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Cubic Ruled Surfaces with Constant Distribution Parameter in E_4

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Abstract. A first order invariant of ruled surfaces of E_3 is the so-called distribution parameter d in a generator. It is defined as the limit of the quotient of the distance and the angle of the generator and its neighbour. Ruled surfaces with constant parameter of distribution are of special interest and have been studied by many authors. H. Brauner could prove that the only nontrivial cubic ruled surface with constant distribution parameter in E_3 is a special type of a Cayley-surface.
This paper is devoted to the investigation of these problems for higher dimensions. We will in fact determine all cubic ruled surfaces of E_n with constant distribution parameter. Surprisingly, there is one class of such surfaces way beyond the 3-dimensional Cayley-surface case.

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1. Ruled surfaces in E_n and the distribution parameter

A C^1-immersion $\mathbf{X}(t,u) : (t,u) \in G \subset \Re^2 \longrightarrow E_n$ $(n \geq 3)$ given by

$$\mathbf{X}(t,u) := \mathbf{L}(t) + u\mathbf{E}(t) \quad t \in T,\ u \in \Re \tag{1}$$

with the C^1-curve $\mathbf{L} : t \in T \subset \Re \longrightarrow E_n$ and the C^1-set of direction vectors $\mathbf{E} : t \in T \subset \Re \longrightarrow \Re^n$ defines a *two-dimensional ruled C^1-surface* in

E_n.[1] $\mathbf{L}(t)$ is called *basic curve* of $\mathbf{X}$. The corresponding generators are given by $t = const$. The tangential behavior of the surface along a generator $t = t_0$ is determined by $\mathbf{E}(t_0)$ and the derivative vectors $\dot{\mathbf{E}}(t_0)$ and $\dot{\mathbf{L}}(t_0)$. The tangent planes at the points of a generator belong to a subspace of E_n with dimension $f(t) := \mathrm{Dim}[\mathbf{E}(t), \dot{\mathbf{E}}(t), \dot{\mathbf{L}}(t)] \leq 3$. Generators with $f(t_0) = 3$ are called *regular*. The tangent planes at the points of a regular generator are contained in a 3-dimensional *tangent space of* $\mathbf{X}$ *at the generator* spanned by the generator and $[\mathbf{E}(t), \dot{\mathbf{E}}(t), \dot{\mathbf{L}}(t)]$.[2]

We are able to measure the distance $\mathrm{dist}(t_0, t)$ and the angle $\phi(t_0, t)$ for a fixed generator $t = t_0$ and the generator given by t. As in the case of a ruled surface imbedded into the 3-dimensional Euclidean space E_3 we define the *distribution parameter*[3] $d(t_0)$ (for short "DP") of the ruled surface $\mathbf{X}(t, u)$ in the generator $t = t_0$ as the limit

$$d(t_0) := \lim_{t \longrightarrow t_0} \frac{\mathrm{dist}(t_0, t)}{\phi(t, t_0)}. \tag{2}$$

This yields

$$d(t) = \frac{\mathbf{E}^2(t)\ \mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \dot{\mathbf{L}}(t))}{\dot{\mathbf{E}}^2(t)\mathbf{E}^2(t) - (\mathbf{E}(t)\dot{\mathbf{E}}(t))^2}. \tag{3}$$

The determinant used in E_3 is replaced by $\mathrm{Vol}(\mathbf{A}, \mathbf{B}, \mathbf{C})$ here. It denotes the volume of the parallelepiped defined by the 3 vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Its square is defined via Gram's determinant

$$\mathrm{Vol}^2(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathrm{Det} \begin{pmatrix} \mathbf{A}^2 & \mathbf{AB} & \mathbf{AC} \\ \mathbf{AB} & \mathbf{BB} & \mathbf{BC} \\ \mathbf{AC} & \mathbf{BC} & \mathbf{C}^2 \end{pmatrix}. \tag{4}$$

This paper is devoted to ruled surfaces of constant distribution parameter $d(t) = const \in \Re - \{0\}\ \forall t \in T$. Cases with $d(t) \equiv 0\ \forall t \in T$ characterize developable surfaces which are excluded here.[4]

We consider the set of two-dimensional algebraic varieties of E_n with a one-parametric set of straight line generators. An element $\mathbf{X}$ with the additional property that any arbitrary $(n-1)$-dimensional subspace H of E_n either contains the whole 2-dimensional ruled surface $\mathbf{X}$ or intersects $\mathbf{X}$ in a cubic curve (in algebraic sense) is called *cubic ruled surface* in E_n $(n \geq 4)$.[5]

Any irreducible non-degenerate two-dimensional variety of degree k of E_n is contained in a subspace of dimension $\leq k + 1$.[6] For our case of two-dimensional

[1] These ruled surfaces are special cases of generalized ruled surfaces with k-dimensional generators treated by several authors (e.g. H. Frank and O. Giering [7] or H. Hagen [8]).

[2] For projective properties of ruled surfaces in the 4-dimensional space see R. Weitzenböck and W. Bos [17].

[3] An overview on line manifolds of E_3 has been given by G. Weiss [15]–[16]. For generalized ruled surfaces there are more such distribution parameters (see H. Frank, O. Giering [7]).

[4] We do not admit $d(t) \equiv \infty$.

[5] The cubic ruled surfaces of the 3-dimensional Euclidean space E_3 are well-known. This is why we restrict the following considerations to the case $n \geq 4$.

[6] For a proof see the textbook by J. Harris [9], p. 231.

cubic ruled surfaces we have: Any non-degenerate cubic ruled surface $\mathbf{X}$ is contained in a 4-dimensional Euclidean space. Therefore we can confine the following considerations to $n = 4$.

2. Cubic ruled surfaces in the real projective space of dimension 4

We embed E_4 into a real projective space P_4 of dimension 4 (if needed with its complex extension). In P_4 there are two different types of cubic ruled surfaces: The so-called 3-*dimensional cases* contained in a 3-dimensional subspace and a second type called *twisted* – its span being 4-dimensional. For results on ruled surfaces with constant DP immersed into a 3-dimensional Euclidean space E_3 we refer to results by H. Brauner [1]–[5]. J. Krames [11] demonstrated that a special cubic Cayley-surface has constant parameter of distribution. H. Brauner [2] was able to prove that these surfaces are the only nontrivial cubic ruled surfaces of E_3 with constant distribution parameter. Generalizations to cubic ruled surfaces with complex generators have been worked out by S. Mick [12].

The aim of this paper is to investigate the genuinely 4-dimensional cases which addresses the so-called *twisted cubic ruled surfaces* of E_4 with constant distribution parameter. They span the 4-dimensional space, but are not contained in a 3-dimensional subspace. Such surfaces can be generated by a rational $(1,1)$-correspondence between a straight line s (called the *directing line*) and a conic section c. There the line s and the plane of the conic section c have to be skew. The straight line s is the only line intersecting all generators of $\mathbf{X}$.

In order to prove this way of generating twisted third order ruled surfaces $\mathbf{X}$ we start with two different non-singular generators e_1, e_2 of $\mathbf{X}$. The 3-dimensional space $[e_1, e_2]$ intersects the cubic surface $\mathbf{X}$ in these two generators and a straight line s. s in turn will intersect e_1 and e_2. Any two general 3-dimensional spaces H_1, H_2 through e_1 intersect $\mathbf{X}$ in the generator e_1 plus a conic section c and k in planes γ and κ, respectively. $H_1 \cap H_2$ is a plane containing e_1 and a point $P \in k$ which will also lie on c. The 3-dimensional spaces $K(t)$ of the pencil through the plane κ intersect the surface $\mathbf{X}$ in k plus some straight line generator $e(t)$ cutting points $C(t)$ out of the conic c. $K(t)$ intersects the plane γ in lines of a pencil with vertex P projectively linked to the points $S(t)$ of intersection of $K(t)$ and, otherwise, points $S(t)$ on the straight line s. As $P \in c$ this projectivity determines a $(1,1)$-correspondence between the points $S(t) \subset s$ and $C(t) \subset c$.

We have: The generators $[S(t), C(t)]$ of $\mathbf{X}$ establish a $(1,1)$-correspondence between the points of s and c.

In E_4 we use Cartesian coordinates (x_1, x_2, x_3, x_4) with corresponding homogenous coordinates $(y_0 : y_1 : y_2 : y_3 : y_4)^t = (1 : x_1 : x_2 : x_3 : x_4)^t$ as counterparts in P_4. The rational $(1,1)$-correspondence always can be achieved by the same parameter values on s and c (see K. Rittenschober, B. Jüttler [13], pp. 27 and [10]). In homogenous coordinates a standard parametrization reads

$$\begin{array}{ll} s(t) \cdots & (1 : t : 0 : 0 : 0)^t \\ c(t) \cdots & (0 : 0 : 1 : t : t^2)^t, \ t \in \Re \cup \{\infty\}. \end{array} \tag{5}$$

The resulting twisted cubic ruled surface $\mathbf{X}$ (algebraically) intersects any 3-dimensional subspace of P_4 in a cubic curve. If the 3-dimensional subspace contains a generator of $\mathbf{X}$ but not the directing line s the cubic splits into the generator and a conic section (which meets the generator).[7] It is remarkable that a twisted cubic ruled surface in P_4 does not contain any singular generator.

Remarks.

- The twisted cubic surface $\mathbf{X}$ is contained in a cone Γ of degree 2 which is defined by the straight line vertex s and the conic section c. For the surface determined by (5) the equation of this cone is given by

$$\Gamma \cdots y_3^2 = y_4 y_2. \tag{6}$$

  The tangent spaces of this quadratic cone Γ are the tangent spaces of the ruled surface $\mathbf{X}$ in its generators.
- P_4 contains the ideal space or space at infinity $\Omega \cdots y_0 = 0$ of E_4.[8] The affine type of $\mathbf{X}$ depends on the ideal curve $\mathbf{X_u} = \mathbf{\Omega} \cap \mathbf{X}$. It can happen that either s or c (but not both!) are parts of $\mathbf{X_u}$.
- The Euclidean structure of E_4 is determined by the polarity of the ideal elements with respect to the absolute quadric $\mathbf{M} \subset \Omega$ given by

$$\mathbf{M} \cdots y_0 = 0,\ y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0. \tag{7}$$

In order to get an affine classification we will classify the twisted cubic ruled surfaces according to their ideal curve $\mathbf{X_u}$. This algebraic curve can be a twisted cubic or it can split into parts: According to the possible shape of $\mathbf{X_u}$ we have 3 different types:

Case A: $\mathbf{X_u}$ is a twisted cubic.

Case B: $\mathbf{X_u}$ splits into a conic section and a real generator line (which intersects the conic section in one point, but is not part of the plane of the conic section).

Case C: $\mathbf{X_u}$ contains the directing line s and two generator lines (in algebraic sense). All generators are parallel to a plane, the surface therefore is a conoidal ruled surface.

To simplify our further investigations we will give standard Euclidean representations of the surfaces of these types:

In the cases A and B we can use the real straight directing line s parametrized by $\mathbf{L}(t) := t\ \mathbf{L}^*$ with constant real vector $\mathbf{L}^*$. A rational linear parameter transformation yields

$$\mathbf{X}(t,u) = [\mathbf{E}_0^* + t\mathbf{E}_1^* + t^2\mathbf{E}_2^* + \varepsilon t^3 \mathbf{L}^*] + ut\mathbf{L}^* \ (t,u) \in \Re^2 \tag{8}$$

with 4 linearly independent vectors $\{\mathbf{E}_0^*, \mathbf{E}_1^*, \mathbf{E}_2^*, \mathbf{L}^*\}$. The case A is characterized by $\varepsilon \neq 0$ while we have $\varepsilon = 0$ for case B. The generators determine a two-dimensional cubic (case A) or quadratic direction cone (case B). In case B there

[7] All has to be seen in algebraic sense. The conic must not be real and can degenerate.

[8] The elements at infinity will be addressed as ideal elements here.

is a real generator line of the surface in the ideal space Ω. In our parametrization it belongs to $t = \infty$. In case A the parameter $t = \infty$ points to the common point on $\mathbf{X_u}$ and the directing line s, but determines a non ideal generator.

In case C we apply a displacement of E_4 transforming the directing line s and the conic section c into

$$\begin{array}{ll} s \cdots & (0:1:t:0:0)^t \\ c \cdots & (1:l_1:l_2:l_3:l_4)^t,\ t \in \Re \cup \{\infty\} \end{array} \tag{9}$$

where c is given by quadratic rational functions $l_i(t)$. We gain a conoidal twisted cubic ruled surface with generators parallel to the 2-dimensional plane $[x_1, x_2]$. A Euclidean parametrization is given by

$$\mathbf{X}(t,u) = \mathbf{E}(t) + u\mathbf{L}(t) = \begin{pmatrix} 1 \\ t \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \\ l_4(t) \end{pmatrix}. \tag{10}$$

Now we are ready to characterize twisted ruled cubic surfaces of E_4 with constant distribution parameter. Our investigations relate to the cases A, B and C.

3. Twisted cubic ruled surfaces of cases A and B with constant DP

We start with the parametric representation (8) and discuss the factors of the formula (3):

$\mathbf{E}^2(t)$ is a polynomial in t, which is of degrees ≤ 6 and ≤ 4 for case A and B, respectively. The denominator $\mathbf{E}^2\dot{\mathbf{E}}^2 - (\mathbf{E}\dot{\mathbf{E}})^2$ will be a polynomial in t, as well. In the ases A and B we have no pole for $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \dot{\mathbf{L}}(t))$. In order to gain constant DP we have as a first necessary condition:[9]

(N1) $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \dot{\mathbf{L}}(t))$ has to be a polynomial in t.

Moreover, the zeros of the numerator and the denominator polynomials of (3) have to coincide. This prompts two further necessary conditions for twisted cubic ruled surfaces of type A or B in order to have constant DP:

(N2) For any isotropic generator $t = t^*$ with $\mathbf{E}^2(t^*) = 0$ we have $\dot{\mathbf{E}}^2(t^*) = 0$, too. The geometric interpretation reads as: Any isotropic generator has an isotropic asymptotic plane. Or, considering the ideal curve $\mathbf{X_u}$: Any intersection of the absolute $\mathbf{M}$ and $\mathbf{X_u}$ counts at least twice in algebraic sense.

(N3) For any generator $t = t^*$ with $\mathrm{Vol}(\mathbf{E}(t^*), \dot{\mathbf{E}}(t^*), \dot{\mathbf{L}}(t^*)) = 0$ (generator with isotropic tangent space) the denominator will also have a zero. Generators with isotropic tangent space must have an isotropic asymptotic plane.

As we are dealing with surfaces with real generators, these zeros cannot be real and have to be pairs of conjugate complex numbers. According to condition (N2) any zero of the polynomial $\mathbf{E}^2$ has to count twice. In the cases A and B this gives

[9] These considerations are similar to those by H. Brauner [2] for E_3.

two conjugate complex zeros $t_1, \bar{t}_1$ of the polynomial $\mathbf{E}^2$. t_1 and $\bar{t}_1$ have to be 3-fold[10] and 2-fold zeros in the case A and B, respectively. A linear transformation in the parametric domain yields $t_1 = i, \bar{t}_1 = -i$, i denoting the complex unit with $i^2 = -1$.

Thus (N2) delivers

$$\mathbf{E}^2(t) = (1+t^2)^k (\mathbf{E}_0^*)^2 \tag{11}$$

with $k = 2, 3$ for the cases A and B, respectively. This implies the following series of conditions for case A:

$$\begin{array}{lll} (\mathbf{E}_0^*)^2 = \varepsilon^2 (\mathbf{L}^*)^2 & \mathbf{E}_0^* \mathbf{E}_1^* = 0 & \varepsilon \mathbf{E}_2^* \mathbf{L}^* = 0 \\ (\mathbf{E}_1^*)^2 + 2\mathbf{E}_0^* \mathbf{E}_2^* = 3(\varepsilon \mathbf{L}^*)^2 & \varepsilon \mathbf{E}_0^* \mathbf{L}^* + \mathbf{E}_1^* \mathbf{E}_2^* = 0 & (\mathbf{E}_2^*)^2 + 2\varepsilon \mathbf{E}_1^* \mathbf{L}^* = 3(\varepsilon \mathbf{L}^*)^2. \end{array} \tag{12}$$

For case B we have

$$\begin{array}{ll} (\mathbf{E}_0^*)^2 = (\mathbf{E}_2^*)^2 & 2\mathbf{E}_0^* \mathbf{E}_2^* + (\mathbf{E}_1^*)^2 = 2(\mathbf{E}_0^*)^2 \\ \mathbf{E}_0^* \mathbf{E}_1^* = 0 & \mathbf{E}_1^* \mathbf{E}_2^* = 0. \end{array} \tag{13}$$

Using conditions (12) we compute the distribution parameter for case A as

$$d(t) = \frac{\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)}{(\mathbf{E}_1^*)^2 + 4t\mathbf{E}_1^* \mathbf{E}_2^* + t^2 (\mathbf{E}_2^*)^2} \tag{14}$$

and for case B

$$d(t) = \frac{\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)}{(\mathbf{E}_1^*)^2}. \tag{15}$$

In both cases we have to discuss the zeros of $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)$: As stated before they belong to generators of $\mathbf{X}$ with isotropic tangent space. The ideal curve $\mathbf{E_u}$ of the generators is given by the homogenous coordinates $(0, \mathbf{E}(t)^t)$. Using the ideal point $(0, (\mathbf{L}^*)^t)$ [11] as the vertex of an ideal cone with the basic curve $\mathbf{E_u}$ we gain a 2-dimensional ideal cone Φ of degree 2 in Ω. In algebraic sense this real quadratic cone Φ in general will have 4 tangent planes touching the absolute quadric $\mathbf{M}$. They are the ideal planes of the isotropic tangent spaces to some generators of $\mathbf{X}$. So we can state: The twisted cubic ruled surfaces of case A and B in algebraic sense have 4 generators with isotropic tangent space. They belong to zeros of $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)$. Therefore $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)$ cannot be constant and according to (15) there is no twisted cubic ruled surface of the case B with constant DP.

Case A will be our next try: As the denominator of (14) is a quadratic polynomial in t, the 4 possible zeros of $\mathrm{Vol}(\mathbf{E}(t), \dot{\mathbf{E}}(t), \mathbf{L}^*)$ have to coincide in pairs of conjugate complex numbers $t_{1,2}$ with $t_2 = \bar{t}_1$. The cone Φ will be tangent to the

[10] In the case A (in algebraic sense) we have 6 intersections of $\mathbf{X_u}$ and $\mathbf{M}$ which are complex and belong to pairs of conjugate complex numbers. The condition (N2) implies that each of the corresponding zeros has to count (at least) twice. Thus the only possibility in case A is the existence of 2 conjugate complex roots, each counting 3 times.

[11] In case A we have the ideal curve $\mathbf{X_u} = \mathbf{E_u}$ and $(0, (\mathbf{L}^*)^t)$ is a point on it. In case B $(0, (\mathbf{L}^*)^t)$ does not belong to $\mathbf{E_u}$.

absolute quadric $\mathbf{M}$ in 2 conjugate complex points. The corresponding generators of Φ are given by $[(0,(\mathbf{E}(t_{1,2})^t),(0,(\mathbf{L}^*)^t)]$ and are tangent to the absolute quadric $\mathbf{M}$. According to condition (N3) the tangent $[(0,(\mathbf{E}(t_{1,2})^t),(0,\dot{\mathbf{E}}(t_{1,2})^t)]$ of $\mathbf{E_u}$ at $t_{1,2}$ has to be tangent to $\mathbf{M}$. As $\mathbf{E_u}$ is a non-degenerate twisted cubic curve in case A the points $(0,\mathbf{E}(t_{1,2})^t)$ must be points on the absolute $\mathbf{M}$. The vertex of the quadratic cone Φ has to be lying within the two tangent planes to $\mathbf{M}$ at $(0,\mathbf{E}(t_{1,2})^t)$. The corresponding generators have to be isotropic generators – the only two on $\mathbf{X}$ are given by $t=\pm i$ and have the direction vectors

$$\mathbf{E}(\pm i)=(\mathbf{E}_0^*-\mathbf{E}_2^*)\pm i(\mathbf{E}_1^*-\mathbf{E}_3^*). \tag{16}$$

The vertex of the cone Φ is determined by the direction $\mathbf{L}^*$ with

$$\begin{aligned}&0=\mathbf{L}^*\mathbf{E}(\pm i)\Leftrightarrow\\ &0=\mathbf{L}^*(\mathbf{E}_0^*-\mathbf{E}_2^*)\text{ and }0=\mathbf{L}^*(\mathbf{E}_1^*-\mathbf{E}_3^*).\end{aligned} \tag{17}$$

With these assumptions $t=\pm i$ are zeros of $\mathrm{Vol}(\mathbf{E}(t),\dot{\mathbf{E}}(t),\dot{\mathbf{L}}(t))$. According to (N3) these zeros $t=\pm i$ must be zeros of $(\mathbf{E}_1^*)^2+4t\mathbf{E}_1^*\mathbf{E}_2*+t^2(\mathbf{E}_2^*)^2$, too. This yields the additional conditions

$$(\mathbf{E}_1^*)^2=(\mathbf{E}_2^*)^2\text{ and }\mathbf{E}_1^*\mathbf{E}_2^*=0. \tag{18}$$

(12), (17) and (18) are conditions for the 4 linearly independent real vectors $\{\mathbf{E}_0^*,\mathbf{E}_1^*,\mathbf{E}_2^*,\mathbf{L}^*\}$, which are summarized in

$$\begin{aligned}(\mathbf{E}_0^*)^2=(\mathbf{E}_1^*)^2&=(\mathbf{E}_2^*)^2=(\varepsilon\mathbf{L}^*)^2\\ \mathbf{L}^*\ \mathbf{E}_0^*=\mathbf{E}_0^*\mathbf{E}_1^*&=\mathbf{E}_1^*\mathbf{E}_2^*=\mathbf{E}_2^*\mathbf{L}^*=0\\ \varepsilon\mathbf{E}_1^*\mathbf{L}^*=\mathbf{E}_0^*\mathbf{E}_2^*&=(\mathbf{E}_0^*)^2.\end{aligned} \tag{19}$$

As we have

$$\cos(\mathbf{E}_0^*,\mathbf{E}_2^*)=\frac{\mathbf{E}_0^*\ \mathbf{E}_2^*}{\|\mathbf{E}_0^*\|\|\mathbf{E}_2^*\|}=1 \tag{20}$$

$\mathbf{E}_0^*$ and $\mathbf{E}_2^*$ have to be linearly dependent. As a consequence (19) cannot be met by real and linearly independent vectors $\{\mathbf{E}_0^*,\mathbf{E}_1^*,\mathbf{E}_2^*,\mathbf{L}^*\}$. Thus there are no twisted cubic ruled surfaces of case A with constant DP. We sum up in

Theorem 1. *There are no twisted cubic ruled surfaces with constant distribution parameter in the cases* A *and* B.

4. Twisted cubic ruled surfaces of case C of E_4 with constant DP

We have to regard to the standard parametric representation (10) of these surfaces with respect to the DP (3). We compute

$$\begin{aligned}&\mathrm{Vol}^2(\mathbf{E}(t),\dot{\mathbf{E}}(t),\dot{\mathbf{L}}(t))=\dot{l}_3^2(t)+\dot{l}_4^2(t)\text{ and}\\ &d^2(t)=(1+t^2)^2(\dot{l}_3^2(t)+\dot{l}_4^2(t)).\end{aligned} \tag{21}$$

The distribution parameter is constant $d(t) := D \neq 0\ \forall t \in \Re$ iff

$$\dot{l}_3^2(t) + \dot{l}_4^2(t) = \frac{D^2}{(1+t^2)^2} \quad \forall t \in \Re. \tag{22}$$

Remark. The situation is different from the one in the cases A and B: The zeros $t = \pm i$ of $\mathbf{E}^2(t)$ have to be compensated by the poles of the volume function and therefore of $\dot{\mathbf{L}}(t)$. We have no direct counterparts to the necessary conditions (N1), (N2) and (N3) here.

(22) is a condition for the orthogonal projection $\mathbf{L}"(t) \cdots (0, 0, l_3(t), l_4(t))^t$ of the conic section $\mathbf{L}(t)$ onto the $[x_3, x_4]$-plane. It characterizes $\mathbf{L}"(t)$ as rational curve with planar rational offsets (in the $[x_3, x_4]$-plane). Therefore $\mathbf{L}"(t)$ has to be a so-called Pythagorean hodograph curve[12] of degree ≤ 2. Thus, $\mathbf{L}"(t)$ has to be a circle, a straight line or has to degenerate into a single point.

In the last two cases the original conic $\mathbf{L}(t)$ is part of a plane or a 3-dimensional space parallel to the $[x_1, x_2]$-plane. The corresponding cubic surfaces are contained in 3-dimensional subspaces – a case excluded here.[13]

If $\mathbf{L}"(t)$ is a circle with radius $R > 0$, it can be transformed into the standard form

$$l_3(t)^2 + l_4(t)^2 = R^2. \tag{23}$$

The resulting ruled surface is contained in a hypercylinder of revolution Γ with the equation

$$\Gamma \cdots x_3^2 + x_4^2 = R^2. \tag{24}$$

This yields

Lemma 1. *The twisted cubic ruled surfaces with constant distribution parameter have to be part of a hypercylinder of revolution Γ with 2-dimensional generators totally orthogonal to the plane of the basic circle.*

Now we are looking for a parametrisation of $\mathbf{L}"(t)$ in line with conditions (22) and (23). A short computation yields standard representations of the form

$$\begin{aligned} l_3(t) &= R(1-t^2)/(1+t^2) \\ l_4(t) &= 2Rt/(1+t^2). \end{aligned} \tag{25}$$

Of course, we can apply any displacement in the plane $[x_3, x_4]$ which keeps the origin invariant. All corresponding ruled surfaces (independent from $l_1(t)$ and $l_2(t)$) have the same constant distribution parameter

$$d^2(t) = 4R^2. \tag{26}$$

[12] Such planar rational curves with planar rational offset curves have been considered by R. Farouki and T. Sakkalis [6].

[13] As mentioned before, cubic ruled surfaces contained in a 3-dimensional space have been studied in detail by H. Brauner [2]. He showed that for the conoidal case C there are no nontrivial counterparts with constant DP contained in a 3-space.

So the case C contains twisted cubic ruled surfaces with constant distribution parameter if we choose the starting conic section $\mathbf{L}(t)$ as a non-degenerate 2-dimensional planar section of Γ.[14] As a standard representation we can use

$$\mathbf{L}(t) = \frac{1}{1+t^2}\begin{pmatrix} (1-t^2)\alpha_1 + 2t\beta_1 \\ (1-t^2)\alpha_2 + 2t\beta_2 \\ R(1-t^2) \\ 2Rt \end{pmatrix} \quad \text{and} \quad \mathbf{E}(t) = \begin{pmatrix} 1 \\ t \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

with arbitrary constants $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Re$. We sum up in

Theorem 2. *The only twisted cubic ruled surfaces of constant distribution parameter are special conoidal ruled surfaces of case* C. *They are contained in quadratic cylinders of revolution with* 2*-dimensional generators. A standard parametrization is given by* (27).

Remarks.

- If we reparametrise by $t := \tan(u/2)$ our parametric representation of the ruled surface can be transformed into the normal form

$$\mathbf{X}(u,v) = \begin{pmatrix} \alpha_1 \cos u + \beta_1 \sin u \\ \alpha_2 \cos u + \beta_2 \sin u \\ R\cos u \\ R\sin u \end{pmatrix} + v \begin{pmatrix} \cos(u/2) \\ \sin(u/2) \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

  with $u \in [0, 2\pi), v \in \Re$.
- These results demonstrate that in E_4 there do exist twisted cubic ruled surfaces with constant distribution parameter which are conoidal – their generators being parallel to a fixed plane of E_4. This is remarkable, as there are no nontrivial counterparts to this case in E_3 (see H. Brauner [2]).
- The surface has two conjugate complex isotropic generators. According to the results by H. Brauner for E_3 (see [2]) we would expect these isotropic generators being generators with isotropic asymptotic planes. Here the two isotropic generators are contained in the absolute quadric $\mathbf{M}$ of E_4 in the ideal space Ω.
- The case $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ in (28) gives a *helical ruled surface* of E_4 which is generated by a one-parameter helical motion. This special motion is generated by the composition of two rotations in totally orthogonal planes of E_4. The ratio of the angular velocities is $1:2$. The corresponding helical paths in general are rational quartics. In the ideal space Ω it induces a one-parameter subgroup of the group of elliptic displacements with respect to the absolute $\mathbf{M}$.[15]

[14] We gain further conoidal ruled surfaces with constant DP immersed into Γ if $l_1(t)$ and $l_2(t)$ are chosen as arbitrary C^1-functions in t.

[15] Such non-Euclidean screws have been considered by K. Strubecker in [14].

Now we sum up in the final

Theorem 3. *There are twisted cubic ruled surfaces of constant distribution parameter in E_4. They are conoidal ruled surfaces contained in a quadratic cylinder of revolution with* 2*-dimensional generators. A standard parametrisation of all these surfaces is given by* (27) *and* (28).

5. Conclusions

In this paper we have been able to determine all twisted cubic ruled surfaces of E_4 with constant distribution parameter. Amazingly, in E_4 there is no counterpart to the only case in E_3 (the special Cayley-surface). So this special Cayley-surface is embedded into a series of forth order ruled surfaces with constant DP of E_3 (see H. Brauner [1] – [5]), but not into a series of third order ruled surfaces with constant DP of E_4.

It comes as a surprise that there is another range of twisted cubic ruled surfaces with constant DP in E_4 which is conoidal. This is astonishing as there is no conoidal cubic ruled surface with constant DP showing up in E_3.

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