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Gaussian Marginals of Convex Bodies with Symmetries

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Abstract. We prove Gaussian approximation theorems for specific k-dimensional marginals of convex bodies which possess certain symmetries. In particular, we treat bodies which possess a 1-unconditional basis, as well as simplices. Our results extend recent results for 1-dimensional marginals due to E. Meckes and the author.

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1. Introduction

Let K be a convex body in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, equipped with its standard inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$, and let μ denote the uniform (normalized Lebesgue) probability measure on K. In this paper we consider k-dimensional marginals of μ, that is, the push-forward $\mu \circ P_E^{-1}$ of μ by the orthogonal projection P_E onto some k-dimensional subspace $E \subset \mathbb{R}^n$.

The question of whether every convex body $K \subset \mathbb{R}^n$ has 1-dimensional marginals which are close to Gaussian measures when n is large is known as the *central limit problem for convex bodies*, and was apparently first explicitly posed in the literature in [1], [6]. A natural extension is to ask, for how large $k \leq n$ does K necessarily possess nearly Gaussian k-dimensional marginals? The latter question can be thought of as asking for a measure-theoretic analogue of Dvoretzky's theorem, which implies the existence of nearly ellipsoidal k-dimensional projections of K when $k \ll \log n$.

*This work was mostly completed while the author was at Stanford University.

Very recently Klartag [12], [13] showed that any convex body has nearly Gaussian k-dimensional marginals when $k \leq cn^{\alpha}$, where $c > 0$ and $0 < \alpha < 1$ are some universal constants; closeness of probability measures is quantified by the total variation metric and also has a power-law dependence on n. This points out an important difference from Dvoretzky's theorem, in which it is known that for an arbitrary convex body k can only be taken to be logarithmically large in n. Klartag's work followed partial results, involving different additional hypotheses and metrics between probability measures, by many authors; we mention [4], [19], [22], [14], [9] among recent contributions and refer to [12] for further references.

In much of the work on this problem, including the main results of [12], [13], the existence of nearly Gaussian marginals $\mu \circ P_E^{-1}$ is proved nonconstructively, so that no concrete such subspace E is exhibited. This is typical of the proofs of Dvoretzky-like results. In [18], E. Meckes and the author used Stein's method of exchangeable pairs to prove Berry-Esseen theorems for specific 1-dimensional marginals of convex bodies which possess certain types of symmetries. Roughly, under some additional hypotheses, [18] shows that a 1-dimensional marginal $\mu \circ P_E^{-1}$ is nearly Gaussian when K possesses many symmetries $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ for which the 1-dimensional subspace $\sigma(E) \subset \mathbb{R}^n$ is very different from E. In [12], [14], another approach is used to study the marginal of a 1-unconditional body on the subspace spanned by $(1, \dots, 1)$; see the remarks at the end of this paper for the relationship between these approaches.

The main purpose of this paper is to prove versions of the results of [18] for k-dimensional marginals with $k \geq 2$, using a new multivariate version of Stein's method of exchangeable pairs due to S. Chatterjee and E. Meckes [7]. Our results show that, in contrast to the situation for Dvoretzky's theorem, in the measure-theoretic setting one can identify specific well-behaved high-dimensional projections for large classes of convex bodies. We consider bodies K which are 1-unconditional, or which possess all the symmetries of a centered regular simplex. Another purpose of this paper is to point out how some of the methods used here improve quantitatively some of the results of [18]. In [18] a symmetry hypothesis was introduced which simultaneously generalizes 1-unconditionality and the symmetries of a regular simplex, described in terms of a normalized tight frame of vectors. For the sake of transparency we have preferred to treat these special cases independently here, although that more general setting could also be treated with the methods of this paper.

Many of the results in this area treat marginals of probability measures μ more general than uniform measures on convex bodies; in particular the methods of [18] apply to completely arbitrary probability measures which satisfy the symmetry hypotheses. One common generalization, treated in [12], [13] for example, is to *log-concave* measures, i.e., measures with a logarithmically concave density with respect to Lebesgue measure. This is a natural setting since marginals of log-concave measures are again log-concave. While some of the methods of this paper apply to general probability measures, we have chosen to restrict to the log-concave case, in which stronger results are possible.

The arguments in this paper are a synthesis of the methods of the papers [18],

[7], [12], [14]. The proofs of the main results generalize the arguments of [18] in order to apply an abstract normal approximation result in [7]. In order to derive stronger results for log-concave measures, we apply a concentration result from [14] and adapt a smoothing argument from [12], [9].

The rest of this paper is organized as follows. After defining some notation and terminology, in Section 2 we state and discuss our main results. Section 3 presents and develops our tools. Finally, in Section 4 we prove our main results and make some final remarks about our methods.

Notation and terminology

It will be convenient to frame our results in terms of random vectors rather than probability measures. We use $\mathbb{P}$ and $\mathbb{E}$ to stand for probability and expectation respectively, and denote by $\mathbb{E}[Y|X]$ the conditional expectation of Y given the value of X.

Throughout this paper $X = (X_1, \dots, X_n)$ will be a random vector in $\mathbb{R}^n$, $n \geq 2$. A random vector is 1-*unconditional* if its distribution is invariant under reflections in the coordinate hyperplanes of $\mathbb{R}^n$. By Z we denote a standard Gaussian random vector in $\mathbb{R}^n$ with density

$$\varphi_1(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$$

with respect to Lebesgue measure, or a standard Gaussian random variable in $\mathbb{R}$; the usage should be clear from context. A random vector X is called *isotropic* if it has mean 0 and identity covariance:

$$\mathbb{E}X = 0, \qquad \mathbb{E}X_iX_j = \delta_{ij}.$$

Observe that if X is isotropic then $\mathbb{E}|X|^2 = n$. Isotropicity is a natural assumption in this setting since it is preserved by orthogonal projections and Z is isotropic; see [19] however for recent work demonstrating that a nonisotropic affine image of X is more useful in some contexts.

The total variation metric on the distributions of random vectors in $\mathbb{R}^n$ may be defined by the two equivalent expressions:

$$\begin{aligned} d_{TV}(X,Y) &= 2\sup\big\{|\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]| : A \subset \mathbb{R}^n \text{ measurable}\big\} \\ &= \sup\big\{|\mathbb{E}f(X) - \mathbb{E}f(Y)| \ : \ f \in C_c(\mathbb{R}^n),\ \|f\|_\infty \leq 1\big\}. \end{aligned}$$

The normalization stated here is the conventional one in analysis and differs by a factor of 2 from a normalization used frequently in probability texts. Note that $d_{TV}(X,Y) = \|f - g\|_1$ if X and Y possess densities f and g respectively. The (L_1)-Wasserstein metric is defined by requiring test functions to be Lipschitz instead of bounded:

$$d_1(X,Y) = \sup\big\{|\mathbb{E}f(X) - \mathbb{E}f(Y)| \ : \forall x, y \in \mathbb{R}^n,\ |f(x) - f(y)| \leq |x - y|\big\}.$$

Note that d_1 metrizes a weaker topology on probability measures than d_{TV}, but quantitative results for these two metrics are not directly comparable. In particular, $d_{TV}(X, Y) \leq 2$ always, but the typical order of magnitude of $d_1(X, Y)$ is $\sqrt{n}$.

For $x \in \mathbb{R}^n$, $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ if $1 \leq p < \infty$. Except where noted, symbols c, C, etc. denote universal constants, independent of n, k, and the distribution of X, which may differ in value from one appearance to another.

2. Statements of the main results

Let $\theta_i = (\theta_i^1, \dots, \theta_i^n)$, $1 \leq i \leq k$, be a fixed collection of k orthonormal vectors. Given an isotropic random vector $X \in \mathbb{R}^n$, define

$$W_i = \langle X, \theta_i \rangle . \tag{2.1}$$

Then $W = (W_1, \dots, W_k) \in \mathbb{R}^k$ is isotropic, and is essentially the same as $P_E(X)$, where E is spanned by $\theta_1, \dots, \theta_k$. More concretely, $W = T(P_E(X))$, where $T : \mathbb{R}^n \to \mathbb{R}^k$ is the partial isometry given by the matrix whose ith row is θ_i. Theorems 2.1 and 2.2 give bounds on the Wasserstein and total variation distance of W from a standard Gaussian random vector $Z \in \mathbb{R}^k$.

Theorem 2.1. *Let $X \in \mathbb{R}^n$ be 1-unconditional, log-concave, and isotropic, and let $W \in \mathbb{R}^k$ be as defined in (2.1). Then*

$$d_1(W, Z) \leq 14 \sqrt{k \sum_{i=1}^k \|\theta_i\|_4^2}$$

and

$$d_{TV}(W, Z) \leq C k^{5/6} \left(\sum_{i=1}^k \|\theta_i\|_4^2 \right)^{1/3} .$$

Before stating our other main results we will make some remarks to put the bounds in Theorem 2.1 in perspective. To begin, assume for the moment that

$$|\theta_i^\ell| = n^{-1/2} \quad \forall\, i, \ell. \tag{2.2}$$

Theorem 2.1 then shows

$$d_1(W, Z) \leq 14 \frac{k}{n^{1/4}} \tag{2.3}$$

and

$$d_{TV}(W, Z) \leq C \frac{k^{7/6}}{n^{1/6}}. \tag{2.4}$$

In particular, $d_1(W, Z) \ll 1$ as soon as $k \ll n^{1/4}$ and $d_{TV}(W, Z) \ll 1$ as soon as $k \ll n^{1/7}$.

If n is a power of 2 then (2.2) will be satisfied if $\{\sqrt{n}\theta_i : 1 \leq i \leq k\}$ are the first k vectors in a Walsh basis for $\mathbb{R}^n$. For arbitrary n it is not necessarily possible to satisfy (2.2) for $\theta_1, \ldots, \theta_k$ orthogonal. However, up to the values of constants, (2.3) and (2.4) will be satisfied by letting $\{\sqrt{m}\theta_i : 1 \leq i \leq k\}$ be the first k vectors in a Walsh basis for $\mathbb{R}^m \subseteq \mathbb{R}^n$, where m is the largest power of 2 not exceeding n (so that $m > n/2$). In fact, a result of de Launey [8] shows that one can also obtain such a so-called partial Hadamard basis (θ_i) when m is the largest multiple of 4 not exceeding n, as long as $k \leq cm$ for some absolute constant $0 < c < 1$. Observe that this latter condition is necessary anyway for the bounds in (2.3) and (2.4) to be nontrivial.

Moreover, at the expense of the value of the constants which appear, (2.3) and (2.4) hold for any orthonormal basis $\theta_1, \ldots, \theta_k$ of most k-dimensional subspaces $E \subseteq \mathbb{R}^n$. This statement can be made precise using a concentration inequality on the Grassmann manifold $G_{n,k}$ due to Gordon [11], cf. [18, Lemma 16], although we do not do so here.

The error bounds in Theorem 2.1 depend on a recent optimal concentration result for 1-unconditional log-concave random vectors due to Klartag [14], given as Proposition 4.1 below. Klartag used a more general version of that result to give a sharp estimate on Gaussian approximation with respect to Kolmogorov distance (maximum difference between distribution functions) in the setting of Theorem 2.1 when $k = 1$; the typical error is of the order $1/n$. Using a smoothing lemma from [5] this implies a total variation estimate (which may not be sharp) of the order $\sqrt{\frac{\log n}{n}}$.

A Wasserstein distance estimate as in Theorem 2.1 can be proved without the assumption of log-concavity, at the expense of explicitly involving $\mathrm{Var}(|X|^2)$ in the bound, and (for technical reasons) making some stronger symmetry assumption on the distribution of X. The smoothing arguments involved in proving the total variation estimate, however, depend more crucially on log-concavity.

We now proceed to our other main results.

Theorem 2.2. *Let $X \in \mathbb{R}^n$ be uniformly distributed in a regular simplex*

$$\Delta_n = \sqrt{n(n+2)}\, \mathrm{conv}\{v_1, \ldots, v_n\}, \tag{2.5}$$

where $|v_i| = 1$ for $1 \leq i \leq n+1$, and let $W \in \mathbb{R}^k$ be as defined in (2.1)*. Then*

$$d_1(W, Z) \leq 20 \sqrt{k \sum_{i=1}^{k} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4}}$$

and

$$d_{TV}(W, Z) \leq C k^{5/6} \left(\sum_{i=1}^{k} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4} \right)^{1/3}.$$

Theorem 2.2 shows that W is approximately normal as long as the vertices of Δ_n are not close to the subspace $E = \mathrm{span}(\theta_1, \ldots, \theta_k)$. By the remarks following

Theorem 2.1 above and [18, Corollary 6], Theorem 2.2 shows that for a typical subspace E, $d_{TV}(W,Z) \leq c\frac{k^{7/6}}{n^{1/6}}$. The same proof as for Theorem 2.2 yields similar results for random vectors with other distributions invariant under the symmetry group of a regular simplex.

Our last main result improves the typical dependence on n of the total variation bound of Theorem 2.2 in the case that $k = 1$.

Theorem 2.3. *Let X be uniformly distributed in a regular simplex Δ_n as defined in (2.5), let $\theta \in S^{n-1}$ be fixed, and let $W = \langle X, \theta \rangle$. Then*

$$d_{TV}(W,Z) \leq C\sqrt{\sum_{i=1}^{n+1} |\langle \theta, v_i \rangle|^3},$$

where Z is a standard Gaussian random variable.

For a typical $\theta \in S^{n-1}$, we obtain here $d_{TV}(W,Z) \leq cn^{-1/4}$. This also improves an error bound given in [18]; see the remarks at the end of this paper for further details.

3. Smoothing and abstract Gaussian approximation theorems

For $n \geq 1$ and $t > 0$, define $\varphi_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_t(x) = \frac{1}{(2\pi t^2)^{n/2}} e^{-|x|^2/2t^2},$$

so φ_t is the density of tZ, where $Z \in \mathbb{R}^n$ is a standard Gaussian random vector. A well-known consequence of the Prékopa-Leindler inequality [16], [21] (or see [10]) is that the convolution of integrable log-concave functions is log-concave; hence in particular $f * \varphi_t$ is log-concave for any log-concave probability density $f : \mathbb{R}^n \to \mathbb{R}_+$. Furthermore it is well-known that

$$\|f * \varphi_t - f\|_1 \to 0 \quad \text{as } t \to 0$$

for any integrable f. Thus log-concave random variables are arbitrarily well approximated, in the total variation metric, by log-concave random vectors with smooth densities. The statements of Section 2 involving the total variation metric rely on a quantitative version of this observation. We say that $f : \mathbb{R}^n \to \mathbb{R}_+$ is isotropic if it is the density of an isotropic random vector in $\mathbb{R}^n$. The following is a sharp version of Lemma 5.1 in [12].

Proposition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}_+$ an isotropic and log-concave probability density.*

1. *If $n = 1$ then*

$$\|f * \varphi_t - f\|_1 \leq 2\sqrt{2}\, t$$

for all $t \geq 0$.

2. (Klartag-Eldan) *If* $n \geq 2$ *then*

$$\|f * \varphi_t - f\|_1 \leq cnt$$

for all $t \geq 0$*, where* $c > 0$ *is an absolute constant.*

Proposition 3.1(2) was conjectured in an earlier version of this paper which also proved a weaker version of that estimate by optimizing over some of the parameters in the proof of [12, Lemma 5.1]. After that version of this paper was posted on arxiv.org, Klartag proved the conjecture; the proof appears in [9, Section 5].

For arbitrary f, Proposition 3.1 is sharp up to the values of the constants $2\sqrt{2}$ and c. For the particular case $f = \varphi_1$, one can show

$$\|\varphi_1 * \varphi_t - \varphi_1\|_1 = \|\varphi_{\sqrt{1+t^2}} - \varphi_1\|_1 < \sqrt{2n}\, t \tag{3.1}$$

for all $t \geq 0$ and any n (cf. [12, Lemma 4.9], or the proof of Proposition 3.1(1) below).

Proof of Proposition 3.1(1). First, we can assume that f is smooth and everywhere positive, for example by convolving f with φ_ε, rescaling for isotropicity, and letting $\varepsilon \to 0$. A special case of a result of Ledoux [15, formula (5.5)] about the heat semigroup on a Riemannian manifold implies that

$$\|f * \varphi_t - f\|_1 \leq \sqrt{2}\, t\, \|f'\|_1 \tag{3.2}$$

for any $t \geq 0$. Since f is log-concave it is unimodal, i.e., there exists an $a \in \mathbb{R}$ such that $f'(s) \geq 0$ for $s \leq a$ and $f'(s) \leq 0$ for $s \geq a$, and since f is also isotropic we have $f(a) \leq 1$ (see e.g. [17, Lemma 5.5(a)]). Therefore

$$\|f'\|_1 = \int_{-\infty}^{a} f'(s)\, ds - \int_{a}^{\infty} f'(s)\, ds = 2f(a) \leq 2,$$

which proves the first claim. □

The following abstract Gaussian approximation theorem was proved by Stein in [23]; the version stated here incorporates a slight improvement in the constants proved in [2]. Recall that a pair of random variables (W, W') is called exchangeable if the joint distribution of (W, W') is the same as the distribution of (W', W).

Proposition 3.2. (Stein) *Suppose that* (W, W') *is an exchangeable pair of random variables such that* $\mathbb{E}W = 0$*,* $\mathbb{E}W^2 = 1$*, and* $\mathbb{E}[W' - W|W] = -\lambda W$*. Then*

$$|\mathbb{E}g(W) - \mathbb{E}g(Z)| \leq \frac{\|g\|_\infty}{\lambda}\sqrt{\operatorname{Var}\mathbb{E}[(W'-W)^2|W]} + \frac{\|g'\|_\infty}{4\lambda}\mathbb{E}|W' - W|^3$$

for any $g \in C_c^\infty(\mathbb{R})$*, where* $Z \in \mathbb{R}$ *denotes a standard Gaussian random variable.*

Stein used a smoothing argument to derive a version of Proposition 3.2 for the Kolmogorov distance, which was the main tool in the proofs of most of the results of [18]. Estimates for total variation distance for log-concave distributions were obtained in [18] by combining the Kolmogorov distance estimates with [5, Theorem 3.3], which entails an additional loss in the error bound. Here we use Proposition 3.1(1) to obtain a version of Proposition 3.2 for total variation distance and log-concave distributions, which matches Stein's bound for Kolmogorov distance used in [18]; this is the main technical tool in the proof of Theorem 2.3.

Corollary 3.3. *Suppose, in addition to the hypotheses of Proposition* 3.2*, that* W *is log-concave. Then*

$$d_{TV}(W,Z) \leq \frac{1}{\lambda}\sqrt{\operatorname{Var}\mathbb{E}[(W'-W)^2|W]} + 2\sqrt{\frac{1}{\lambda}\mathbb{E}|W'-W|^3}.$$

It follows from the proof of Proposition 3.1(1) that Corollary 3.3 only requires W to have a bounded unimodal density with respect to Lebesgue measure; the coefficient 2 in the r.h.s. should be replaced by a constant depending on the maximum value of the density.

Proof. Let $g \in C_c(\mathbb{R})$ with $\|g\|_\infty \leq 1$, and let f denote the density of W. Assume for now that f is smooth. Given $t > 0$, define $h = g * \varphi_t$. To begin, observe that

$$\|h\|_\infty \leq \|g\|_\infty \|\varphi_t\|_1 \leq 1,$$

$$\|h'\|_\infty = \|g * (\varphi_t')\|_\infty \leq \|g\|_\infty \|\varphi_t'\|_1 \leq \frac{1}{t}\sqrt{\frac{2}{\pi}}.$$

Proposition 3.2 applied to h implies that

$$|\mathbb{E}h(W) - \mathbb{E}h(Z)| \leq \frac{1}{\lambda}\sqrt{\operatorname{Var}\mathbb{E}[(W'-W)^2|W]} + \frac{1}{2\sqrt{2\pi}\,\lambda t}\mathbb{E}|W'-W|^3. \tag{3.3}$$

Next, by Proposition 3.1(1),

$$\begin{aligned}|\mathbb{E}h(W) - \mathbb{E}g(W)| &= \left|\int [g * \varphi_t(s) - g(s)] f(s)\, ds\right| = \left|\int_{\mathbb{R}} [f * \varphi_t(s) - f(s)] g(s)\, ds\right| \\ &\leq \|g\|_\infty \|f * \varphi_t - f\|_1 \leq 2\sqrt{2}\, t.\end{aligned} \tag{3.4}$$

Similarly, by (3.2)

$$|\mathbb{E}h(Z) - \mathbb{E}g(Z)| \leq \|g\|_\infty \|\varphi_1 * \varphi_t - \varphi_1\|_1 < \sqrt{2}\|\varphi_1'\|_1\, t \ \leq \frac{2}{\sqrt{\pi}}\, t. \tag{3.5}$$

The only reason for using (3.2) directly instead of applying Proposition 3.1(1) here is to obtain a slightly better constant. Combining (3.3), (3.4), and (3.5) yields

$$\begin{aligned}|\mathbb{E}g(W) - \mathbb{E}g(Z)| \leq& \frac{1}{\lambda}\sqrt{\operatorname{Var}\mathbb{E}[(W'-W)^2|W]} \\ &+ \frac{1}{2\sqrt{2\pi}\,\lambda t}\mathbb{E}|W'-W|^3 + 2\left(\sqrt{2} + \frac{1}{\sqrt{\pi}}\right) t.\end{aligned} \tag{3.6}$$

The corollary, under the assumption that f is smooth, now follows by optimizing in t. The coefficient of 2 given in the second term in the statement of the corollary is not optimal and is given as such for simplicity.

The corollary can be reduced to the smooth case with a convolution argument as for Proposition 3.1, although it is slightly more complicated because it is necessary to smooth not only f but the exchangeable pair (W, W'). To do this, let Z_1, Z_2 be standard Gaussian random variables independent of each other and of (W, W'), set $Z = Z_1$, and set

$$Z' = (1-\lambda)Z_1 + \sqrt{2\lambda - \lambda^2}\, Z_2.$$

Then (Z, Z') is an exchangeable pair and $\mathbb{E}[Z' - Z|Z] = -\lambda Z$. Now for $\varepsilon > 0$ let

$$W_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}}(W + \varepsilon Z), \quad W'_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}}(W' + \varepsilon Z').$$

Then $(W_\varepsilon, W'_\varepsilon)$ is an exchangeable pair that satisfies all the hypotheses of the corollary (log-concavity follows from the Prékopa-Leindler inequality), and W_ε has a smooth density. Applying the corollary to $(W_\varepsilon, W'_\varepsilon)$ and letting $\varepsilon \to 0$ yields the general case. □

The main technical tool in the proofs of Theorems 2.1 and 2.2 is the following multivariate version of Proposition 3.2, recently proved by S. Chatterjee and E. Meckes in [7]. For a smooth function $f : \mathbb{R}^k \to \mathbb{R}$, we denote by $M_1(f) = \||\nabla f|\|_\infty$ the Lipschitz constant of f and

$$M_2(f) = \|\|\nabla^2 f\|\|_\infty$$

the maximum value of the operator norm of the Hessian of f, or equivalently the Lipschitz constant of $\nabla f : \mathbb{R}^k \to \mathbb{R}^k$.

Proposition 3.4. (Chatterjee and E. Meckes) *Let W and W' be identically distributed random vectors in $\mathbb{R}^k$ defined on a common probability space. Suppose that for some constant $\lambda > 0$ and random variables E_{ij}, $1 \le i, j \le k$,*

$$\begin{aligned} \mathbb{E}\big[W_i' - W_i \big| W\big] &= -\lambda W_i, \\ \mathbb{E}\big[(W_i' - W_i)(W_j' - W_j)\big| W\big] &= 2\lambda\delta_{ij} + E_{ij}. \end{aligned}$$

Then

$$\big|\mathbb{E}f(W) - \mathbb{E}f(Z)\big| \le \frac{M_1(f)}{\lambda}\mathbb{E}\sqrt{\sum_{i,j=1}^k E_{ij}^2} + \frac{\sqrt{2\pi}M_2(f)}{24\lambda}\mathbb{E}|W' - W|^3$$

for any smooth $f : \mathbb{R}^k \to \mathbb{R}$, where $Z \in \mathbb{R}^k$ is a standard Gaussian random vector.

Note that the normalization for E_{ij} used here differs from that in the statement of [7, Theorem 4] by a factor of λ. An earlier version of this paper was posted to arxiv.org which was based on an earlier version of Proposition 3.4. The version given above allows improved estimates in Theorem 2.1.

Convolution arguments similar to those in the proof of Corollary 3.3 yield bounds on Wasserstein and total variation distances.

Corollary 3.5. *Under the same hypotheses as Proposition* 3.4*,*

$$d_1(W,Z) \leq \frac{1}{\lambda}\mathbb{E}\sqrt{\sum_{i,j=1}^{k} E_{ij}^2} + k^{1/4}\sqrt{\frac{2}{3\lambda}\mathbb{E}|W'-W|^3}.$$

If moreover W is log-concave, then

$$d_{TV}(W,Z) \leq C\left(\frac{k}{\lambda}\mathbb{E}\sqrt{\sum_{i,j=1}^{k} E_{ij}^2} + \frac{k^2}{\lambda}\mathbb{E}|W'-W|^3\right)^{1/3},$$

where $C > 0$ is an absolute constant.

Proof. To prove the first claim, let $g : \mathbb{R}^k \to \mathbb{R}$ be 1-Lipschitz, and define $h = g*\varphi_t$ for $t > 0$. Standard calculations show

$$M_1(h) \leq M_1(g)\|\varphi_t\|_1 \leq 1 \tag{3.7}$$

and

$$M_2(h) \leq M_1(g) \sup_{\theta\in S^{n-1}} \|\langle\nabla\varphi_t, \theta\rangle\|_1 \leq \sqrt{\frac{2}{\pi}}\frac{1}{t}. \tag{3.8}$$

Note that $\mathbb{E}h(W) = \mathbb{E}g(W + tZ)$, where $Z \in \mathbb{R}^k$ is a standard Gaussian random vector independent of W, which implies

$$\left|\mathbb{E}h(W) - \mathbb{E}g(W)\right| \leq \mathbb{E}|tZ| \leq t\sqrt{k} \tag{3.9}$$

since g is 1-Lipschitz, and similarly

$$\left|\mathbb{E}h(Z) - \mathbb{E}g(Z)\right| \leq t\sqrt{k}. \tag{3.10}$$

The claim follows by applying Proposition 3.4 to h, using (3.7), (3.8), (3.9), and (3.10), and optimizing in t.

The proof of the second claim is similar. As in the proof of Corollary 3.3 we may assume that W has a smooth density f. Let $g \in C_c(\mathbb{R}^k)$ with $\|g\|_\infty \leq 1$, and again define $h = g * \varphi_t$ for $t > 0$. By standard calculations,

$$M_1(h) \leq \|g\|_\infty \sup_{\theta\in S^{n-1}} \|\langle\nabla\varphi_t, \theta\rangle\|_1 \leq \sqrt{\frac{2}{\pi}}\frac{1}{t} \tag{3.11}$$

and

$$M_2(h) \le \|g\|_\infty \sup_{\theta \in S^{n-1}} \| \left\langle (\nabla^2 \varphi_t)\theta, \theta \right\rangle \|_1 \le \frac{\sqrt{2}}{t^2}. \tag{3.12}$$

Proposition 3.1(2) and the identity

$$\mathbb{E}h(W) = \int g * \varphi_t(x)\ f(x)\ dx = \int f * \varphi_t(x)\ g(x)\ dx$$

imply that

$$\left|\mathbb{E}h(W) - \mathbb{E}g(W)\right| \le \|g\|_\infty \|f * \varphi_t - f\|_1 \le ckt. \tag{3.13}$$

Similarly,

$$\left|\mathbb{E}h(Z) - \mathbb{E}g(Z)\right| \le \|g\|_\infty \|\varphi_1 * \varphi_t - \varphi_1\|_1 \le ckt. \tag{3.14}$$

This last estimate can be improved to $\sqrt{2k}t$ using (3.1), but there is no advantage to doing so here.

Applying Proposition 3.4 to h and using (3.11), (3.12), (3.13), and (3.14) yields

$$d_{TV}(W, Z) = \sup_{g \in C_c(\mathbb{R}^k),\ \|g\|_\infty \le 1} \left|\mathbb{E}g(W) - \mathbb{E}g(Z)\right| \le \frac{A}{t^2} + \frac{B}{t} + ckt \tag{3.15}$$

for any $t > 0$, where

$$A = \frac{\sqrt{\pi}}{12\lambda}\mathbb{E}|W' - W|^3, \qquad B = \frac{\sqrt{2}}{\pi}\frac{1}{\lambda}\mathbb{E}\sqrt{\sum_{i,j=1}^k E_{ij}^2}.$$

Although it is not straightforward to optimize the r.h.s. of (3.15) precisely, this is simplified by noting that $d_{TV}(W, Z) \le 2$ always. Therefore (3.15) is vacuously true for $t \ge 2/(ck)$, and so

$$d_{TV}(W, Z) \le \frac{A + 2B/(ck)}{t^2} + ckt$$

for any $t > 0$. Optimizing this latter expression in t yields

$$d_{TV}(W, Z) \le C\left(k^2 A + kB\right)^{1/3},$$

from which the result follows. □

4. Proofs of the main results

In this section we prove Theorems 2.1 and 2.2 from Corollary 3.5, and indicate how Theorem 2.3 may be proved from Corollary 3.3. The arguments mostly generalize the proofs of [18].

First, observe that the Prékopa-Leindler theorem [21], [16] implies that marginals of log-concave measures are log-concave. Therefore when X is log-concave, W is log-concave as well, and the second estimate of Corollary 3.5 may be applied. This fact will be used without further comment in all the proofs in this section.

Second, we state a version of Klartag's concentration result for unconditional convex bodies.

Proposition 4.1. (Klartag) *If X is isotropic, unconditional, and log-concave, and $a_1, \dots, a_n \in \mathbb{R}$, then*

$$\operatorname{Var}\left(\sum_{\ell=1}^n a_\ell X_\ell^2\right) \le 32 \sum_{\ell=1}^n a_\ell^2.$$

Proposition 4.1 is essentially a special case of [14, Lemma 4], which is stated with the additional assumption that $a_1, \dots, a_n \ge 0$. For the precise constants which appear here see the comments following the proof in [14]; an extra factor of 2 is introduced to allow negative coefficients by observing that

$$\operatorname{Var}(X+Y) \le 2(\operatorname{Var} X + \operatorname{Var} Y)$$

for any pair of random variables X and Y.

We now proceed with the proofs of our main results.

Proof of Theorem 2.1. To construct W' appropriately coupled with W, we first define X' by reflecting X in a randomly chosen coordinate hyperplane, and then let $W_i' = \langle X', \theta_i \rangle$. By the 1-unconditionality of X, X and X' are identically distributed and hence so are W and W'.

More precisely, let I be a random variable chosen uniformly from $\{1, \dots, n\}$ and independently from the random vector X. Then

$$X' = X - 2X_I e_I,$$

where e_i is the ith standard basis vector in $\mathbb{R}^n$, and

$$W_i' = \langle X - 2X_I e_I, \theta_i \rangle = W_i - 2\theta_i^I X_I.$$

It follows that

$$\mathbb{E}\big[W_i' - W_i \big| X\big] = -\frac{2}{n} \sum_{\ell=1}^n \theta_i^\ell X_\ell = -\frac{2}{n} W_i$$

and

$$\mathbb{E}\big[(W_i' - W_i)(W_j' - W_j) \big| X\big] = \frac{4}{n} \sum_{\ell=1}^n \theta_i^\ell \theta_j^\ell X_\ell^2.$$

Therefore we may apply Corollary 3.5 with $\lambda = \frac{2}{n}$ and

$$E_{ij} = \frac{4}{n} \left(\sum_{\ell=1}^n \theta_i^\ell \theta_j^\ell X_\ell^2 - \delta_{ij} \right).$$

Now by Jensen's inequality, Proposition 4.1, and the Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{1}{\lambda} \mathbb{E} \sqrt{\sum_{i,j=1}^k E_{ij}^2} \le \frac{1}{\lambda} \sqrt{\sum_{i,j=1}^k \mathbb{E} E_{ij}^2} &= 2 \sqrt{\sum_{i,j=1}^k \operatorname{Var}\left(\sum_{\ell=1}^n \theta_i^\ell \theta_j^\ell X_\ell^2 \right)} \\ &\le 8 \sqrt{2 \sum_{i,j=1}^k \sum_{\ell=1}^n (\theta_i^\ell)^2 (\theta_j^\ell)^2} \le 8\sqrt{2} \sum_{i=1}^k \|\theta_i\|_4^2. \end{aligned}$$

By the triangle inequality for the $L_{3/2}$ norm and a precise version of Borell's lemma (found, e.g., in [20]),

$$\begin{aligned}\mathbb{E}|W-W|^3 &= \mathbb{E}\left(\sum_{i=1}^k |W_i'-W_i|^2\right)^{3/2} \leq \left(\sum_{i=1}^k \left(\mathbb{E}|W_i'-W_i|^3\right)^{2/3}\right)^{3/2} \\ &= \frac{8}{n}\left(\sum_{i=1}^k \left(\sum_{\ell=1}^n |\theta_i^\ell|^3 \mathbb{E}|X_\ell|^3\right)^{2/3}\right)^{3/2} \leq \frac{12\sqrt{2}}{n}\left(\sum_{i=1}^k \|\theta_i\|_3^2\right)^{3/2}.\end{aligned} \tag{4.1}$$

By the standard estimates between ℓ_p^k norms and the fact that $\|\theta_i\|_3^3 \leq |\theta_i|\|\theta_i\|_4^2 = \|\theta\|_4^2$,

$$\left(\sum_{i=1}^k \|\theta_i\|_3^2\right)^{3/2} \leq \sqrt{k}\sum_{i=1}^k \|\theta_i\|_3^3 \leq \sqrt{k}\sum_{i=1}^k \|\theta_i\|_4^2. \tag{4.2}$$

Proposition 3.5 now implies the stated bound for d_{TV} immediately. For the bound on d_1 observe also that $\|\theta_i\|_4^2 \leq |\theta_i|^2 = 1$, and so $\sum_{i=1}^k \|\theta_i\|_4^2 \leq k$. □

By using Proposition 3.4 directly, the proof of Theorem 2.1 above yields better bounds on the distance

$$d_2(W,Z) = \sup\left\{|\mathbb{E}f(W) - \mathbb{E}f(Z)| : M_1(f), M_2(f) \leq 1\right\}.$$

In particular, as in the remarks following the statement of Theorem 2.1, under the conditions of that theorem, typical k-dimensional marginals are nearly Gaussian with respect to d_2 if $k \ll n^{1/3}$. The same remark applies to the proof of Theorem 2.2 below. While we have preferred here to work with the more classical Wasserstein and total variation metrics, metrics like d_2 based on smooth test functions are commonly used in quantifying multivariate Gaussian approximation.

It is also worth pointing out here that [18] does prove multivariate Gaussian approximation results, but with respect to a weak metric referred to as T-distance which captures only the behavior of 1-dimensional marginals. Using T-distance yields misleadingly good results in terms of how large k may be for an approximately Gaussian marginal, cf. the remarks at the very end of [12]. Metrics like d_{TV}, d_1, and d_2 based on regular test functions better capture high-dimensional behavior.

Proof of Theorem 2.2. In this case X' is obtained by reflecting X in a hyperplane spanned by $(n-1)$ vertices of Δ_n; alternatively one may think of this operation as transposing two vertices.

We will need the well-known facts about vertices of centered regular simplices (which may be seen e.g. as consequences of John's theorem, cf. [3]) that

$$\sum_{i=1}^{n+1} v_i = 0 \tag{4.3}$$

and

$$\sum_{i=1}^{n+1} \langle x, v_i \rangle v_i = \frac{n+1}{n} x \tag{4.4}$$

for any $x \in \mathbb{R}^n$. It will be convenient to use the notation

$$u_{ij} = \sqrt{\frac{n}{2(n+1)}}(v_i - v_j), \quad 1 \le i, j \le n+1,$$

and $x^{ij} = \langle x, u_{ij} \rangle$ for $x \in \mathbb{R}^n$. It follows from (4.3) and (4.4) that $|u_{ij}| = 1$ for $i \neq j$ and

$$\sum_{\ell \neq m} x^{\ell m} u_{\ell m} = (n+1)x \quad \forall x \in \mathbb{R}^n. \tag{4.5}$$

To define W' precisely, first pick a pair (I, J) of distinct elements of $\{1, \dots, n+1\}$ uniformly and independently of X. Let

$$X' = X - 2X^{IJ} u_{IJ},$$

and $W_i' = \langle X', \theta_i \rangle$ as before. Using (4.5), one obtains

$$\mathbb{E}\big[W_i' - W_i \big| X\big] = -\frac{2}{n} \langle X, \theta_i \rangle = -\frac{2}{n} W_i,$$
$$\mathbb{E}\big[(W_i' - W_i)(W_j' - W_j)\big| X\big] = \frac{4}{n(n+1)} \sum_{\ell,m=1}^{n+1} \theta_i^{\ell m} \theta_j^{\ell m} (X^{\ell m})^2,$$

and so Corollary 3.5 applies with $\lambda = \frac{2}{n}$ and

$$E_{ij} = \frac{4}{n} \left(\frac{1}{n+1} \sum_{\ell,m=1}^{n+1} \theta_i^{\ell m} \theta_j^{\ell m} (X^{\ell m})^2 - \delta_{ij}. \right)$$

The relevant moments were calculated in [18]: for $\ell \neq m$ and $p \neq q$,

$$\mathbb{E}(X^{\ell m})^2 (X^{pq})^2 = \frac{(n+1)(n+2)}{(n+3)(n+4)} \begin{cases} 1 & \text{if } \{\ell, m\} \cap \{p, q\} = \emptyset, \\ 3 & \text{if } |\{\ell, m\} \cap \{p, q\}| = 1, \\ 6 & \text{if } |\{\ell, m\} \cap \{p, q\}| = 2, \end{cases} \tag{4.6}$$

and

$$\mathbb{E}|X^{\ell m}|^3 < 3\sqrt{2}. \tag{4.7}$$

In order to estimate $\mathbb{E}E_{ij}^2$, decompose the resulting sum of terms involving $\mathbb{E}(X^{\ell m})^2 (X^{pq})^2$ according to the size of $\{\ell, m\} \cap \{p, q\}$ and use (4.6):

$$\begin{aligned} &\frac{(n+3)(n+4)}{(n+1)(n+2)} \sum_{\ell,m,p,q} \theta_i^{\ell m} \theta_i^{pq} \theta_j^{\ell m} \theta_j^{pq} \mathbb{E}(X^{\ell m})^2 (X^{pq})^2 \\ &\quad = \left(\sum_{\ell,m} \theta_i^{\ell m} \theta_j^{\ell m} \right)^2 + 2 \sum_{|\{\ell,m\} \cap \{p,q\}|=1} \theta_i^{\ell m} \theta_i^{pq} \theta_j^{\ell m} \theta_j^{pq} + 10 \sum_{\ell,m} (\theta_i^{\ell m})^2 (\theta_j^{\ell m})^2. \end{aligned} \tag{4.8}$$

In all of the sums in (4.8) the indices range from 1 to $n+1$, with $\ell \neq m$ and $p \neq q$ and in the last term we have also used that $u_{\ell m} = -u_{m\ell}$. It follows from (4.5) that

$$\sum_{\ell \neq m} \theta_i^{\ell m} \theta_j^{\ell m} = (n+1) \langle \theta_i, \theta_j \rangle = (n+1)\delta_{ij}. \tag{4.9}$$

By the Cauchy-Schwarz inequality, the definition of $u_{\ell m}$, and the ℓ_4 triangle inequality,

$$\sum_{\ell \neq m} (\theta_i^{\ell m})^2 (\theta_j^{\ell m})^2 \leq \sqrt{\sum_{\ell \neq m} (\theta_i^{\ell m})^4} \sqrt{\sum_{\ell \neq m} (\theta_j^{\ell m})^4} \leq 4n \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_j, v_\ell \rangle^4}. \tag{4.10}$$

Also by the Cauchy-Schwarz inequality,

$$\sum_{\substack{|\{\ell,m\} \cap \{p,q\}|=1 \\ \ell \neq m,\ p \neq q}} \theta_i^{\ell m} \theta_i^{pq} \theta_j^{\ell m} \theta_j^{pq} \leq \sum_{\substack{|\{\ell,m\} \cap \{p,q\}|=1 \\ \ell \neq m,\ p \neq q}} (\theta_i^{\ell m})^2 (\theta_j^{\ell m})^2 \leq 4n \sum_{\ell \neq m} (\theta_i^{\ell m})^2 (\theta_j^{\ell m})^2. \tag{4.11}$$

Combining (4.8), (4.9), (4.10), and (4.11),

$$\begin{aligned} \mathbb{E} E_{ij}^2 &= \frac{16}{n^2} \left(\frac{1}{(n+1)^2} \sum_{\ell,m,p,q} \theta_i^{\ell m} \theta_i^{pq} \theta_j^{\ell m} \theta_j^{pq} \mathbb{E} (X^{\ell m})^2 (X^{pq})^2 - \delta_{ij} \right) \\ &\leq \frac{512}{n^2} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_j, v_\ell \rangle^4}, \end{aligned}$$

and therefore the first error term in Corollary 3.5 is bounded by

$$\frac{1}{\lambda} \mathbb{E} \sqrt{\sum_{i,j=1}^k E_{ij}^2} \leq 8\sqrt{2} \sum_{i=1}^k \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4}.$$

To bound the second error term, we begin as in (4.1), using (4.7), the definition of $u_{\ell m}$, and the ℓ_3 triangle inequality to obtain

$$\begin{aligned} \mathbb{E}|W - W|^3 &\leq \left(\sum_{i=1}^k \left(\mathbb{E}|W_i' - W_i|^3 \right)^{2/3} \right)^{3/2} \\ &= \frac{8}{n(n+1)} \left(\sum_{i=1}^k \left(\sum_{\ell \neq m} |\theta_i^{\ell m}|^3 \mathbb{E}|X^{\ell m}|^3 \right)^{2/3} \right)^{3/2} \\ &\leq \frac{12\sqrt{n}}{(n+1)^{5/2}} \left(\sum_{i=1}^k \left(\sum_{\ell \neq m} | \langle \theta_i, v_\ell \rangle - \langle \theta_i, v_m \rangle |^3 \right)^{2/3} \right)^{3/2} \\ &\leq \frac{96\sqrt{n}}{(n+1)^{3/2}} \left(\sum_{i=1}^k \left(\sum_{\ell=1}^{n+1} | \langle \theta_i, v_\ell \rangle |^3 \right)^{2/3} \right)^{3/2}. \end{aligned}$$

The error bounds from Proposition 3.5 are now simplified similarly as in the proof of Theorem 2.1. For each i define $x_i = (x_i^1, \ldots, x_i^{n+1})$ by $x_i^\ell = \langle \theta_i, v_\ell \rangle$. By (4.4),

$$|x_i|^2 = \sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^2 = \frac{n+1}{n} |\theta_i|^2 = \frac{n+1}{n}.$$

Therefore $\|x_i\|_3^3 \leq |x_i| \|x_i\|_4^2 \leq \sqrt{\frac{n+1}{n}} \|x_i\|_4^2$, and so by the same reasoning as in (4.2),

$$\mathbb{E}|W' - W|^3 \leq \frac{96\sqrt{k}}{n+1} \sum_{i=1}^{k} \sqrt{\sum_{\ell=1}^{n+1} \langle \theta_i, v_\ell \rangle^4};$$

finally observe also that $\|x_i\|_4^2 \leq |x|^2$ to simplify the bound on d_1. □

Theorem 2.3 may be proved by following the proof of Theorem 2.2 in the case $k = 1$, applying Corollary 3.3 in place of Corollary 3.5. Alternatively, one can follow the proof of [18, Corollary 6], using Corollary 3.3 in place of the Stein's Kolmogorov distance version of Proposition 3.2; this amounts to the same thing.

In [18] Stein's Kolmogorov distance version of Proposition 3.2 was applied for arbitrary isotropic X (under various symmetry assumptions). Total variation estimates for the log-concave case were then deduced using [5, Theorem 3.3], which allows Gaussian approximation estimates for log-concave random variables to be transferred from Kolmogorov distance to total variation distance. The present approach entails less loss in the final total variation bound since it uses only one smoothing argument instead of two. In general, the second approach described above to prove Theorem 2.3 can be used to deduce total variation bounds of the same order as the Kolmogorov distance bounds in most of the results of [18] for log-concave random vectors. In particular, this applies to Theorem 1, Corollary 4(2), and parts of Corollary 5 of [18].

In a similar fashion, using Proposition 3.2 directly yields versions of many of the results of [18] for the bounded Lipschitz metric

$$d_{BL}(X, Y) = \sup \left\{ |\mathbb{E}f(X) - \mathbb{E}f(Y)| : \|f\|_\infty, |f|_L \leq 1 \right\}.$$

In general, the dominant error term for the results of [18] using d_{BL} is typically of the order $n^{-1/2}$, as opposed to the order $n^{-1/4}$ for the Kolmogorov distance in most of the results of [18] and for d_{TV} in the present Theorem 2.3.

In [12], Klartag used another approach, based on an application of the classical Berry-Esseen theorem, to prove a univariate estimate in the setting of Theorem 2.1. Since Stein's method can be used to prove the Berry-Esseen theorem, the approach taken here and in [18] is arguably more direct, and the total variation bounds which can be derived in this way are better than those derived by the method of [12]. However, since the original version of this paper was written, in [14] Klartag has given a proof of an optimal result for Kolmogorov distance, which, as discussed after the statement of Theorem 2.1 above, implies sharper total variation bounds (when $k = 1$) than the methods used here. Klartag's proof is based partly on the optimal concentration result proved in [14], and also on careful arguments similar to those in classical proofs of the Berry-Esseen theorem. It is not clear whether the Stein's method approach can achieve these optimal error bounds.

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