJZ}(U + iV) = \sqrt{\lambda^2 - \mu^2}(Z - iJZ).$$

Theorem 4.1. *On $SL(2, \mathbb{C})$ the vector field ξ_1 is a holomorphic Anosov flow and the stable and unstable subbundles, E^s and E^u, agree with the subbundles determined by the special directions corresponding to solutions* (24) *and* (25) *respectively.*

Proof. The complex flow associated to the holomorphic vector field

$$2\xi_1 = U + iV = \sqrt{\lambda^2 - \mu^2}\Big(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\Big)$$

is

$$\psi_z = \begin{pmatrix} z_1 e^{\sqrt{\lambda^2-\mu^2}\,z} & z_2 e^{-\sqrt{\lambda^2-\mu^2}\,z} \\ z_3 e^{\sqrt{\lambda^2-\mu^2}\,z} & z_4 e^{-\sqrt{\lambda^2-\mu^2}\,z} \end{pmatrix}$$

and its differential with respect to $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}\}$ is given by

$$\psi_{z*} = \begin{pmatrix} e^{\sqrt{\lambda^2-\mu^2}\,z} & 0 & 0 & 0 \\ 0 & e^{-\sqrt{\lambda^2-\mu^2}\,z} & 0 & 0 \\ 0 & 0 & e^{\sqrt{\lambda^2-\mu^2}\,z} & 0 \\ 0 & 0 & 0 & e^{-\sqrt{\lambda^2-\mu^2}\,z} \end{pmatrix}.$$

Now for the solution (24)

$$\begin{aligned} Y - iJY &= (a+ib)(E_2 - iJE_2) + \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}(a+ib)(E_3 - iJE_3) \\ &= 2(a+ib)\xi_2 + 2\sqrt{\frac{\lambda+\mu}{\lambda-\mu}}(a+ib)\xi_3 \\ &= -4\sqrt{\frac{\lambda+\mu}{2}}(a+ib)\Big(z_1\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_4}\Big). \end{aligned}$$

Applying ψ_{z*} to $Y - iJY$ at the point p, we have

$$\psi_{z*}(Y-iJY)_p = e^{-\sqrt{\lambda^2-\mu^2}z}(Y-iJY)_p = e^{-2\sqrt{\lambda^2-\mu^2}z}(Y-iJY)_{\psi_z(p)}$$

and

$$||\psi_{z*}(Y-iJY)_p|| = e^{-\sqrt{\lambda^2-\mu^2}\Re(z)}||(Y-iJY)_p||.$$

Therefore the special direction of solution (24) determines the stable subbundle.

Similarly solution (25) yields the holomorphic special direction

$$Z - iJZ = 2(a+ib)\xi_2 - 2\sqrt{\frac{\lambda+\mu}{\lambda-\mu}}(a+ib)\xi_3$$

$$= 4\sqrt{\frac{\lambda+\mu}{2}}(a+ib)\Big(z_2\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_3}\Big).$$

Therefore

$$\psi_{z*}(Z-iJZ)_p = e^{2\sqrt{\lambda^2-\mu^2}z}(Z-iJZ)_{\psi_z(p)}$$

and

$$||\psi_{z*}(Z-iJZ)_p|| = e^{\sqrt{\lambda^2-\mu^2}\Re(z)}||(Z-iJZ)_p||$$

giving the unstable subbundle. □

We now study the real special directions associated to the vector field U.

Theorem 4.2. *If $\lambda > 1$, then there exist real special directions associated to the vector field U on $SL(2,\mathbb{C})$. Moreover when $\mu = 1$, U determines a partially hyperbolic flow with 2-dimensional central subbundle E^c spanned by U and V.*

Proof. The first statement is clear from Theorem 4.1. In terms of the coordinates $z_j = x_j + iy_j$, when $\mu = 1$,

$$U = \frac{1}{2}\sqrt{\lambda^2 - 1}\sum_{j=1}^{4}(-1)^{j+1}\Big(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\Big).$$

The corresponding flow ψ_t maps a point $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ to the point

$$e^{\frac{\sqrt{\lambda^2-1}}{2}t}(x_1, y_1, 0, 0, x_3, y_3, 0, 0) + e^{-\frac{\sqrt{\lambda^2-1}}{2}t}(0, 0, x_2, y_2, 0, 0, x_4, y_4).$$

Consider the particular eigenvector E_2 of h_U, eigenvalue λ, and the solution in the proof of Theorem 3.1. Then

$$Y = aE_2 + bE_3 = \sqrt{\frac{\lambda^2 - 1}{\lambda}}\Big(- x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2} - x_3\frac{\partial}{\partial x_4} - y_3\frac{\partial}{\partial y_4}\Big).$$

Applying ψ_{t*} at a point p we have

$$\psi_{t*}Y_p = e^{-\frac{\sqrt{\lambda^2-1}}{2}t}Y_p = e^{-\sqrt{\lambda^2-1}t}Y_{\psi_t(p)}$$

and

$$||\psi_{t*}Y_p|| = e^{-\frac{\sqrt{\lambda^2-1}}{2}t}||Y_p||.$$

Now consider the vector

$$JY = aJE_2 + bJE_3 = \sqrt{\frac{\lambda^2 - 1}{\lambda}}\Big(y_1\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial y_4}\Big).$$

Applying ψ_{t*} to this vector we have

$$\psi_{t*}JY_p = e^{-\frac{\sqrt{\lambda^2-1}}{2}t}JY_p = e^{-\sqrt{\lambda^2-1}t}JY_{\psi_t(p)}$$

and

$$||\psi_{t*}JY_p|| = e^{-\frac{\sqrt{\lambda^2-1}}{2}t}||JY_p||.$$

Thus Y and JY give a subbundle, E^s, which ψ_{t*} leaves invariant and for which ψ_{t*} shortens lengths exponentially. We remark that since JE_3 is also an eigenvector of h_U with eigenvalue λ, $aJE_3 + bJE_2$ is a special direction but the corresponding vector field is not invariant under ψ_{t*}.

Turning to the solution in the paragraph following the proof of Theorem 3.1, we have the special direction

$$Z = aE_2 + bE_3 = -\sqrt{\frac{\lambda^2 - 1}{\lambda}}\Big(x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} + x_4\frac{\partial}{\partial x_3} + y_4\frac{\partial}{\partial y_3}\Big)$$

which satisfies

$$\psi_{t*}Z_p = e^{\sqrt{\lambda^2-1}t}Z_{\psi_t(p)}$$

and together with JZ defines an unstable bundle E^u.

Finally consider the vector field

$$V = \frac{1}{2}\sqrt{\lambda^2 - 1}\sum_{j=1}^{4}(-1)^{j+1}\Big(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\Big).$$

Applying ψ_{t*} we have

$$\psi_{t*}V_p = V_{\psi_t(p)}.$$

Thus U defines a partially hyperbolic flow whose central bundle is 2-dimensional and spanned by U and V; E^s and E^u are the stable and unstable bundles. □

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