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On Shadow Boundaries of Centrally Symmetric Convex Bodies

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Abstract. We discuss the concept of the so-called shadow boundary belonging to a given direction $\mathbf{x}$ of Euclidean n-space R^n lying in the boundary of a centrally symmetric convex body K. Actually, K can be considered as the unit ball of a finite dimensional normed linear (= Minkowski) space. We introduce the notion of the general parameter spheres of K corresponding to the above direction $\mathbf{x}$ and prove that if all of the non-degenerate general parameter spheres are topological manifolds, then the shadow boundary itself becomes a topological manifold as well. Moreover, using the approximation theorem of cell-like maps we obtain that all these parameter spheres are homeomorphic to the $(n-2)$-dimensional sphere $S^{(n-2)}$. We also prove that the bisector (i.e., the equidistant set with respect to the norm) belonging to the direction $\mathbf{x}$ is homeomorphic to $R^{(n-1)}$ iff all of the non-degenerate general parameter spheres are $(n-2)$-manifolds. This implies that if the bisector is a homeomorphic copy of $R^{(n-1)}$, then the corresponding shadow boundary is a topological $(n-2)$-sphere.

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1. Introduction

1.1. Notation and terminology

R, R^n, S^n: The real line, the n-dimensional Euclidean space and the n-dimensional unit sphere, respectively.

bd(K), int(K), cl(K): The boundary, interior and closure of the set K, respectively.

dim(K): The topological (covering) dimension of the set K.

ANR: Absolute neighbourhood retract. (See Section 2 after Lemma 1.)

concepts without definition: connectivity (arcwise, locally), contractibility (locally), manifold, manifold with boundary, retract, compact metric space, inverse limit of topological spaces, bonding maps, standard hyperplane.

topological hyperplane: A homeomorphic copy H of the space $R^{(n-1)}$ is a topological hyperplane if there is a homeomorphism of R^n onto itself which sends H onto a standard hyperplane of R^n. We recall such a homeomorphism as a standard embedding of H into R^n.

topological n-sphere: is a homeomorphic copy of an n-dimensional Euclidean sphere. The embedding of a topological n-sphere S into the unit $(n+1)$-sphere S^{n+1} is standard, if $S \subset S^{n+1}$ and there is a homeomorphism of S^{n+1} onto itself which sends S onto an equator n-sphere of S^{n+1}.

cellular set, map: The definition can be seen in the Section 2 after Theorem 1.

cell-like set, map: The definition can be seen in the Section 2 before Theorem 1.

near homeomorphism: The definition can be seen in the Section 2 before Theorem 1.

$S(K, \mathbf{x})$: The shadow boundary of the body K in direction $\mathbf{x}$ (see at the beginning of Section 2).

K^+, K^-: The positive and negative part of bd K, respectively (see in Section 2).

P^+, P^-: The positive and negative pole of bd(K), respectively (see in Section 2).

longitudinal parameter curve: The two dimensional intersection curve of bd(K) with a plane through the poles (see in Section 2).

λ_0: The smallest value λ for which λK and $\lambda K + \mathbf{x}$ are intersecting.

$\gamma_\lambda(K, \mathbf{x})$: the generalized parameter sphere of K corresponding to the direction $\mathbf{x}$ and to the parameter $\lambda \geq \lambda_0$ (see Definition 2 in Section 3).

$H_\mathbf{x}$: The bisector of the vector $\mathbf{x}$. It is the equidistant set belonging to the starting point and endpoint of the vector $\mathbf{x}$.

$p_\mathbf{x}$: The orthogonal projection mapping of the space R^n onto a hyperplane orthogonal to the vector $\mathbf{x}$.

1.2. Historical remarks and the results

If K is a 0-symmetric, bounded, convex body in the Euclidean n-space R^n (with fixed origin O) then it defines a norm whose unit ball is K itself (see [12]). Such a space is called *Minkowski space* or *normed linear space* of finite dimension. Many results on this topic are collected in the surveys [18], [19] and [16]. In fact, the norm is a continuous function which is considered (in geometric terminology, as in [12]) as a gauge function. The metric (the so-called Minkowski metric), i.e., the distance of two points induced by this norm, is invariant with respect to translations.

The unit ball is said to be *strictly convex* if its boundary contains no line segment.

In some previous papers on this topic (see [10] and [11]), we examined the boundary related to the unit ball of the norm and gave two theorems (Theorem 2 and Theorem 4) similar to the characterization of the Euclidean norm investigated by H. Mann, A. C. Woods and P. M. Gruber in [15], [25], [6], [7] and [8], respectively. We proved that if the unit ball of a Minkowski space is strictly convex, then every *bisector* (which is the collection of those points of the embedding Euclidean space which have the same distance, with respect to the norm, to two given points of the space) is a topological hyperplane (Theorem 2). Example 3 in [10] showed that strict convexity does not follow from the fact that all bisectors are topological hyperplanes.

We examined the connection between the shadow boundaries of the unit ball and the bisectors of the Minkowskian space. We were sure that the following statement is true:

A bisector is a topological hyperplane if and only if the corresponding shadow boundary is a topological $(n-2)$-dimensional sphere.

However, we proved the conjecture only for the three-dimensional case (Theorem 2 and Theorem 4). We also examined basic properties of the shadow boundary (Section 2) and defined a useful class of sets – the so-called general parameter spheres.

In this paper we discuss some further topological observations on shadow boundaries and general parameter spheres. We prove that the general parameter spheres and the shadow boundary are in general not ANRs (see [4] or [21]), but are still compact metric spaces, containing $(n-2)$-dimensional closed, connected subsets separating the boundary of K. We investigate the manifold case and (using the approximation theorem of cell-like mappings) prove that the general parameter spheres and the corresponding shadow boundary are homeomorphic to the $(n-2)$-dimensional sphere. A consequence of this result (if the bisector is a homeomorphic copy of $R^{(n-1)}$ then the shadow boundary is a topological $(n-2)$-sphere) yields the proof of the first direction of the above mentioned conjecture. Two more questions concerning the same conjecture are left: Is the converse statement true? Is it possible that in the manifold case the embeddings of the bisector and of the shadow boundary are not standard ones? In the fourth section we prove that the embedding of the examined sets (in the manifold case) are always standard ones; the first question remains open.

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2. Once more on the shadow boundary of the unit ball

There are several well-known properties of the shadow boundary of a convex body with respect to a given direction in n-space (see in [16]), but there is no comprehensive list of its topological properties. Of course, shadow boundaries have been considered frequently in convexity theory. Two interesting results in context of Baire categories should be mentioned; see [9] and [24]. In [9] the authors proved that a typical shadow boundary of a convex body under parallel illumination has infinite $(n-2)$-dimensional Hausdorff measure, while having Hausdorff dimension $(n-2)$. In [24] it is shown that, in the sense of Baire categories, most of the n-dimensional convex bodies have infinitely long shadow boundaries if the light vector comes along one of the $(n-2)$-dimensional subspaces.

Definition 1. *Let K be a centrally symmetric, compact, convex body in n-dimensional Euclidean space E^n, and let $S^{(n-1)}$ denote the $(n-1)$-dimensional unit sphere in E^n. For $\mathbf{x} \in S^{(n-1)}$ the shadow boundary $S(K,\mathbf{x})$ of K in direction $\mathbf{x}$ consists of all those points P in $\mathrm{bd}(K)$ for which the line $\{P+\lambda\mathbf{x} : \lambda \in R\}$ supports K, i.e. it meets K but does not meet the interior of K. The shadow boundary $S(K,\mathbf{x})$ is sharp if any of the above supporting lines of K intersects K exactly in the point P. If $S(K,\mathbf{x})$ is not sharp, in general, it may have a sharp point for which the above uniqueness holds.*

To make this paper more self-contained, we list and show some topological properties of the shadow boundary. (Of course, some of these are well-known facts.) First we introduce some notation:

$$\begin{aligned} K^+ &:= \{\mathbf{y} \in \mathrm{bd}(K) | \text{ there is } \tau > 0 \text{ such that } \mathbf{y} - \tau \cdot \mathbf{x} \in \mathrm{int}(K)\}, \\ K^- &:= \{\mathbf{y} \in \mathrm{bd}(K) | \text{ there is } \tau > 0 \text{ such that } \mathbf{y} + \tau \cdot \mathbf{x} \in \mathrm{int}(K)\}. \end{aligned} \tag{1}$$

We call the congruent (thus homeomorphic) sets K^+ and K^- the *positive* and *negative part* of $\mathrm{bd}(K)$, respectively. The line passing through the origin and parallel to the vector $\mathbf{x}$ intersects the boundary of K at the points $P^+ \in K^+$ and $P^- \in K^-$ showing that the positive and negative part of $\mathrm{bd}(K)$ are not empty, respectively. We call the points P^+ and P^- the *positive* and *negative pole* of K, respectively. The intersection of $\mathrm{bd}(K)$ with a 2-plane containing the poles is called a *longitudinal parameter curve* of K.

Statement 1. *The shadow boundary decomposes the boundary of K into three disjoint sets: $S(K,\mathbf{x})$, K^+, and K^-. $S(K,\mathbf{x})$ is an at least $(n-2)$-dimensional closed (and therefore compact) set in $\mathrm{bd}(K)$ which is connected for $n \geq 3$, the sets K^+ and K^- are homeomorphic copies of $R^{(n-1)}$ giving two arcwise connected components of their union.*

Proof. The first statement is obvious. Let $p_{\mathbf{x}}$ be the orthogonal projection of the embedding space R^n onto a hyperplane orthogonal to the vector $\mathbf{x}$. Since the orthogonal projection is a contraction then it is continuous (i.e. it is a mapping of the space). $p_{\mathbf{x}}(K)$ is a convex body of the image hyperplane. The interior of $p_{\mathbf{x}}(K)$ is the image of the sets K^+ and K^-, respectively and its boundary is the image of $S(K,\mathbf{x})$. Since $p_{\mathbf{x}}$ restricting for K^+ is a bijection, there exists a homeomorphism on K^+ to $R^{(n-1)}$. Using the same argument for K^- we proved the validity of the first part of the statement on K^+ and K^-. Of course their union is open therefore the shadow boundary is closed.

Since $R^{(n-1)}$ is arcwise connected the second part of the statement on K^+ follows from the fact that an arc connecting two points of K^+ and K^- should be decomposed into two relative open sets by K^+ and K^-, which is a contradiction. (Arcwise connectivity of a set implies its connectivity, too.) Thus the shadow boundary separates the boundary of K. By a theorem of Alexandrov (Theorem 5.12 in Vol. I of [1]) we get that the topological dimension of $S(K,\mathbf{x})$ is at least $(n-2)$, as we stated.

We now prove that (for $n \geq 3$) the set $S(K,\mathbf{x})$ is connected. Assume that K_1 and K_2 are two closed disjoint subsets of the shadow boundary for which $K_1 \cup K_2 = S(K,\mathbf{x})$. First we observe that each of the metric segments lying on a longitudinal parameter curve and parallel to $\mathbf{x}$ is a connected subset of $S(K,\mathbf{x})$, thus its points (by the "basic lemma of connectivity" see Vol. I, p. 13 in [1]) belong either to the set K_1 or to the set K_2. Let C_1 and C_2 the sets defined by the union of those longitudinal parameter curves which intersect the sets K_1 and K_2. In this case $C_1 \cup C_2 = \mathrm{bd}(K)$ and $C_1 \cap C_2 = \{P^+, P^-\}$ hold. The sets C_i are closed in $\mathrm{bd}(K)$, meaning that the sets $C_i \setminus \{P^+, P^-\}$ give a decomposition of $\mathrm{bd}(K)c \setminus \{P^+, P^-\}$ into disjoint relative closed subsets, too. Since the latter set is connected it follows that either K_1 or K_2 is empty. $\square$

In general, the dimension of $S(K,\mathbf{x})$ is $(n-2)$ or $(n-1)$. We prove that there is an $(n-2)$-dimensional closed, connected subset of $S(K,\mathbf{x})$ separating $\mathrm{bd}(K)$, too.

Lemma 1. *The boundary (frontier) of the closure of the set K^+ (denoted by $\mathrm{bd}(\mathrm{cl}(K^+))$) is a closed, connected $(n-2)$ dimensional subset of $S(K,\mathbf{x})$ separating the boundary of K.*

Proof. By definition it is closed. Since $\mathrm{cl}(K^+) \supset K^+$ and $\mathrm{cl}(K^+) \cap K^- = \emptyset$ we have $K^+ \subset \mathrm{cl}(K^+) \subset K^+ \cup S(K,\mathbf{x})$. On the other hand $\mathrm{bd}(\mathrm{cl}(K^+)) \cap K^+ = \emptyset$ (K^+ is an open subset of $\mathrm{cl}(K^+)$), thus we get that $\mathrm{bd}(\mathrm{cl}(K^+)) \subset S(K,\mathbf{x})$.

The separating property follows from the fact that the union of the pairwise disjoint sets $\mathrm{bd}(K) \setminus \mathrm{cl}(K^+)$, $\mathrm{int}(\mathrm{cl}(K^+))$, $\mathrm{bd}(\mathrm{cl}(K^+))$ fills the boundary of K and the first two sets are open.

Now the separating property implies (again by the Alexandrov theorem above) the inequality $dim(\mathrm{bd}(\mathrm{cl}(K^+))) \geq (n-2)$. On the other hand a closed connected set of dimension $(n-1)$ on $\mathrm{bd}(K)$ contains an interior point relative to $\mathrm{bd}(K)$ (see p. 174 in Vol. I of [1]) which contradicts the definition of $\mathrm{bd}(\mathrm{cl}(K^+))$. $\square$

Before proving the main statement of this section, we consider three examples which clearly show the strange attitude of the sets defined above. In the first example we construct such a centrally symmetric convex body whose shadow boundary is neither locally connected nor locally contractible. This implies that the shadow boundary (in general) is not an absolute neighbourhood retract (ANR) (especially a topological manifold). We recall that the space Y is an *ANR* if, whenever it is embedded as a closed subset of a separable metric space, then it is a retract of some its neighbourhood in that space. An ANR is always a locally contractible space.

Examples. 1. Consider the following sequence of segments of R^3 (with respect to a fixed orthonormal coordinate system): $s_n = \{(t, \frac{1}{n}, \frac{\sqrt{n^2-1}}{n}) | -1 \leq t \leq 1\}, n \in N$, with limit segment $s = \{(t, 0, 1) | -1 \leq t \leq 1\}$. Connect the point $(1, \frac{1}{n}, \frac{\sqrt{n^2-1}}{n})$ with the point $(-1, \frac{1}{n+1}, \frac{\sqrt{(n+1)^2-1}}{n+1})$ by an arc, which is the intersection of the 2-plane containing the points and orthogonal to the [x,y]-plane with the cylinder $C := \{(t, r, s) | -1 \leq t \leq 1, r^2 + s^2 = 1, r, s \geq 0\}$. The union of these curves forms a connected, closed set lying on the cylinder C. This set is neither arcwise nor locally connected; moreover, it is not a locally contractible one. It is easy to see that if we add this curve to its reflected images with respect to the plane $[x, z]$, and the resulting union curves we add to its images reflected in the plane $[x, y]$, finally we have a single centrally symmetric closed curve γ belonging to the cylinder $\{(t, r, s) | -1 \leq t \leq 1, r^2 + s^2 = 1\}$. The convex hull of γ (similar to the so-called "topologist's sine curve") is a centrally symmetric convex body. If the direction of the light is parallel to the x-axis, then we have $S(K, \mathbf{x}) = \mathrm{bd}(\mathrm{cl}(K^+)) = \mathrm{bd}(\mathrm{cl}(K^-)) = \gamma$. Since it is not locally contractible, it can not be an ANR.

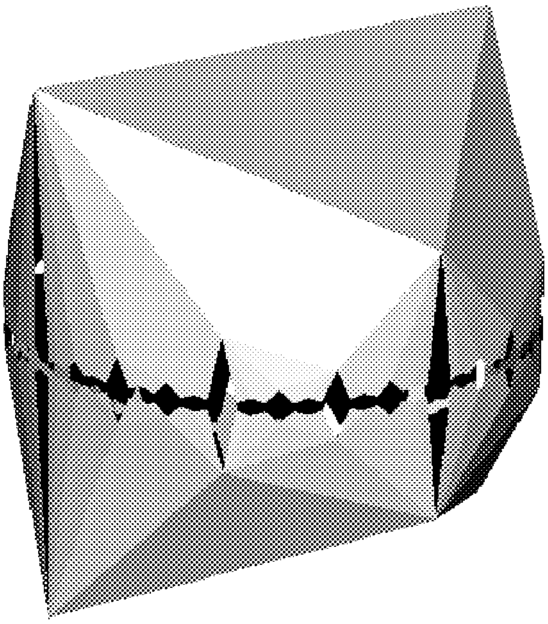

Figure 1. Shadow boundary which is not a topological manifold

2. Secondly we refer to Example 1 in the paper [11], which is a presentation of a shadow boundary as it is shown in Figure 1. This body K is also closed, for which $S(K,\mathbf{x})$ is an ANR, but not a manifold. The sets $\mathrm{bd}(\mathrm{cl}(K^+)) = \mathrm{bd}(\mathrm{cl}(K^-))$ coincide with the same metric circle.

3. In this example the sets $\mathrm{bd}(\mathrm{cl}(K^+))$, $\mathrm{bd}(\mathrm{cl}(K^-))$ form the common boundary of the sets $S(K,\mathbf{x})$ and K^+, $S(K,\mathbf{x})$ and K^-, respectively. They are homeomorphic to S^1 but $S(K,\mathbf{x})$ is neither a 1-manifold nor 2-manifold with boundary. Consider the regular octahedron as K and let the direction of the light be parallel to an edge of K. The shadow boundary is the polyhedron containing four faces of K that are connected to each other either with a common edge (parallel to $\mathbf{x}$) or with a common vertex. The sharp points of the shadow boundary are these two vertices. Of course, there is no neighbourhood of this points homeomorphic to a segment or a plane.

Now we focus to the cases when the above sets are topological manifolds. We need to recall some useful definitions and theorems on the topic of cell-like mappings. There are several good papers on this important chapter of geometric topology (e.g. [13], [14] or [22]). We follow here the setting up of the paper of W. J. R. Mitchell and D. Repovs [20].

A non-empty compactum K is said to be *cell-like* if for some embedding of K in an ANR M the following property holds: For every neighbourhood U of K in M, there exists a neighbourhood V such that $K \subset V \subset U$ and the inclusion $i : V \longrightarrow U$ is nullhomotopic. Given a map (by definition it is a continuous function) $f : X \longrightarrow Y$, we say that f is *cell-like*, if for each $y \in Y$ the inverse image $f^{-1}(y)$ is cell-like. We will use the following theorem:

Theorem 1. (Cell-like approximation theorem for manifolds) *Let $n \neq 3$ be a positive integer. For every cell-like map $f : M \longrightarrow N$ between topological n-manifolds and every $\varepsilon > 0$, there is a homeomorphism $h : M \longrightarrow N$ such that $d(f,h) < \varepsilon$ in the sup-norm metric on the space of all continuous maps (so f is a so-called near homeomorphism).*

The long history of this result can be read in [20]. We note that in the 3-dimensional case there is an analogous approximation theorem for a subset of the class of cell-like mappings called the class of *cellular maps*. A set of the manifold M is called *cellular*, if it is an intersection of a sequence of closed cells B_i of M with the properties $K \subset B_i$ and $B_{i+1} \subset \mathrm{int}(B_i)$. A map is cellular if the inverse images are cellular sets. Cellularity goes back to the work of M. Brown [2] while the concept of cell-likeness has been introduced by R. C. Lacher in [13]. The concept of cellularity depends on the embedding of the examined metric space K in M; this dependence on embedding was eliminated in the concept of cell-likeness. In fact, (in the manifold case) every cellular map is a cell-like map, since every cellular set is a cell-like one. Conversely, if we consider a wild arc in R^3 which has non-simply-connected complement, it is non-cellular set, while the standard embedding manifestly is cellular in R^3, showing that it will be a cell-like set.

We also remark that a cellular (or cell-like) map (in a general case) is not a near homeomorphism, since there is a cellular map on $S^1 \times [0,1]$ to S^1 (here $[0,1]$ means the unit interval of the real line) which is not a near homeomorphism (if we have a near homeomorphism between compact metric spaces, then these two spaces should be homeomorphic to each other).

Now we can prove the main theorem of this section.

Theorem 2. *If the shadow boundary $S(K,\mathbf{x})$ is a topological manifold of dimension $(n-2)$, then it is homeomorphic to the $(n-2)$-sphere $S^{(n-2)}$. If it is an $(n-1)$-dimensional manifold with boundary, then it is homeomorphic to the cylinder $S^{(n-2)} \times [0,1]$.*

Proof. Consider first the projection $p_{\mathbf{x}}$ (which was defined in the proof of Statement 1), and restrict it to the shadow boundary of K parallel to $\mathbf{x}$. It is a cell-like map because the inverse images are points or segments, respectively. In this way, by the approximation theorem above we have for $n \neq 5$ that this restricted map is a near homeomorphism on $S(K,\mathbf{x})$ to a homeomorphic copy $\tilde{S}^{(n-2)}$ of $S^{(n-2)}$, implying that they are homeomorphic to each other. On the other hand, this map is also cellular, since the metric segments and points of $S(K,\mathbf{x})$ are cellular sets in $S(K,\mathbf{x})$. To prove this, let $s = p_{\mathbf{x}}^{-1}(\mathbf{v})$ be a segment in $S(K,\mathbf{x})$ for some $\mathbf{v} \in \tilde{S}^{(n-2)}$. If now $Q \in s$ is a point, consider a metric ball $B_\epsilon(Q) \subset \mathrm{bd}(K)$ with center Q and radius $\epsilon > 0$ for which $\mathrm{int}(B_\epsilon(Q)) \cap S(K,\mathbf{x})$ is homeomorphic to $R^{(n-2)}$. Such an $\epsilon > 0$ surely exists. In fact, Q has a neighbourhood N_Q in $S(K,\mathbf{x})$ homeomorphic to $R^{(n-2)}$. If for every ϵ we can choose a point $P_\epsilon \in B_\epsilon(Q) \cap S(K,\mathbf{x})$ which does not belong to N_Q, then we have a sequence of points (P_ϵ) having the same property and tending to Q. Since N_Q is open in $S(K,\mathbf{x})$, this is impossible. Thus there is an $\epsilon > 0$ for which $B_\epsilon(Q) \cap S(K,\mathbf{x}) = B_\epsilon(Q) \cap N_Q$. This implies that $\mathrm{int}(B_\epsilon(Q)) \cap S(K,\mathbf{x})$ is an open subset of N_Q relative to the topology of $S(K,\mathbf{x})$. Of course, ϵ depends on Q, but s is a compact set. Thus there is a finite number of points Q_i and positive real numbers ϵ_i, such that for the minimal value ϵ^* of ϵ_i's we have $\cup\, \mathrm{int}(B_{\epsilon^*}(Q_i)) \supset s$. Here $\cup\, \mathrm{int}(B_{\epsilon^*}(Q_i))$ is the interior of the closed cell $\cup(B_{\epsilon^*}(Q_i))$. Since $B_\epsilon(Q) \cap S(K,\mathbf{x}) = B_\epsilon(Q) \cap N_Q$ also holds for every ϵ' which is less or equal to ϵ, we have an infinite sequence of sets of the form $\cup(B_{\epsilon^*}(Q_i))$ the property needed to prove the cellularity of s.

Observe now that if $S(K,\mathbf{x})$ is an $(n-1)$-manifold with boundary, then its boundary has two connected components which are equal to $\mathrm{bd}(\mathrm{cl}(K^+))$ and $\mathrm{bd}(\mathrm{cl}(K^-))$, respectively.

First we can see that $\mathrm{bd}(\mathrm{cl}(K^+))$ is the set of the common boundary points of $\mathrm{cl}(K^+)$ and $S(K,\mathbf{x})$ yielding $\mathrm{bd}(\mathrm{cl}(K^+)) \subset \mathrm{bd}(S(K,\mathbf{x}))$. (Analogously we have that $\mathrm{bd}(\mathrm{cl}(K^-)) \subset \mathrm{bd}(S(K,\mathbf{x}))$.)

Secondly we note that there is no point of $\mathrm{int}(\mathrm{cl}(K^+))$ belonging to $S(K,\mathbf{x})$. Indirectly assume that the point P is in $\mathrm{int}(\mathrm{cl}(K^+)) \cap S(K,\mathbf{x})$. Then

- either one can find a neighbourhood U of P in $S(K,\mathbf{x})$ which is homeomorphic to the $(n-1)$-dimensional half-space and therefore P is a boundary point of $\mathrm{cl}(K^+)$ (in U there exists a point Q with a neighbourhood

$V \subset S(K, \mathbf{x})$ homeomorphic to $R^{(n-1)}$ such that $Q \in V \subset U$; this means that Q is a point of the complement of $\mathrm{cl}(K^+)$),

- or there is a neighbourhood U homeomorphic to the space $R^{(n-1)}$ for which $P \in U \subset S(K, \mathbf{x})$. In this case P is in the interior of $S(K, \mathbf{x})$ contradicting the assumption that it is a point of $\mathrm{int}(\mathrm{cl}(K^+))$.

In this way $\mathrm{int}(\mathrm{cl}(K^+)) = K^+$, and then $\mathrm{bd}(\mathrm{cl}(K^+)) = \mathrm{bd}(K^+)$ is the common boundary of K^+ and $S(K, \mathbf{x})$. Applying Lemma 1 we obtain that $\mathrm{bd}(\mathrm{cl}(K^+))$ is a connected closed subset of the boundary of $S(K, \mathbf{x})$.

Using the fact that $\mathrm{bd}(\mathrm{cl}(K^-))$ is the image of $\mathrm{bd}(\mathrm{cl}(K^+))$ by a central projection, we have a similar result for $\mathrm{bd}(\mathrm{cl}(K^-))$, too. (It is the common boundary of K^- and $S(K, \mathbf{x})$.) We will prove that the boundary of $S(K, \mathbf{x})$ is the disjoint union of these two sets.

The relation $\mathrm{bd}(S(K, \mathbf{x})) \subset \mathrm{bd}(\mathrm{cl}(K^-)) \cup \mathrm{bd}(\mathrm{cl}(K^+))$ is obvious. Consider a point P from the intersection $\mathrm{bd}(\mathrm{cl}(K^-)) \cap \mathrm{bd}(\mathrm{cl}(K^+))$. Let U be a neighbourhood of P in $S(K, \mathbf{x})$. (It is homeomorphic to a half-space of $R^{(n-1)}$.) Let B be a metric $(n-1)$-ball around P with such a sufficiently small radius $\epsilon > 0$ that the sets $B \cap U$ and $B \setminus (B \cap U)$ serve as topological images of a closed and the complementary open half-space of $R^{(n-1)}$, respectively. (Similarly as the proof of the cellularity property of a segment goes one can show that such an $\epsilon > 0$ and ball B exist.) Since B contains points from each of the sets K^+ and K^-, we have a contradiction by the separating property of $S(K, \mathbf{x})$. (There is no point of $S(K, \mathbf{x})$ in the complementary domain $B \setminus (B \cap U)$.)

This implies that the boundary of $S(K, \mathbf{x})$ has two connected components which are the common boundaries of $S(K, \mathbf{x})$ and K^+, $S(K, \mathbf{x})$ and K^-, respectively. Of course, these sets are also $(n-2)$-manifolds connected with straight line segments through all their points. So we have that $S(K, \mathbf{x}) = \mathrm{bd}(\mathrm{cl}(K^+)) \times [0, 1]$ holds. We still have to prove that in this case $\mathrm{bd}(\mathrm{cl}(K^+))$ is homeomorphic to $S^{(n-2)}$, too. Since $p_{\mathbf{x}}$ on $\mathrm{bd}(\mathrm{cl}(K^+))$ into $S^{(n-2)}$ is also a cell-like (and cellular) mapping, $\mathrm{bd}(\mathrm{cl}(K^+))$ is an $(n-2)$-dimensional manifold, and this restricted map is one to one. The last statement of the theorem follows from Theorem 1, too. □

3. General parameter spheres

We now recall the definition of general parameter spheres (see [11]).

Definition 2. *Let*

$$\lambda_0 := \sup\{t | tK \cap (tK + \mathbf{x}) = \emptyset\}$$

be the smallest value λ for which λK and $\lambda K + \mathbf{x}$ are intersecting. Then the generalized parameter sphere of K corresponding to the direction $\mathbf{x}$ and to the parameter $\lambda \geq \lambda_0$ is the following set:

$$\gamma_\lambda(K, \mathbf{x}) := \frac{1}{\lambda}(\mathrm{bd}(\lambda K) \cap \mathrm{bd}(\lambda(K) + \mathbf{x})).$$

In [11] we mentioned that in general the above sets are not topological spheres of dimension $(n-2)$ and are not homeomorphic to each other. E.g., the dimension of $\gamma_{\lambda_0}(K,\mathbf{x})$ may be $0,1,\dots,n-1$, while the topological dimension of $\gamma_\lambda(K,\mathbf{x})$ is at least $(n-2)$, because this set divides the surface of K. We remark that the interiors of the given two caps of the boundary are also homeomorphic to each other, as in the case of a shadow boundary. In fact, a central projection from $\frac{1}{\lambda}\mathbf{x}$ sending the left half of $\mathrm{bd}(K)$ onto the left one of $\frac{1}{2\lambda}(\mathrm{bd}(\lambda K)+\mathbf{x})$ is an appropriate homeomorphism. (The latter set is congruent to the right half of $\mathrm{bd}(K)$, since the body $\lambda K\cap\lambda K+\mathbf{x}$ is a centrally symmetric one.) We also proved that the shadow boundary $S(K,\mathbf{x})$ is the limit of the generalized parameter spheres $\gamma_\lambda(K,\mathbf{x})$, with respect to the Haussdorff metric, when λ tends to infinity.

We have shown (in the proof of Lemma 1 in [11]) that the general parameter sphere $\gamma_\lambda(K,\mathbf{x})$ is the shadow boundary of the convex body $\frac{1}{\lambda}(\lambda K\cap\lambda K+\mathbf{x})$. Thus the statements of the previous section can be adapted to them.

Before presenting our result, we recall a nice theorem of M. Brown on the projective limit of compact metric spaces and corresponding near homeomorphisms (see [3] or [23]). The concept of the near homeomorphism of topological manifolds can be adapted to the case of compact metric spaces, too. A map from X to Y between compact metric spaces is a near homeomorphism if it is in the closure of the set of all homeomorphisms from X onto Y, with respect to the sup-norm metric on the space $C(X,Y)$ of all maps from X to Y. Here is the announced theorem.

Theorem 3. (M. Brown) *Let (X_n) be an inverse sequence of compact metric spaces with limit X_∞. If all bonding maps $X_k\longrightarrow X_n$ are near homeomorphisms, then so are the limit projections $X_k\longrightarrow X_\infty$.*

The purpose of this section is to examine the manifold case. We prove the following theorem:

Theorem 4. I. *The shadow boundary $S(K,\mathbf{x})$ is an $(n-2)$-dimensional manifold if all of the non-degenerate general parameter spheres $\gamma_\lambda(K,\mathbf{x})$ with $\lambda>\lambda_0$ are $(n-2)$-dimensional manifolds. Conversely, if $S(K,\mathbf{x})$ is an $(n-2)$-dimensional manifold, then all of the general parameter spheres are ANRs.*

II. *The shadow boundary $S(K,\mathbf{x})$ is an $(n-1)$-dimensional manifold with boundary iff there is a λ for which the general parameter sphere $\gamma_\lambda(K,\mathbf{x})$ is an $(n-1)$-dimensional manifold with boundary.*

Before the proof, we give an example showing that we should distinguish the above two cases.

Example. Consider the union of the six connecting rectangles $\pm\{(r,1,t)|-1\le r,t\le 1\}$, $\pm\{(r,s,t)|r+s=2,1\le r\le 2,-1\le t\le 1\}$, $\pm\{(r,s,t)|r-s=2,1\le r\le 2,-1\le t\le 1\}$ and the segments $\pm\{(r,0,2)|-\frac{3}{2}\le r\le\frac{3}{2}\}$. The convex hull K of this set is a convex polyhedron. If now the vector $\mathbf{x}$ is the position vector directed to the point $(4,0,0)$, we have three important values for the parameters of the generalized parameter spheres. For $\lambda_0=1$ the degenerate sphere $\gamma_{\lambda_0}(K,\mathbf{x})$ is a segment. For $1<\lambda\le\frac{5}{4}$ the general parameter spheres $\gamma_\lambda(K,\mathbf{x})$ are homeomorphic to S^1. In the range $\frac{5}{4} < \lambda \leq \frac{3}{2}$ the general parameter sphere $\gamma_\lambda(K, \mathbf{x})$ is a simplicial complex containing one or two-dimensional simplices, respectively. (This space is an ANR, but it is not a topological manifold.) Finally, in the last parameter domain $\lambda > \frac{3}{2}$ the set $\gamma_\lambda(K, \mathbf{x})$ is homeomorphic to the cylinder $S^1 \times [0, 1]$. Since $S(K, \mathbf{x})$ is the union of six quadrangles, parallel to the x-axis it is also a cylinder.

We also remark that if $S(K, \mathbf{x})$ is an $(n-2)$-dimensional manifold, then probably all of the non-degenerate parameter spheres are the same, too. Unfortunately we could not prove this statement.

Now we are ready for the proof of Theorem 4.

Proof. First we note that, for every $\lambda_0 < \lambda' < \infty$, $S(K, \mathbf{x})$ can be considered as the inverse limit space X_∞ of the metric spaces $X_\lambda := \gamma_\lambda(K, \mathbf{x})$ for $\lambda' < \lambda$. In fact, by Lemma 1 in [11], if for $\lambda > \lambda_0$ the intersection of $\gamma_\lambda(K, \mathbf{x})$ by a longitudinal parameter curve, say r, is a segment, then $r \cap \gamma_\mu(K, \mathbf{x})$ with $\mu > \lambda$ is also a segment containing the segment $r \cap \gamma_\lambda(K, \mathbf{x})$. So in this case the union of the sets $r \cap \gamma_\mu(K, \mathbf{x})$ is the segment $r \cap S(K, \mathbf{x})$. On the other hand we have two possibilities for $r \cap \gamma_\lambda(K, \mathbf{x})$ being a point. First, $r \cap S(K, \mathbf{x})$ is a point, too, meaning that for all $\mu > \lambda$, $r \cap \gamma_\lambda(K, \mathbf{x})$ is also a point. If now $r \cap S(K, \mathbf{x})$ is a segment, then we have a value $\lambda' > \lambda$ with the property that if $\mu > \lambda'$ then $r \cap \gamma_\mu(K, \mathbf{x})$ is a segment, too. In this latter case $r \cap S(K, \mathbf{x}) = \cup_{\mu \geq \lambda'} \{r \cap \gamma_\mu(K, \mathbf{x})\}$. Define now the left end of a segment parallel to $\mathbf{x}$ as the end having the smaller parameter in the usual parametrization with respect to $\mathbf{x}$ (meaning that a general point of a line parallel to $\mathbf{x}$ is written in the form $P + \tau\mathbf{x}$ where P is a point of this line). Let us define the bonding map $p_{\lambda,\mu}$ for $\gamma_\mu(K, \mathbf{x})$ to $\gamma_\lambda(K, \mathbf{x})$ $(\mu > \lambda)$ in the following way:

For a point P of $\gamma_\mu(K, \mathbf{x})$

$$p_{\lambda,\mu}(P) = \begin{cases} r \cap \gamma_\lambda(K, \mathbf{x}) & \text{if } r \cap \gamma_\lambda(K, \mathbf{x}) \text{ is a point} \\ P & \text{if } r \cap \gamma_\lambda(K, \mathbf{x}) \text{ is a segment and} \\ & \quad P \in r \cap \gamma_\lambda(K, \mathbf{x}) \\ \text{the left end of } r \cap \gamma_\lambda(K, \mathbf{x}) & \text{if } P \in r \cap \gamma_\mu(K, \mathbf{x}) \setminus r \cap \gamma_\lambda(K, \mathbf{x}) . \end{cases}$$

The continuity of this function (with respect to the relative metric) is obvious and the inverse (projective) limit space X_∞ can be identified with $S(K, \mathbf{x})$ by the limit mappings p_μ (defined in an analogous way from $S(K, \mathbf{x})$ to $\gamma_\mu(K, \mathbf{x})$ as the above functions $p_{\lambda,\mu}(P)$). (Of course, we have the sufficient equality $p_{\mu',\mu''} \circ p_{\mu'} = p_{\mu''}$ for $\mu'' > \mu'$.)

Using Theorems 1 and 3 above, the proof of the first direction of the first statement is an easy consequence. In fact, if for $\lambda > \lambda_0$ the space $\gamma_\lambda(K, \mathbf{x})$ is an $(n-2)$-manifold, then (using Theorem 1) we know that the bonding maps $p_{\mu',\mu''} : \gamma_{\mu''}(K, \mathbf{x}) \longrightarrow \gamma_{\mu'}(K, \mathbf{x})$ are near homeomorphisms. By Theorem 3 we obtain that the limit projections p_λ are also near homeomorphisms. This implies that the space $S(K, \mathbf{x})$ is also an $(n-2)$ manifold.

Conversely, if now $S(K, \mathbf{x})$ is an $(n-2)$-dimensional manifold then it is locally contractible. By Lemma 1 in [11] this also implies that all general parameter spheres are locally contractible manifolds, too. On the other hand, the general parameter spheres can be considered as compact subsets of $R^{(n-1)}$ meaning that they are ANRs (see Theorem 8, p. 117 in [5]).

The proof of both parts of the second statement uses Theorem 2. If first we have a general parameter sphere $\gamma_\lambda(K, \mathbf{x})$ which is an $(n-1)$-dimensional manifold with boundary, then by Theorem 2 it is a cylinder with boundaries homeomorphic to $S^{(n-2)}$. In this case the shadow boundary contains this general parameter sphere showing that all point-inverses with respect to $p_{\mathbf{x}}$ are segments (with non-zero lengths). On the other hand, the sets $\mathrm{bd}(K^+) \cap S(K, \mathbf{x})$ and $\mathrm{bd}(K^+) \cap \gamma_\lambda(K, \mathbf{x})$ coincide, showing that $S(K, \mathbf{x})$ is a cylinder based on an $(n-2)$-manifold homeomorphic to $S^{(n-2)}$. Since $\mathrm{bd}(K^-) \cap S(K, \mathbf{x})$ is homeomorphic to $S^{(n-2)}$ (by central symmetry) and these two sets are disjoint, we get that $S(K, \mathbf{x})$ is homeomorphic to $S^{(n-2)} \times [0, 1]$, as we stated.

Conversely, if $S(K, \mathbf{x})$ is an $(n-1)$-manifold with boundary, then it is (by Theorem 2) homeomorphic to $S^{(n-2)} \times [0, 1]$. Since this cylinder is compact, there is a positive value ε less than or equal to the length of any segment intersected by the shadow boundary in a longitudinal parameter curve. This fact implies that there exists a $\lambda < \infty$ such that $\gamma_\lambda(K, \mathbf{x}) \subset S(K, \mathbf{x})$. The intersection $\gamma_\lambda(K, \mathbf{x}) \cap K^+$ is the same as the intersection $S(K, \mathbf{x}) \cap K^+$, which is one of the two components of the boundary of $S(K, \mathbf{x})$ homeomorphic to $S^{(n-2)}$. For such a λ it is possible to find a trivial point-inverse with respect to the map $p_{\mathbf{x}}$ as we saw it in the example of this section, but for every $\lambda' > \lambda$ the general parameter sphere $\gamma_{\lambda'}(K, \mathbf{x})$ is a cylinder. Using now the fact that it is also the shadow boundary of a centrally symmetric convex body whose positive part is the set K^+, we have proved that it is also a manifold with boundary homeomorphic to $S^{(n-2)} \times [0, 1]$. $\square$

4. On the bisector and its embedding

In this section we investigate the bisector $H_{\mathbf{x}}$ (which is the set being equidistant to the starting point and endpoint of the vector $\mathbf{x}$) using the system $\lambda\gamma_\lambda(K, \mathbf{x})$ of compact metric spaces. Our goal is to prove the following theorem.

Theorem 5. *The set $H_{\mathbf{x}}$ is an $(n-1)$-dimensional manifold if and only if the non-degenerate general parameter spheres $\gamma_\lambda(K, \mathbf{x})$ are manifolds of dimension $(n-2)$.*

Since the neighbourhoods of the point $\frac{1}{2}\mathbf{x}$ (with respect to $H_{\mathbf{x}}$) cannot be homeomorphic to either R^n or a half space, this is the only manifold case for $H_{\mathbf{x}}$.

Proof. First we prove that if the non-degenerate general parameter spheres $\gamma_\lambda(K, \mathbf{x})$ are manifolds of dimension $(n-2)$, then $H_{\mathbf{x}}$ is an $(n-1)$-dimensional manifold. From Theorem 2 we know that the general parameter spheres are homeomorphic copies of $S^{(n-2)}$. Let us construct now the bisector $H_{\mathbf{x}}$ as the disjoint union of the sets $\lambda\gamma_\lambda(K, \mathbf{x})$ for $\lambda \geq \lambda_0$. The set $H_{\mathbf{x},\mu} = \{\lambda\gamma_\lambda(K, \mathbf{x}) | \mu \geq \lambda \geq \lambda_0\}$ is obviously homeomorphic to $\gamma_\lambda(K, \mathbf{x}) \cup K^+$ meaning that it is a homeomorphic copy of the closed $(n-1)$-dimensional ball. Thus $\mathrm{int}(H_{\mathbf{x},\mu})$ is homeomorphic to $R^{(n-1)}$ for each $\mu \geq \lambda_0$. Applying now a theorem of M. Brown (see in [23] or [2]) saying that if a topological space is the union of an increasing sequence of open subsets, which are homeomorphic to $R^{(n-1)}$, resp., then it is also homeomorphic to $R^{(n-1)}$. Thus we get the required result.

Conversely, if $H_{\mathbf{x}}$ is homeomorphic to $R^{(n-1)}$, then the projection $p_{\mathbf{x}} : H_{\mathbf{x}} \longrightarrow R^{(n-1)}$ is a cellular map between two manifolds of the same dimension. Thus it is a near homeomorphism yielding that its restriction to the compact metric space $\lambda\gamma_\lambda(K,\mathbf{x})$ is a near homeomorphism, too. But its image is the boundary of a compact, convex $(n-1)$-dimensional body, so we get at once that it is a homeomorphic copy of $S^{(n-2)}$. Hence the general parameter spheres $\gamma_\lambda(K,\mathbf{x})$ are, for $\lambda > \lambda_0$, manifolds of dimension $(n-2)$, as we stated. □

Corollary. The proof of the first direction of the conjecture follows from Theorems 2, 4 and 5. In fact, if $H_{\mathbf{x}}$ is a topological hyperplane then each of the non-degenerate general parameter spheres is a homeomorphic copy of $S^{(n-2)}$ by Theorem 5 and Theorem 2. So by Theorem 4 we get that the shadow boundary is also a homeomorphic copy of $S^{(n-2)}$, which is the statement of the mentioned direction of our conjecture.

On the other hand we could only prove in Theorem 4 that if $S(K,\mathbf{x})$ is a homeomorphic copy of $S^{(n-2)}$ then the non-degenerate general parameter spheres are ANRs. Thus the manifold property of the bisector does not immediately follow from our theorems. Furthermore, in the manifold case we prove only that the bisector is a homeomorphic copy of $R^{(n-1)}$ which is a weaker property as the required one. Consequently we have to investigate the question of embedding. In fact, all the examples in geometric topology aiming a non-standard (wild) embedding of a set into R^n are based on the observation that the connectivity properties of the complement (with respect to R^n) of the set can change if we apply a homeomorphism to it. In our case, for example, the complement of the bisector (which is now a homeomorphic copy of $R^{(n-1)}$) is the disjoint union of homeomorphic copies of R^n. It gives the chance that a homeomorphism on R^n to itself exists sending the bisector to a hyperplane. It is a well-known fact that a manifold homeomorphic to $S^{(n-1)}$ in S^n is unknotted if and only if the closures of the components of its complement are homeomorphic copies of the closed n-cell B^n. This implies that in the manifold case the embedding of the shadow boundary and the general parameter spheres are always standard. From this the existence of a homeomorphism of the boundary of K into itself follows which sends these sets into a standard $(n-1)$-dimensional sphere of $\mathrm{bd}(K)$. Considering bisectors, we have to carry out the proof in a way which is a bit more sophisticated. Let φ be a homeomorphism sending $H_{\mathbf{x}}$ into $R^{(n-1)}$ (which is now a hyperplane H of R^n). We consider the compactification of the embedding space by an element denoted by ∞. Extend first the map φ to the compact space $H_{\mathbf{x}} \cup \{\infty\}$ by the condition $\varphi(\infty) = \infty$. Of course, this extended map gives a homeomorphism between the sets $H_{\mathbf{x}} \cup \{\infty\}$ and $H \cup \{\infty\}$. Since the closure of the components of the complement of $H_{\mathbf{x}} \cup \{\infty\}$ in $R^n \cup \{\infty\}$ are closed n-cells, the homeomorphism φ can be extended to a homeomorphism $\Phi : R^n \cup \{\infty\} \longrightarrow R^n \cup \{\infty\}$. Since by our method we have that $\Phi(\infty) = \varphi(\infty) = \infty$ and $\Phi(H_{\mathbf{x}}) = H$, we get that the bisector is a topological hyperplane as we stated. Thus the following statement has been proved:

Theorem 6. *In the manifold case the embeddings of $H_{\mathbf{x}}$, $S(K, \mathbf{x})$ and $\gamma_\lambda(K, \mathbf{x})$ are standard, respectively. This means that if the bisector is homeomorphic to $R^{(n-1)}$, then it is a topological hyperplane.*

Finally we mention the very recent contribution [17] where it is shown that already radial projections of bisectors in Minkowski spaces have sufficiently interesting geometric properties yielding many new results, including characterizations of the Euclidean case.

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