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On Gaussian Polynomials and Content Ideal

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Abstract. Loper and Roitman proved that every integral domain satisfies the property "each Gaussian polynomial has locally principal content". In this paper, we study this property in a ring with zero-divisors, and give then a class of such rings which does not satisfying this property and another class of rings with zero-divisors satisfying this property.

Keywords: Gaussian polynomial, content ideal, locally principal, trivial ring extension, direct product

1. Introduction

Throughout this paper all rings are assumed to be commutative with identity elements and all modules are unital.

Let R be a commutative ring. We say that an ideal is regular if it contains a regular element, i.e., a non-zerodivisor element. A ring R is called locally has a property (P) if each localisation of R by a maximal ideal of R has the property (P).

The content $C(f)$ of a polynomial $f \in R[X]$ is the ideal of R generated by all coefficients of f. One of its properties is that $C(.)$ is semi-multiplicative, that is $C(fg) \subseteq C(f)C(g)$, and a polynomial $f \in R[X]$ is said to be Gaussian over R if $C(fg) = C(f)C(g)$, for every polynomial $g \in R[X]$. A polynomial $f \in R[X]$ is Gaussian provided $C(f)$ is locally principal by [7, Remark 1.1]. Our guiding question is the converse of this property, that is "each regular Gaussian polynomial has locally principal content". Notice for convenience that the conjecture has a local character since the Gaussian condition is a local property (i.e., a polynomial is Gaussian over a ring R if and only if its image is Gaussian over R_M for each maximal ideal M of R), [12, Lemma 5].

Significant progress has been made on this conjecture. Glaz and Vasconcelos proved it for normal Noetherian domains [6]. Then Heinzer and Huneke established this conjecture over locally approximately Gorenstein rings and over locally Noetherian domains [7, Theorem 1.5 and Corollary 3.4]. Recently, Loper and Roitman established the conjecture for (locally) domains [11, Theorem 4], and then Lucas extended their result to arbitrary rings by restricting to polynomials with regular content [12, Theorem 3.6]. Finally, in [1], by using pullbacks, the authors construct a new class of rings that are not locally domains, nor locally Noetherians, and satisfy this conjecture. Let us note that Heinzer and Huneke, in [7, Remark 1.6], give an example showing that the conjecture is false in general.

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Note that a prime (respectively, maximal) ideal of R has the form $P \propto E$ (respectively, $M \propto E$) where P (respectively, M) is a prime (respectively, maximal) ideal of A (by [9, Theorem 25.1]).

Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [4] and Huckaba's book (where R is called the idealization of E by A) [9].

The goal of this work is to exhibit a class of rings (with zero-divisors) that does not satisfy the property "each Gaussian polynomial has locally principal content" and a second class of rings with zero-divisors satisfying this property. For this purpose, we study the transfer of this property to trivial ring extension and direct product.

2. Main results

This section develops a result of the transfer of the property "each Gaussian polynomial has locally principal content" for a particular context of trivial ring extensions. And so, we will construct a class of rings (with zero-divisors) that does not satisfy the property "each Gaussian polynomial has locally principal content".

Theorem 2.1. *Let (A, M) be a local ring which is not a field such that $M^2 = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E, where E is a free A-module. Then, R does not satisfy the property "each Gaussian polynomial has locally principal content" in the following cases:*

(1) *$rank_A(E) = 1$ and M is not principal,*

(2) *$rank_A(E) \geq 2$.*

Before proving Theorem 2.1, we establish the following Lemma.

Lemma 2.2. *Let (A, M) be a local ring which is not a field such that $M^2 = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E, where E is an A-module. Assume that there exists $a, b \in E - \{0\}$ such that $Ma = Mb = 0$ and the A-module generated by $\{a, b\}$ is not principal. Then, R does not satisfy the property "each Gaussian polynomial has locally principal content".*

Proof. Under the hypothesis of Lemma 2.2, remark that R is local with maximal ideal $M \propto E$ by [9, Theorem 25.1] since (A, M) is local. Let $f = (0, a) + (0, b)X \in R[X]$. Our aim is to show that f is Gaussian and $C(f)$ is not (locally) principal.

We claim that f is Gaussian. Indeed, let $g \in R[X]$. We may assume that $g \in (M \propto E)[X]$ (since if $g \notin (M \propto E)[X]$, then $C(g) = R$ and so $C(fg) = C(f)C(g)$). Hence, $C(f)C(g) = [R(0, a) + R(0, b)]C(g) \subseteq [R(0, a) + R(0, b)][M \propto E] = 0$ since $aM = bM = 0$ and so $C(fg) = C(f)C(g) = 0$ which means that f is Gaussian.

We claim that $C(f)$ is not principal. Deny. There exists $(c, d) \in R$ such that $C(f) := R(0, a) + R(0, b)(= 0 \propto (Aa + Ab)) = R(c, d)$ since R is local. Hence, $c = 0$ and so $Aa + Ab = Ad$, a contradiction since the A-module generated by $\{a, b\}$ is not principal. Therefore, $C(f)$ is not locally principal and this completes the proof of Lemma 2.2.

Proof of Theorem 2.1. Let (A, M) be a local ring which is not a field such that $M^2 = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E, where E is a free A-module.

1) Assume that $rank_A(E) = 1$ and M is not principal. We may assume that $E = A$. Hence, there exists $m_1, m_2 \in M - \{0\}$ such that $m_2 \notin Am_1$. Set $f = (m_1, 0) + (0, m_2)X \in R[X]$. Our aim is to show that f is Gaussian and $C(f)$ is not (locally) principal.

We claim that f is a Gaussian polynomial. Let $g = \sum_{i=0}^{n} (n_i, a_i)X^i \in R[X]$.

If $n_i \notin M$ for some $i = 0, \ldots, n$, then n_i is invertible in A and so (n_i, a_i) is invertible in R. Hence, $C(g) = R$ and so g is Gaussian; thus $C(fg) = C(f)C(g)$.

Assume that $n_i \in M$ for each $i = 0, \ldots, n$. We have $fg = \sum_{i=0}^{n} (m_1, 0)(n_i, a_i)X^i + \sum_{i=0}^{n} (0, m_2)(n_i, a_i)X^{i+1} = \sum_{i=0}^{n} (0, m_1 a_i)X^i$ since $m_1, m_2, n_i \in M$ and $M^2 = 0$. Hence $C(fg) = \sum_{i=0}^{n} R(0, m_1 a_i)$. On the other hand, $C(f)C(g)$

$= [R(m_1, 0) + R(0, m_2)][\sum_{i=0}^{n} R(n_i, a_i)] = \sum_{i=0}^{n} R(m_1, 0)(n_i, a_i) = \sum_{i=0}^{n} R(0, m_1 a_i)$ since $m_1, m_2, n_i \in M$ and $M^2 = 0$. Hence, $C(fg) = C(f)C(g)$ and so f is Gaussian.

We claim that $C(f)$ is not principal (since R is local). Deny. Then $C(f)(= R(m_1, 0) + R(0, m_2)) = R(n, e)$ for some $n \in M$ and $e \in A$ and so $(m_1, 0) = (a, b)(n, e)$ for some $(a, b) \in R$.

If $a \in M$, then $m_1 = an = 0$ since $n \in M$, a contradiction.

If $a \notin M$, then a is invertible in A and so (a, b) is invertible in R. Hence, $R(m_1, 0) = R(n, e)(= R(m_1, 0) + R(0, m_2))$ and so $(0, m_2) \in R(m_1, 0)$. Therefore, $(0, m_2) = (a, b)(0, m_1) = (0, am_1)$ and so $m_2 \in Am_1$, a contradiction. Therefore, $C(f)$ is not (locally) principal.

2) Now, assume that $rank_A(E) \geq 2$. Let $m \in M - \{0\}$, $(e_i)_{i\in I}$ be a basis of the free A-module E, $a_i = me_i \in E$ for $i = 1, 2$. We have $Ma_i = 0$ for each $i = 1, 2$ since $M^2 = 0$. We claim that the A-module generated by $(a_i)_{i=1,2}$ is not principal.

Assume that the A-module generated by $(a_i)_{i=1,2}$ is principal, that is $Aa_1 + Aa_2 = Ame_1 + Ame_2 = Af$ for some $f \in E$. Hence, $f = b_1me_1 + b_2me_2$ for some $b_i \in A$, where $i = 1, 2$. But, $a_1 \in Af$ implies that $a_1(:= me_1) = cf = cb_1me_1 + cb_2me_2$ for some $c \in A$. Thus, $m = cb_1m$ and $cb_2m = 0$ since $(e_i)_{i\in I}$ is a basis of the free A-module E. Therefore, $(1 - cb_1)m = 0$ and so $1 - cb_1 \in M$ since (A, M) is a local ring and $m \neq 0$. Hence, $cb_1 \notin M$ and so cb_1 is invertible; in particular, c is invertible.

Hence, the equation $cb_2m = 0$ implies that $b_2m = 0$ (as c is invertible) and so $b_2 \in M$ (since (A, M) is a local ring and $m \neq 0$). Hence, $f = b_1me_1 + b_2me_2 = b_1me_1$ as $b_2, m \in M$ and $M^2 = 0$. But $a_2(:= me_2) \in Aa_1 + Aa_2 = Af$ implies that $me_2 = df = db_1me_1$ for some $d \in A$; so $m = 0$ since $(e_i)_{i\in I}$ is a basis of the free A-module E, a contradiction as $m \neq 0$.

Therefore, the A-module generated by $(a_i)_{i=1,2}$ is not principal and Lemma 2.2 completes the proof of Theorem 2.1.

Remark 2.3. The hypothesis "M is not principal" in Theorem 2.1(1) is necessary. Indeed, let K be a field considered as a local ring with maximal ideal $M = 0$. Hence, $R := K \propto K$ is a local ring with unique proper ideal $R(0, 1)(= 0 \propto K)$. So, R satisfies the property "each Gaussian polynomial has locally principal content".

The goal of the following result is to construct a second class of rings that does not satisfy the property "each Gaussian polynomial has locally principal content".

Proposition 2.4. *Let (A, M) be a local ring which is not a Bézout ring such that $M^2 = 0$. Then A does not satisfy the property "each Gaussian polynomial has locally principal content".*

Proof. It suffices to show that there exists a polynomial $f \in A[X]$ such that $C(f)$ is not principal since A is local and Gaussian. For that, let's consider $a, b \in A$ such that the ideal generated by $\{a, b\}$ is not principal (since A is not a Bézout ring). Set $f = a + bX \in A[X]$. Therefore, $C(f) := aA + bA$ is not principal and this completes the proof of Proposition 2.4.

The next result gives a second wide class of rings that does not satisfy the property "each Gaussian polynomial has locally principal content".

Corollary 2.5. *Let (A, M) be a local ring which is not a field such that $M^2 = 0$ and let $R := A \propto E$, where E is a nonzero A-module such that $ME = 0$. Then R does not satisfy the property "each Gaussian polynomial has locally principal content".*

Proof. Remark that the ring R is local with maximal ideal $M \propto E$ which satisfies $(M \propto E)^2 = 0$ since $M^2 = 0$ and $ME = 0$. Hence, each polynomial in R is Gaussian. It remains to show that R is not Bézout by Proposition 2.4 (since $(M \propto E)^2 = 0$).
Let $a \in M - \{0\}$ and $e \in E - \{0\}$. We claim that the ideal I generated by $\{(a, 0), (0, e)\}$ is not principal. Deny. Assume that $I := R(a, 0)+R(0, e) = R(b, h)$, where $(b, h) \in M \propto E$. We claim that $b \neq 0$.
Indeed, if $b = 0$, then $(a, 0) \in I = R(0, h)$ which implies that $a = 0$, a contradiction. Hence $b \neq 0$.
But $(0, e) \in I = R(b, h)$. Hence, $(0, e) = (c, l)(b, h) = (cb, ch)$ for some $(c, l) \in R$ (since $b \in M$). Then, $cb = 0$ and $c \in M$ (since $c \notin M$ implies that c is invertible and $b = 0$, a contradiction). Therefore, $e = ch = 0$, a contradiction. Then I is not a principal ideal of R and so R is not a Bézout ring and this completes the proof of Corollary 2.5.

Now, we will construct a wide class of rings satisfying the property "each Gaussian polynomial has locally principal content". For this, we study the transfer of this property to finite direct products.

Theorem 2.6. *Let $(R_i)_{i=1,...,n}$ be a family of rings. Then $\prod_{i=1}^{n} R_i$ satisfies the property "each Gaussian polynomial has locally principal content" if and only if so does R_i for each $i = 1, \ldots, n$.*

Before proving Theorem 2.6, we establish the following Lemma.

Lemma 2.7. *Let R be a ring and, $h : R \to h(R)$ be a ring homomorphism, and f be a Gaussian polynomial in $R[X]$. Then the homomorphic image of f is a Gaussian polynomial in $h(R)[X]$.*

Proof. Let $h : R \to h(R)$ be a ring homomorphism, f be a Gaussian polynomial, and let $g = \sum_{i=0}^{n} a_i X^i \in R[X]$. Let's remark first that,

$$\begin{aligned} C_{h(R)}(h(g)) &= \sum_{i=0}^{n} h(R)h(a_i) \\ &= \sum_{i=0}^{n} h(Ra_i) \\ &= h(\sum_{i=0}^{n} Ra_i) \\ &= h(C_R(g)). \end{aligned}$$

Hence, we have (since f is Gaussian):

$$\begin{aligned} C_{h(R)}(h(f)h(g)) &= C_{h(R)}(h(fg)) \\ &= h(C_R(fg)) \\ &= h(C_R(f)C_R(g)) \\ &= h(C_R(f))h(C_R(g)) \\ &= C_{h(R)}(h(f))C_{h(R)}(h(g)). \end{aligned}$$

As desired.

Proof of Theorem 2.6. We will prove the result for $i = 1, 2$, and the theorem will be established by induction on n. Let $f = (f_1, f_2)$ be a Gaussian polynomial in $(R_1 \times R_2)[X]$, and M be a maximal ideal of $R_1 \times R_2$. Then $M = m_1 \times R_2$ or $M = R_1 \times m_2$ where $m_i \in Max(R_i)$ for $i = 1, 2$.
We may assume that $M = m_1 \times R_2$ (the case $M = R_1 \times m_2$ is similar). We wish prove that $C_{R_1 \times R_2}(f)_M$ is principal. But $(R_1 \times R_2)_M$ is naturally isomorphic to $(R_1)_{m_1}$ and $C_{R_1 \times R_2}(f)_M$ is isomorphic to $C(f_1)_{m_1}$. Therefore, $C_{R_1 \times R_2}(f)_M$ is principal since f_1 is supposed Gaussian by Lemma 2.7 and so $R_1 \times R_2$ satisfies the property "each Gaussian polynomial has locally principal content".

Conversely, assume that the polynomial f_1 is Gaussian in $R_1[X]$ (it is the same for f_2). We easily check that $f := (f_1, 0)$ is Gaussian in $(R_1 \times R_2)[X]$. Let $m_1 \in Max(R_1)$. Therefore, $(C_{R_1 \times R_2}(f))_{m_1 \times R_2}$ is principal since $R_1 \times R_2$ satisfies the property "each Gaussian polynomial has locally principal content". Hence, $(C_{R_1}(f_1))_{m_1}$ is principal (since $(R_1 \times R_2)_M$ is naturally isomorphic to $(R_1)_{m_1}$ and $C_{R_1 \times R_2}(f)_M$ is isomorphic to $C(f_1)_{m_1}$), which means that $C_{R_1}(f_1)$ is locally principal and this completes the proof of Theorem 2.6.

Now, we are able to construct a wide class of rings satisfying the property "each Gaussian polynomial has locally principal content".

Corollary 2.8. *Let $(R_i)_{i=1,\dots,n}$ be a family of domains. Then $\prod_{i=1}^{n} R_i$ satisfies the property "each Gaussian polynomial has locally principal content".*

Proof. By Theorem 2.5 and since every domain satisfies the property "each Gaussian polynomial has locally principal content" (by [11, Theorem 4]]).

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