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The Intersection of Convex Transversals is a Convex Polytope

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Abstract. It is proven that the intersection of convex transversals of finitely many sets in $\mathbb{R}^n$ is a convex polytope, possibly empty. Related results on support hyperplanes and complements of finite unions of convex sets are given.

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Introduction

Geometric transversal theory mainly studies affine subspaces of some dimension k, $0 \leq k \leq n-1$, that intersect every member of a given family of convex sets in $\mathbb{R}^n$ (see, e. g., the handbook [2] for general references). Significantly less attention in the literature is given to more general, convex transversals, whose study is related to matroids and combinatorial optimization (see, for example, [3],[5],[6]). We recall that a set $X \subset \mathbb{R}^n$ is a *transversal* of a given family $\mathcal{F} = \{S_1, \ldots, S_k\}$ of nonempty sets provided $X \cap S_i \neq \varnothing$ for all $i = 1, \ldots, k$. A transversal is called *convex* if it is a convex set.

Our interest in convex transversals is partly motivated by the paper [6], whose main geometric ingredient is the assertion that the intersection of all convex transversals of a given family of $n+1$ linear segments in $\mathbb{R}^n$ is an n-simplex, provided that it has nonempty interior. The following theorem shows that the nature of this assertion is far more general.

Theorem 1. *For any family $\mathcal{F} = \{S_1, \ldots, S_k\}$ of nonempty sets in $\mathbb{R}^n$, the intersection, $T(\mathcal{F})$, of convex transversals of $\mathcal{F}$ is a convex polytope.*

Equivalently, Theorem 1 states that the intersection of convex polytopes of the form conv $\{x_1, \dots, x_k\}$, where $x_1 \in S_1, \dots, x_k \in S_k$, is again a convex polytope. (See [10] for some other classes of convex polytopes whose intersections are again convex polytopes.) It would be interesting to establish an upper bound for the number of vertices (or j-faces) of the polytope $T(\mathcal{F})$ as a function of n and k. We will show that for $k = n+1$, the intersection of the convex transversals is a simplex provided it is n-dimensional. We would certainly like to be able to describe $T(\mathcal{F})$ as an intersection of closed halfspaces that support some subfamilies of $\mathcal{F}$. The difficulty of this problem, even when $T(\mathcal{F})$ is n-dimensional, is partly caused by lack of results on support hyperplanes of finite families of convex sets in $\mathbb{R}^n$. Using existing results on support properties of small families of convex bodies (see [8]), we are able to characterize the hyperplanes spanned by the facets of the simplex as those hyperplanes which support n of the sets $S_1, \dots, S_{n+1}$ and separate those n sets from the remaining one.

Theorem 2. *Let $\mathcal{F} = \{S_1, \dots, S_{n+1}\}$ be a family of nonempty sets in $\mathbb{R}^n$ such that the intersection $T(\mathcal{F})$ of convex transversals of $\mathcal{F}$ has dimension n. Then $T(\mathcal{F})$ is a simplex. If, in addition, the sets S_i are bounded, then this simplex is the intersection of $n+1$ closed halfspaces supporting, respectively, the subfamilies $\mathcal{F} \setminus \{S_i\}$, $i = 1, \dots, n+1$.*

The proof of Theorem 1 is based on the following result of proper interest. In what follows, by *polyhedron* we mean a finite union of intersections of the form $\cap Q_i$, where each Q_i is either a closed or an open halfspace in $\mathbb{R}^n$. A *convex polyhedron* is a polyhedron which is a convex set. Finally, a *convex polytope* is the convex hull of finitely many points.

Theorem 3. *If $C_1, \dots, C_k$ are any convex sets in $\mathbb{R}^n$, then the set*

$$S = \operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \dots \cup C_k))$$

is a convex polyhedron.

In particular, if the complement of a convex set $C \subset \mathbb{R}^n$ is also convex, then both C and $\mathbb{R}^n \setminus C$ are convex polyhedra as defined above (see [4]).

We conclude this section with the list of necessary notation. In what follows, the usual abbreviations aff, bd, conv, dim, int, pos, and rint, are used for affine hull, boundary, convex hull, dimension, interior, positive hull, and relative interior (taken in the affine hull), respectively. The notations $[x, y]$, $]x, y[$, $\langle x, y\rangle$, $[x, y\rangle$ mean, respectively, closed line interval, open line interval, the line passing through different points x, y, and the closed halfline with apex x passing through the point y. For any vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, the dot product $x \cdot y$ denotes the sum $x_1 y_1 + \dots + x_n y_n$ (also called the scalar product of x and y). To distinguish similarly looking elements, we write 0 for number zero, and *0* for zero point (also called the *origin*) of $\mathbb{R}^n$: $\mathit{0} = (0, \dots, 0)$.

Proof of Theorem 1

Clearly,
$$T(\mathcal{F}) = \cap\{\operatorname{conv}\{x_1,\dots,x_k\} \mid x_1 \in S_1,\dots,x_k \in S_k\}. \tag{1}$$
Assuming that $T(\mathcal{F}) \neq \varnothing$, let $\mathcal{H}$ be the collection of closed halfspaces of $\mathbb{R}^n$ which are transversals of the family $\mathcal{F}$.

Lemma 1. *The following equality holds:*

$$T(\mathcal{F}) = \cap\{X \subset \mathbb{R}^n \mid X \in \mathcal{H}\}. \tag{2}$$

Proof. Since the inclusion $\cap\{X \subset \mathbb{R}^n \mid X \in \mathcal{H}\} \subset T(\mathcal{F})$ trivially holds, it remains to establish the opposite inclusion. From (1) it follows that $T(\mathcal{F})$ is a compact convex set. If $x \notin T(\mathcal{F})$, then we may choose a closed halfspace $H \subset \mathbb{R}^n$ such that $x \notin H$ and $T(\mathcal{F}) \subset H$. Because H is a transversal of $\mathcal{F}$, we have $x \notin \cap\{X \subset \mathbb{R}^n \mid X \in \mathcal{H}\}$. Thus $T(\mathcal{F}) \subset \cap\{X \subset \mathbb{R}^n \mid X \in \mathcal{H}\}$. $\square$

In view of Lemma 1, we may assume that $S_1,\dots,S_k$ are convex sets, because a closed halfspace is a transversal of $\mathcal{F}$ if and only if it is a transversal of the family $\{\operatorname{conv} S_1,\dots,\operatorname{conv} S_k\}$. Also, we may assume that the sets $S_1,\dots,S_k$ are closed. Indeed, let H be a closed halfspace which is a transversal of $\{\operatorname{cl} S_1,\dots,\operatorname{cl} S_k\}$. Expressing H as the intersection H_α of closed halfspaces properly containing H, we observe that each H_α is a transversal of $\mathcal{F}$.

Given a point $x \in \mathbb{R}^n$, let

$$H_x = \{(u,t) \in \mathbb{R}^n \times \mathbb{R} \mid u\cdot x \le t\}.$$

For $(u,t) \in \mathbb{R}^{n+1}$, with $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$, put

$$G(u,t) = \{x \in \mathbb{R}^n \mid u\cdot x \le t\}.$$

Clearly,
$$x \in G(u,t) \text{ if and only if } (u,t) \in H_x. \tag{3}$$
Also, H_x is a closed halfspace of $\mathbb{R}^{n+1}$, and $G(u,t)$ is a closed halfspace of $\mathbb{R}^n$ if $u \neq 0$. For every $i = 1,\dots,k$, let

$$Y_i = \{(u,t) \in \mathbb{R}^{n+1} \mid G(u,t) \cap S_i \neq \varnothing\}.$$

From (3), it is clear that
$$Y_i = \cup\{H_x \mid x \in S_i\}. \tag{4}$$
For $\lambda > 0$ and $(u,t) \in Y_i$, we have $(\lambda u, \lambda t) \in Y_i$. Thus the sets $Y_1,\dots,Y_k$ are closed cones with common apex *0*.

Lemma 2. *The sets*

$$\mathbb{R}^{n+1} \setminus Y_i = \cap\{\mathbb{R}^{n+1} \setminus H_x \mid x \in S_i\},\ i = 1,\dots,k,$$

are convex.

Proof. That the equality holds is clear from (4). The set is convex, being an intersection of open halfspaces. $\square$

Put $Y = Y_1 \cap \cdots \cap Y_k$. By the above, Y is a cone with apex 0, and

$$Y = \{(u,t) \in \mathbb{R}^{n+1} \mid G(u,t) \text{ is a transversal of } \mathcal{F}\},$$

which, together with Lemma 1, implies that

$$T(\mathcal{F}) = \cap \{G(u,t) \mid (u,t) \in Y\}. \tag{5}$$

Lemma 3. *The set $\mathbb{R}^{n+1} \setminus Y$ is the union of k convex sets.*

Proof. It is clear that $\mathbb{R}^{n+1} \setminus Y$ coincides with

$$(\mathbb{R}^{n+1} \setminus Y_1) \cup \cdots \cup (\mathbb{R}^{n+1} \setminus Y_k),$$

and the convexity of these sets is given by Lemma 2. $\square$

Lemma 3 shows that $\mathbb{R}^{n+1} \setminus Y$ is a union of k convex cones with common apex 0, and Theorem 3 (see the proof below) implies that the set $Z = \operatorname{cl} \operatorname{conv} Y$ is a closed convex polyhedral cone with apex 0.

Lemma 4. *We have*

$$T(\mathcal{F}) = \cap \{G(u,t) \mid (u,t) \in Z\}.$$

Proof. By (5), since $Y \subset Z$, we obtain

$$\cap \{G(u,t) \mid (u,t) \in Z\} \subset T(\mathcal{F}).$$

Assume for a moment the existence of a point

$$x_0 \in T(\mathcal{F}) \setminus \cap \{G(u,t) \mid (u,t) \in Z\}.$$

Then there is a point $(u_0, t_0) \in Z$ such that $x_0 \in T(\mathcal{F}) \setminus G(u_0, t_0)$, which implies the inequality $u_0 \cdot x_0 > t_0$. Since $\mathbb{R}^n \setminus G(u_0, t_0)$ is an open halfspace, there is a point $(u_0', t_0') \in \operatorname{conv} Y$ such that $x_0 \in \mathbb{R}^n \setminus G(u_0', t_0')$; that is, $u_0' \cdot x_0 > t_0'$. Write

$$(u_0', t_0') = \lambda_1 (u_1, t_1) + \cdots + \lambda_k (u_k, t_k)$$

as a convex combination of some points $(u_i, t_i) \in Y_i$, $i = 1, \dots, k$. Then there is an index $j \in \{1, \dots, k\}$ such that $u_j \cdot x_0 > t_j$. Indeed, otherwise we would have

$$u_0' \cdot x_0 = (\lambda_1 u_1 + \cdots + \lambda_k u_k) \cdot x_0 \le \lambda_1 t_1 + \cdots + \lambda_k t_k = t_0',$$

a contradiction. Thus $x_0 \in T(\mathcal{F}) \setminus G(u_j, t_j)$, which is impossible by the definition of Y_j. Hence the statement holds. $\square$

Since Z is a closed convex polyhedral cone, there is a finite set $W \subset Z$ such that $Z = \operatorname{pos} W$.

Lemma 5. *We have*

$$T(\mathcal{F}) = \cap\{G(u,t) \mid (u,t) \in W\}.$$

Proof. Indeed, $T(\mathcal{F}) \subset \cap\{G(u,t) \mid (u,t) \in W\}$ because of $W \subset Z$. Assume for a moment the existence of a point

$$x_0 \in \cap\{G(u,t) \mid (u,t) \in W\} \setminus T(\mathcal{F}).$$

By Lemma 4, there is a point $(u_0, t_0) \in Z$ such that $x_0 \notin G(u_0, t_0)$, which implies the inequality $u_0 \cdot x_0 > t_0$. Write

$$(u_0', t_0') = \mu_1(u_1, t_1) + \cdots + \mu_m(u_m, t_m)$$

as a positive combination of points $(u_i, t_i) \in W$, $i = 1, \ldots, m$. Put

$$\mu = \mu_1 + \cdots + \mu_m, \quad (u_i', t_i') = \mu_i(u_i, t_i), \quad \lambda_i = \mu_i/\mu, \quad i = 1, \ldots, m.$$

Then (u_0', t_0') is a convex combination of the points (u_i', t_i'), $i = 1, \ldots, m$:

$$(u_0', t_0') = \lambda_1(u_1', t_1') + \cdots + \lambda_m(u_m', t_m').$$

Since

$$x_0 \in G(u_i, t_i) = G(\mu_i u_i, \mu_i t_i) = G(u_i', t_i'),$$

we have $u_i' \cdot x_0 \le t_0'$ for all $i = 1, \ldots, m$. Hence

$$x_0 \cdot u_0 = (\lambda_1 u_1' + \cdots + \lambda_k u_k') \cdot x_0 \le \lambda_1 t_1' + \cdots + \lambda_k t_k' = t_0,$$

a contradiction. Thus the equality holds. □

To finalize the proof of Theorem 1, we observe that $T(\mathcal{F})$ is a convex polytope as a compact convex set which is an intersection of finitely many closed halfspaces (see Lemma 5).

Proof of Theorem 2

Suppose now that $k = n+1$, so that $\mathcal{F} = \{S_1, \ldots, S_{n+1}\}$. By the remarks that follow Lemma 1, the sets $S_1, \ldots, S_{k+1}$ are assumed to be closed and convex. Since $\dim T(\mathcal{F}) = n$, any transversal of $\mathcal{F}$ is n-dimensional. Put

$$\widetilde{T} = \cap\{\operatorname{int} C \mid C \text{ is a convex transversal of } \mathcal{F}\}.$$

Clearly $\widetilde{T} \subset T(\mathcal{F})$. Since each convex transversal C of $\mathcal{F}$ satisfies the inclusion $\operatorname{int} T(\mathcal{F}) \subset \operatorname{int} C$, we have $\operatorname{int} T(\mathcal{F}) \subset \widetilde{T}$ and $T(\mathcal{F}) = \operatorname{cl} \widetilde{T}$. Let

$$Z = \operatorname{conv}(S_1 \cup \cdots \cup S_{n+1})$$

and

$$Z_j = \operatorname{conv}(S_1 \cup \cdots S_{j-1} \cup S_{j+1} \cdots \cup S_{n+1}), \quad j = 1, \ldots, n+1.$$

Lemma 6. *We have*

$$\widetilde{T} = Z \setminus (Z_1 \cup \cdots \cup Z_{n+1}).$$

Proof. Let $x \in \widetilde{T}$. Choosing any points $p_1 \in S_1, \ldots, p_{n+1} \in S_{m+1}$, we have $x \in \operatorname{int conv} \{p_1, \ldots, p_{n+1}\} \subset Z$. Hence $\widetilde{T} \subset Z$. Suppose, for contradiction, that $x \in Z_j$ for some index $j \in \{1, \ldots, n+1\}$. Then there are some points

$$u_i \in S_i,\ i = 1, \ldots, n+1, \quad i \neq j,$$

such that $x \in \operatorname{conv} U$, where

$$U = \{u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n+1}\}.$$

Since $|U| \leq n$, there is a hyperplane H containing U. Then one of the two closed halfspaces bounded by H, say P, is a transversal of $\mathcal{F}$. Then $x \in H = \operatorname{bd} P$ in contradiction with the assumption $x \in \widetilde{T}$. It follows that

$$\widetilde{T} \subset Z \setminus (Z_1 \cup \cdots \cup Z_{n+1}).$$

To prove the opposite inclusion, choose any point $x \notin \widetilde{T}$. We are going to show that $x \notin Z \setminus (Z_1 \cup \cdots \cup Z_{n+1})$. Assume that this is not the case, so that $x \in Z \setminus (Z_1 \cup \cdots \cup Z_{n+1})$. The inclusion $x \in Z = \operatorname{conv}(S_1 \cup \cdots \cup S_{n+1})$ and Carathéodory's theorem (see [1]) imply the existence of a set $Q \subset S_1 \cup \cdots \cup S_{n+1}$ such that $|Q| \leq n+1$ and $x \in \operatorname{conv} Q$. Since $x \notin Z_1 \cup \cdots \cup Z_{n+1}$, it follows that $|Q| = n+1$ and x lies in the convex hull of no proper subset of Q. Thus Q is contained in no hyperplane. Furthermore, $|Q \cap S_i| = 1$ for all $i = 1, \ldots, n+1$. Hence $\operatorname{conv} Q$, being of dimension n, is an n-simplex.

Since $x \notin \widetilde{T}$, there exists a convex polytope

$$P = \operatorname{conv} \{p_1, \ldots, p_{n+1}\}, \quad p_1 \in S_1, \ldots, p_{n+1} \in S_{n+1},$$

such that $x \notin \operatorname{int} P$. Then $x \notin P$ because of $P \subset \operatorname{int} P \cup Z_1 \cup \cdots \cup Z_{n+1}$. By Radon's theorem (see [9]), there is a partition of $\{p_1, \ldots, p_{n+1}\}$ into disjoint subsets, say $\{A, B\}$, such that $\operatorname{conv}(x \cup A) \cap \operatorname{conv} B \neq \varnothing$. Choose a point

$$y \in \operatorname{conv}(x \cup A) \cap \operatorname{conv} B.$$

We observe that $x \neq y$ because of $y \in \operatorname{conv} B \subset P$. Since $x \in \operatorname{conv} Q$ and $x \neq y$, there is a point $z \in \operatorname{bd conv} Q$ such that $x \in [y, z]$. Let $w \in \operatorname{conv} A$ be such that $y \in [x, w]$. Let $k \in \{1, \ldots, n+1\}$ be an index such that $z \in Z_k$. If $p_k \in A$, then $y \in Z_k$ because of $y \in \operatorname{conv} B$, and if $p_k \in B$, then $w \in Z_k$ because of $w \in \operatorname{conv} A$. It follows that $x \in [y, z] \cap [w, z] \subset Z_k$, contradicting the assumption $x \in Z \setminus (Z_1 \cup \cdots \cup Z_{n+1})$. □

Lemma 7. *If $T(\mathcal{F})$ has nonempty interior, then $T(\mathcal{F})$ is an n-simplex. Furthermore, $T(\mathcal{F})$ can be expressed as the intersection of $n+1$ closed halfspaces $H_1, \ldots, H_{n+1}$ such that each H_i is the (unique) closed halfspace whose boundary hyperplane supports Z_i and separates Z_i from T. Finally,*

$$S_i \subset \mathbb{R}^n \setminus (\cup \{ \operatorname{int} H_j \mid i = 1, \ldots, n+1,\ j \neq i\}), \quad i = 1, \ldots, n+1.$$

Proof. As it is shown in the proof of Lemma 6, $\widetilde{T}$ and Z_j are disjoint convex sets. Choose a closed halfspace H_j that contains $\widetilde{T}$ such that $Z_j \subset \mathbb{R}^n \setminus \operatorname{int} H_i$, $j = 1, \dots, n+1$. Hence

$$\begin{aligned} S_i &\subset \cap \{Z_j \mid i = 1, \dots, n+1,\, j \neq i\} \\ &\subset \mathbb{R}^n \setminus (\cup \{\operatorname{int} H_j \mid i = 1, \dots, n+1,\, j \neq i\}),\ i = 1, \dots, n+1. \end{aligned}$$

Since every halfspace H_i, $i = 1, \dots, n+1$, contains $\widetilde{T}$, we have

$$T(\mathcal{F}) = \operatorname{cl} \widetilde{T} \subset H_1 \cap \dots \cap H_{n+1}.$$

To prove the opposite inclusion, choose any points $x \notin T(\mathcal{F})$ and $u \in \operatorname{int} T \subset \widetilde{T}$. Then the halfline $[u, x)$ intersects the boundary of $T(\mathcal{F})$, which lies in $\operatorname{cl}(Z_1 \cup \dots \cup Z_{n+1})$. Then there is an index $k \in \{1, \dots, n+1\}$ such that $[u, x)$ intersects $\operatorname{cl} Z_k$ at some point w. In this case, $u \in \operatorname{int} T \subset \operatorname{int} H_k$ and $w \in \mathbb{R}^n \setminus \operatorname{int} H_k$, which implies that $x \notin H_k$. Hence

$$T(\mathcal{F}) = H_1 \cap \dots \cap H_{n+1}.$$

Since $\operatorname{int} T(\mathcal{F}) \neq \varnothing$, we conclude that $T(\mathcal{F})$ is an n-simplex and the halfspaces $H_1, \dots, H_{n+1}$ are uniquely determined. □

To complete the proof of Theorem 2, it remains to consider the case when the sets $S_1, \dots, S_{n+1}$ are compact. We will say that a closed halfspace $Q \subset \mathbb{R}^n$ *supports* a set C provided $\operatorname{cl} C$ has nonempty intersection with Q but not with the interior of Q. Furthermore, a closed halfspace Q *supports* a family of sets provided each of the sets is supported by Q. The next lemma is a particular case of a result on supporting hyperplanes of a family of n convex bodies in $\mathbb{R}^n$ proved in [8].

Lemma 8. *Let $\mathcal{K} = \{K_1, \dots, K_n\}$ be a family of n compact convex sets in $\mathbb{R}^n$ having no convex transversal of dimension less than $n-1$. Then there are exactly two closed halfspaces each supporting $\mathcal{K}$.* □

Lemma 9. *For every $i = 1, \dots, n+1$ there is a unique closed halfspace P_i containing S_i and supporting the family $\mathcal{F} \setminus \{S_i\}$.*

Proof. Let, for example, $i = n+1$. If $S_1 \cup \dots \cup S_n$ lies in a hyperplane H, then the closed halfspace P_{n+1} bounded by H and containing S_{n+1} satisfies the conclusion. Hence we may assume, in what follows, that $S_1 \cup \dots \cup S_n$ does not lie in a hyperplane.

By Lemma 8, there are two closed halfspaces, P' and P'', both supporting the family $\mathcal{F} \setminus \{S_{n+1}\}$. Since $\operatorname{int} T(\mathcal{F}) \neq \varnothing$, the set S_{n+1} is disjoint from $\operatorname{bd} P' \cup \operatorname{bd} P''$. We claim that S_{n+1} lies in the interior of one of the halfspaces P', P''. Indeed, assume for a moment that $S_{n+1} \subset \mathbb{R}^n \setminus (\operatorname{int} P' \cup \operatorname{int} P'')$. Since $S_{n+1} \cap (\operatorname{bd} P' \cup \operatorname{bd} P'') = \varnothing$, we have $S_{n+1} \subset \mathbb{R}^n \setminus (P' \cup P'')$. Choose a point $x_{n+1} \in S_{n+1}$ and denote by H the hyperplane through x_{n+1} such that:

(a) H is parallel to $\operatorname{bd} P'$ if $\operatorname{bd} P'$ and $\operatorname{bd} P''$ are parallel,

(b) H contains $\operatorname{bd} P' \cap \operatorname{bd} P''$ if $\operatorname{bd} P'$ and $\operatorname{bd} P''$ are not parallel.

Because the family $\mathcal{F} \setminus \{S_{n+1}\}$ is supported by both $\operatorname{bd} P'$ and $\operatorname{bd} P''$, the hyperplane H intersects every set of this family. Hence H is a transversal of $\mathcal{F}$, contradicting $\operatorname{int} T(\mathcal{F}) \neq \varnothing$. The obtained contradiction shows that S_{n+1} lies in the interior of (exactly) one of the halfspaces P', P''. □

Lemma 10. *If $P_1, \ldots, P_{n+1}$ are the halfspaces determined by Lemma 9, then $T(\mathcal{F}) = P_1 \cap \cdots \cap P_{n+1}$.*

Proof. Let $Q = P_1 \cap \cdots \cap P_{n+1}$. Since each of $P_1, \ldots, P_{n+1}$ is a transversal of $\mathcal{F}$, we have $T(\mathcal{F}) \subset Q$. Therefore Q is a closed convex polyhedron with nonempty interior.

To prove the opposite inclusion $Q \subset T(\mathcal{F})$, choose any points $z_1 \in S_1, \ldots, z_{n+1} \in S_{n+1}$. Clearly, $z_1, \ldots, z_{n+1}$ are affinely independent and their convex hull $C = \operatorname{conv}\{z_1, \ldots, z_{n+1}\}$ is an n-simplex. As it is shown in the proof of Lemma 9, the halfspace P_i contains z_i in its interior, and the $(n-1)$-face

$$F_i = \operatorname{conv}\{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1}\},$$

of C has empty intersection with $\operatorname{int} P_i$, $i = 1, \ldots, n+1$. This implies the inclusion, $Q \subset C$. Since $T(\mathcal{F})$ is the intersection of the simplices $\operatorname{conv}\{x_1, \ldots, x_{n+1}\}$ with an arbitrary choice of points $x_1 \in S_1, \ldots, x_{n+1} \in S_{n+1}$, we have $Q \subset T(\mathcal{F})$. □

Proof of Theorem 3

We study the convex hull of the complement of the convex sets $C_1, \ldots, C_k$. Given a point $x \in \mathbb{R}^n$, put

$$\lambda(x) = \{i \in \{1, \ldots, k\} \mid x \in C_i\}.$$

We begin with the special case in which the sets C_i are assumed to be closed.

Lemma 11. *Let $C_1, \ldots, C_k \subset \mathbb{R}^n$ be closed convex sets such that the set*

$$S = \operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \cdots \cup C_k))$$

is nonempty and bounded. If y and z are distinct exposed point of the set $P = \operatorname{cl} S$, then $\lambda(y) \not\subset \lambda(z)$.

Proof. The statement is trivial for $n = 1$, so we may put $n \geq 2$. Assume, for contradiction, that $\lambda(y) \subset \lambda(z)$ for a pair of distinct exposed points y and z of P. Let $H \subset \mathbb{R}^n$ be a closed halfspace with the properties $P \subset H$ and $P \cap \operatorname{bd} H = \{y\}$. Choose an open ball U centered at y such that $U \cap C_i = \varnothing$ for all $i \notin \lambda(z)$.

Put $T = \operatorname{conv}(\{z\} \cup (U \setminus H))$. We claim that $T \setminus \{z\} \subset C_1 \cup \cdots \cup C_k$. Indeed, let $w \in T \setminus \{z\}$. Then there is a point $u \in U \setminus H$ such that $w \in [u, z]$, and we may assume (since $U \setminus H$ is open) that $u \neq w$. Because $u \notin P$, there is an index $j \in \{1, \ldots, k\}$ such that $u \in C_j$. Thus $j \in \lambda(y)$ by the choice of U. From $\lambda(y) \subset \lambda(z)$ we conclude that $z \in C_j$. Hence $w \in [u, z] \subset C_j \subset C_1 \cup \cdots \cup C_k$.

Since $T \setminus \{z\} \subset C_1 \cup \cdots \cup C_k$ and the union $C_1 \cup \cdots \cup C_k$ is a closed set, we have $T \subset \operatorname{cl}(T \setminus \{z\}) \subset C_1 \cup \cdots \cup C_k$. Furthermore, $z \in \operatorname{int} H$ because $z \in P \setminus \{y\}$. Then $y \in \operatorname{int} T \subset \operatorname{int}(C_1 \cup \cdots \cup C_k)$. In this case, y cannot be an exposed point of P, a contradiction. □

Lemma 12. *There exists a function $\alpha(k)$ such that, whenever $C_1, \ldots, C_k \subset \mathbb{R}^n$ are closed convex sets for which the set*

$$S = \operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \cdots \cup C_k))$$

is bounded, the set $P = \operatorname{cl} S$ is a convex polytope with $\alpha(k)$ or fewer vertices.

Proof. This is an immediate corollary of the preceding lemma. Using Sperner's Lemma, we may put $\alpha(k) = \binom{k}{\lfloor \frac{k}{2} \rfloor}$. □

Lemma 13. *There exists a function $\beta(m)$ for which the following holds. Let $X_1, \ldots, X_m$ be convex sets such that $\operatorname{cl} X_1 \cup \cdots \cup \operatorname{cl} X_m = \mathbb{R}^n$. Then there exists a family of $\beta(m)$ or fewer proper affine subspaces of $\mathbb{R}^n$ whose union covers $\mathbb{R}^n \setminus (X_1 \cup \cdots \cup X_m)$.*

Proof. We show that $\beta(m) = 2^m - 1$ is such a function. For each nonempty set $\Omega \subset \{1, \ldots, m\}$, denote by A_Ω the affine hull of the convex set $Y_\Omega = \cap\{\operatorname{cl} X_i \mid i \in \Omega\}$ (so that $A_\Omega = \varnothing$ if $Y_\Omega = \varnothing$). Let

$$\mathcal{H} = \{\Omega \subset \{1, \ldots, m\} \mid \varnothing \neq A_\Omega \neq \mathbb{R}^n\}.$$

We claim that

$$\mathbb{R}^n \setminus (X_1 \cup \cdots \cup X_m) \subset \cup\{A_\Omega \mid \Omega \in \mathcal{H}\}. \tag{6}$$

Indeed, choose a point $x \in \mathbb{R}^n \setminus (X_1 \cup \cdots \cup X_m)$ and put

$$\Omega' = \{i \in \{1, \ldots, m\} \mid x \in \operatorname{cl} X_i\}.$$

Clearly, $\Omega' \neq \varnothing$ and $x \in Y_{\Omega'} \subset A_{\Omega'}$. We are going to show that $A_{\Omega'} \neq \mathbb{R}^n$. For this, it is sufficient to verify that $\operatorname{int} Y_{\Omega'} = \varnothing$. Suppose this is not the case and choose a point $y \in \operatorname{int} Y_{\Omega'}$. Let U be an open ball centered at x and disjoint from $\cup\{\operatorname{cl} X_i \mid i \notin \Omega'\}$.

Choose a point $z \in U$ such that $x \in [y, z[$. Clearly, $z \notin \cup\{\operatorname{cl} X_i \mid i \notin \Omega'\}$ by the choice of U. If there existed an index $i \in \Omega'$ with $z \in \operatorname{cl} X_i$, then $x \in [y, z[\subset \operatorname{int} X_i$ due to $y \in \operatorname{int} Y_{\Omega'} \subset \operatorname{int} X_i$. But the inclusion $x \in \operatorname{int} X_i$ is impossible due to the assumption $x \in \mathbb{R}^n \setminus (X_1 \cup \cdots \cup X_m)$. Hence $z \notin \cup\{\operatorname{cl} X_i \mid i \in \Omega'\}$. Summing up, $z \in \mathbb{R}^n \setminus (\operatorname{cl} X_1 \cup \cdots \cup \operatorname{cl} X_m)$, which contradicts the hypothesis $\operatorname{cl} X_1 \cup \cdots \cup \operatorname{cl} X_m = \mathbb{R}^n$. Thus $A_{\Omega'} \neq \mathbb{R}^n$ and (6) holds.

Since there are fewer than 2^m nonempty subsets of $\{1, \ldots, m\}$, the family $\mathcal{H}$ has fewer than 2^m elements. □

Using Sperner's Lemma, it is easy to show that the function $\beta(m)$ above may be taken to be $\binom{k}{\lfloor \frac{k}{2} \rfloor}$.

Lemma 14. *Let $C_1, \dots, C_k \subset \mathbb{R}^n$ be convex sets such that the set*

$$S = \operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \dots \cup C_k))$$

is bounded. Then the set $P = \operatorname{cl} S$ is a convex polytope. Furthermore, there exists a function $\eta(n,k)$ such that the number of vertices of P is bounded by $\eta(n,k)$.

Proof. We proceed by induction on n. The statement is clearly true for $n = 1$. Suppose that $n > 1$ and that the result holds for $n-1$. Let $\eta = \eta(n-1,k)$ be a bound on the number of vertices of P in the lower-dimensional case. Put

$$Y_0 = \operatorname{conv}(\mathbb{R}^n \setminus (\operatorname{cl} C_1 \cup \dots \cup \operatorname{cl} C_k)).$$

Lemma 12 implies that $\operatorname{cl} Y_0$ is a convex polytope with $\alpha(k)$ or fewer vertices. Obviously, $Y_0 \subset P$ and $\operatorname{cl} Y_0 \cup \operatorname{cl} C_1 \cup \dots \cup \operatorname{cl} C_k = \mathbb{R}^n$. By Lemma 13, there are proper affine subspaces $A_1, \dots, A_m \subset \mathbb{R}^n$, $m \le \beta(k)$, such that

$$\mathbb{R}^n \setminus (Y_0 \cup C_1 \cup \dots \cup C_k) \subset A_1 \cup \dots \cup A_m.$$

Without loss of generality, we may assume that all $A_1, \dots, A_m$ are hyperplanes. Put

$$Y_i = \operatorname{conv}(A_i \setminus (C_1 \cup \dots \cup C_k)), \; i = 1, \dots, m.$$

By the inductive assumption, each set $\operatorname{cl} Y_i$, $i = 1, \dots, m$, is a convex polytope with $\eta(n-1,k)$ or fewer vertices. Hence the set

$$Q = \operatorname{cl}\operatorname{conv}(Y_0 \cup Y_1 \cup \dots \cup Y_m) = \operatorname{conv}(\operatorname{cl} Y_0 \cup \operatorname{cl} Y_1 \cup \dots \cup \operatorname{cl} Y_m)$$

is a convex polytope with at most $\alpha(k) + \beta(k)\eta(n-1,k)$ vertices.

Put $\eta(n,k) = \alpha(k) + \beta(k)\eta(n-1,k)$. To complete the proof, we show that $P = Q$. Indeed, since $Y_i \subset P$ for all $i = 0, 1, \dots, m$, we have $Q \subset P$. Conversely, let $x \in \mathbb{R}^n \setminus (C_1 \cup \dots \cup C_k)$. Then either $x \in Y_0$, or there is an index $i \in \{1, \dots, m\}$ such that $x \in Y_i$. Hence

$$\mathbb{R}^n \setminus (C_1 \cup \dots \cup C_k) \subset Y_0 \cup Y_1 \cup \dots \cup Y_m,$$

and $P = \operatorname{cl}\operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \dots \cup C_k)) \subset \operatorname{cl}\operatorname{conv}(Y_0 \cup Y_1 \cup \dots \cup Y_m) = Q$. □

Lemma 15. *Let $Q_1 \subset Q_2 \subset \cdots$ be an ascending sequence of convex polytopes in $\mathbb{R}^n$, each with m or fewer facets. Then $Q = \operatorname{cl}(Q_1 \cup Q_2 \cup \cdots)$ is a closed convex polyhedron with m or fewer facets.*

Proof. Since $\dim Q_1 \le \dim Q_2 \le \dots \le n$, we may assume, without loss of generality, that all polytopes $Q_1, Q_2, \dots$ have dimension n. Assume also that the origin 0 of $\mathbb{R}^n$ lies in $\operatorname{int} Q_1$. Then the polar polytopes form a descending sequence, $Q_1^\circ \supset Q_2^\circ \supset \cdots$, each of them having m or fewer vertices. Clearly, $Q^\circ = Q_1^\circ \cap Q_2^\circ \cap \cdots$. Hence Q° is compact.

Assuming, for contradiction, the existence of $m+1$ distinct exposed points of Q°, we can choose $m+1$ pairwise disjoint neighborhoods $U_1, \dots, U_{m+1}$ of these points. Then there should be an index i_0 such that any polytope Q_i°, $i \geq i_0$, has an exposed point in each of the sets $U_1, \ldots, U_{m+1}$, which is impossible by the above. Hence Q° is a convex polytope with m or fewer exposed points. So Q° is a convex polytope with m or fewer vertices, and Q is a convex polyhedron with m or fewer facets. □

Lemma 16. *For any convex sets $C_1, \ldots, C_k \subset \mathbb{R}^n$ the set*

$$P = \operatorname{cl}\operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \cdots \cup C_k))$$

is a convex polyhedron.

Proof. Choose an ascending sequence of simplices $T_1 \subset T_2 \subset \cdots$ whose union is $\mathbb{R}^n$. Express every simplex T_i, $i = 1, 2, \ldots$, as the intersection of $n+1$ closed halfspaces $H_{i,1}, \ldots, H_{i,n+1}$. By Lemma 14, the sets

$$P_i = \operatorname{cl}\operatorname{conv}\big(\mathbb{R}^n \setminus (C_1 \cup \cdots \cup C_k \cup H_{i,1} \cup \cdots \cup H_{i,n+1})\big), \; i = 1, 2, \ldots,$$

are convex polytopes, and there is a common bound $\eta(n,k)$ on the number of vertices of every P_i. The same is true for the numbers of facets of the P_i's, albeit with a different upper bound. Since $P_1 \subset P_2 \subset \cdots$ and $P = \operatorname{cl}(P_1 \cup P_2 \cup \cdots)$, Lemma 15 implies that P is a convex polyhedron. □

Lemma 17. *For any convex sets $C_1, \ldots, C_k \subset \mathbb{R}^n$ the set*

$$S = \operatorname{conv}(\mathbb{R}^n \setminus (C_1 \cup \cdots \cup C_k))$$

is a convex polyhedron.

Proof. We proceed by induction on the dimension of S. The statement is trivial when $\dim S = 0$. Suppose that $\dim S > 0$ and that the result holds in all smaller-dimensional cases. By Lemma 16, $\operatorname{cl} S$ is a convex polyhedron. Since S is convex, its relative interior coincides with that of $\operatorname{cl} S$. Clearly, S is the union of its relative interior and the sets $A \cap S$, where A is an affine subspace spanned by one of the finitely-many facets of $\operatorname{cl} S$. For any such affine subspace A, we have

$$A \cap S = \operatorname{conv}(A \setminus (A \cap C_1) \cup \cdots \cup (A \cap C_k)),$$

so, by the inductive assumption, $A \cap S$ is a finite union of relatively open convex polyhedra. Thus S is a convex polyhedron. □

Notes

Several combinatorial questions are raised but not treated in the foregoing. In particular, we note the problems of finding the best (smallest) functions α, β, and η from Lemmas 12–14. The paper [7] contains some material related to this. In particular, it can be seen that, unlike the situation for α and β, the dependence of η upon the dimension is necessary, for, in each dimension $n \geq 1$, it is possible to find three convex sets in $\mathbb{R}^n$ such that the complement of their union is the set of vertices of an n-simplex.

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