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Sharply 2-transitive Sets of Permutations and Groups of Affine Projectivities

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Abstract. Using new results on sharply transitive subsets, we determine the groups of projectivities of finite affine planes, apart from (unknown) planes of order 23 or 24.

The group of all projectivities of a geometry $\mathcal{G}$ is a measure for the complexity of $\mathcal{G}$: this group tends to be rather large if $\mathcal{G}$ is far from being a classical geometry. See [29] for more information on the role of projectivities in geometry.

In Section 1 we consider almost simple finite permutation groups which contain a sharply 2-transitive subset. The results of this section yield Theorem 2.3 on affine projectivities of finite affine planes that are not translation planes.

1. Sharply transitive subsets

Let Ω be a set. A set S of permutations of Ω is said to be *sharply transitive* if for all $\alpha, \beta \in \Omega$ there is exactly one $s \in S$ with $\alpha^s = \beta$. If t is any permutation of Ω, then S is sharply transitive if and only if this holds for tS. Thus, if we study the (non-)existence of sharply transitive sets S of permutations, we may assume that $1 \in S$. Then all elements in $S \setminus \{1\}$ are fixed-point-free.

Let $\Omega^{(2)}$ be the set of pairs of distinct elements from Ω. Then S is said to be *sharply* 2*-transitive* on Ω if S is sharply transitive on $\Omega^{(2)}$.

If this holds, then each stabilizer $S_\omega = \{ s \in S \mid \omega^s = \omega \}$ is sharply transitive on $\Omega \setminus \{\omega\}$. Some 2-transitive groups G do not contain any sharply 2-transitive subset for the simple reason that a point-stabilizer G_ω does not contain a subset which is sharply transitive on $\Omega \setminus \{\omega\}$. Examples for this are the symplectic groups appearing in Theorem 1.5, the Conway group $G = \mathrm{Co}_3$ (see 1.7), the Higman-Sims group $G = \mathrm{HS}$ and the Mathieu group $G = M_{11}$ of degree 11 (see 1.9, cases (5) and (7) of the proof) and the groups in 1.10.

The Bruck-Ryser theorem says that the symmetric group S_n contains no sharply 2-transitive subset provided that $n \equiv 1, 2 \pmod 4$ and n is not a sum of two squares; see [8] 3.2.13 and 3.2.6.

In order to obtain further non-existence results for sharply 2-transitive subsets, we first extend Lemma 1 of [27] to arbitrary characteristic.

Proposition 1.1. *Let G be a permutation group on the finite set Ω, let $F[\Omega]$ be the permutation module for the group algebra $F[G]$ over a field F, and let L be the one-dimensional submodule spanned by $\sum_{\omega\in\Omega} \omega$. Suppose that $S \subseteq G$ is sharply transitive on Ω. Then $\sigma := \sum_{s\in S} s \in F[G]$ acts as zero on each composition factor of $F[\Omega]/L$.*

Proof. Let $\alpha \in \Omega$. By sharp transitivity of S we obtain $\alpha^\sigma = \sum_{s\in S} \alpha^s = \sum_{\omega\in\Omega} \omega \in L$, so σ is zero on $F[\Omega]/L$. Then σ is zero on each composition factor of $F[\Omega]/L$. □

Applying the trace map we obtain the following result.

Corollary 1.2. *Assume in addition that $1 \in S$. Let $C_1, C_2, \dots, C_h$ be the conjugacy classes of fixed-point-free elements of G. Set $a_i = |C_i \cap S|$, and let $\chi(C_j)$ be the trace of $g \in C_j$ in its action on a fixed composition factor of $F[\Omega]/L$. Then*

$$\sum_{j=1}^{h} a_j \chi(C_j) = -\chi(1).$$

In contrast to the situation in characteristic 0, the principal module (i.e. the trivial one-dimensional module) may occur more than once as a composition factor in the permutation module of a transitive permutation group in positive characteristic. In fact, the following result allows to prove the non-existence of sharply transitive subsets in many cases.

Proposition 1.3. *Let G be a permutation group on the finite set Ω, let $S \subseteq G$ be sharply transitive on Ω, and assume that the characteristic of the field F does not divide $|\Omega|$. Then the principal module appears at most once among the composition factors of the $F[G]$-module $F[\Omega]$.*

Proof. With notation as above, we have $a_1 + a_2 + \cdots + a_h = |S| - 1 = |\Omega| - 1$. Suppose that the principal module appears twice in $F[\Omega]$. Then it appears at least once in $F[\Omega]/L$, and Corollary 1.2 yields $a_1 + a_2 + \cdots + a_h = -1$. Thus $|\Omega|$ is zero in F, contrary to our assumption on F. □

Our first application uses Corollary 1.2 in characteristic 0.

Theorem 1.4. *The symplectic group $G = \mathrm{Sp}(6,2)$, in its doubly transitive action of degree 28 or 36, does not contain a sharply 2-transitive subset S.*

Proof. We may assume that $1 \in S$. First we consider the action of G of degree 28, say on Ω. Note that each element from $S \setminus \{1\}$ has at most one fixed point. There are 14 conjugacy classes of such elements in G. If $\pi = 1 + \chi$ is the permutation character of G on Ω, then $\pi(\pi - 1) = \chi^2 + \chi$ is the permutation character of G on $\Omega^{(2)}$. In order check whether an irreducible character ψ of G appears in $\chi^2 + \chi$, one has to compute the scalar product $(\chi^2 + \chi, \psi)$; this can be done using the character table of G (see [7]). We find that three irreducible characters of degrees 27, 120 and 210 appear in $\chi^2 + \chi$.

Then Corollary 1.2 yields the following system of linear equations:

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 \\ -6 & 0 & -2 & -2 & -2 & 1 & 0 & 1 & 0 & 0 \\ 3 & -2 & -1 & 2 & 2 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \cdot \mathbf{a} = \begin{pmatrix} -27 \\ -120 \\ -210 \end{pmatrix},$$

where $\mathbf{a}$ is the transpose of the row vector $(a_1, a_3, a_4, a_5, a_6, a_8, a_9, a_{12}, a_{13}, a_{14})$; we have omitted those a_i where $\psi(C_i) = 0$ for all three irreducible characters ψ used here. Left multiplication with $(\,-2\ -1\ 1\,)$ gives

$$9a_1 + a_4 + 4a_5 + 4a_6 + a_8 + 2a_9 + a_{13} + 2a_{14} = -36,$$

which is a contradiction since the a_i are non-negative integers.

Analogously, the action of G of degree 36 yields

$$\begin{pmatrix} -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ 3 & -2 & -1 & -1 & 2 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathbf{a} = \begin{pmatrix} -35 \\ -210 \end{pmatrix},$$

where $\mathbf{a}$ is the transpose of the row vector $(a_1, a_2, a_4, a_5, a_6, a_9, a_{10}, a_{12}, a_{13})$. Left multiplication with $(\,-2\ 1\,)$ gives

$$5a_1 + a_4 + a_5 + 2a_6 + 2a_9 + 2a_{10} + a_{12} + 3a_{13} = -140,$$

which is again a contradiction. □

Theorem 1.5. *Let $G = \mathrm{Sp}(2d,2)$, $d \geq 4$, be the symplectic group in one of its two doubly transitive actions on sets $\Omega_\pm$ with degrees $|\Omega_\pm| = 2^{2d-1} \pm 2^{d-1}$. Then the stabilizer G_ω of $\omega \in \Omega_\pm$ has no subset which is sharply transitive on $\Omega_\pm \setminus \{\omega\}$. Therefore, G has no sharply 2-transitive subset.*

Proof. Suppose that G_ω contains a sharply transitive subset. Since $|\Omega_\pm| - 1 = 2^{2d-1} \pm 2^{d-1} - 1$ is odd, the permutation module $\mathbb{F}_2[\Omega_\pm \setminus \{\omega\}]$ for the action of G_ω on $\Omega_\pm \setminus \{\omega\}$ contains the principal module at most once as a composition factor; see Proposition 1.3. Therefore, the permutation module $\mathbb{F}_2[\Omega_\pm]$ for the action of G on $\Omega_\pm$ contains the principal module at most twice, contrary to the following lemma. □

Lemma 1.6. *Let $G = \mathrm{Sp}(2d, 2)$, $d \geq 4$, be the symplectic group in one of its two doubly transitive actions on sets $\Omega_\pm$ with degrees $|\Omega_\pm| = 2^{2d-1} \pm 2^{d-1}$. Then the principal module appears at least three times in the permutation module $\mathbb{F}_2[\Omega_\pm]$.*

Proof. The multisets of the composition factors of $\mathbb{F}_2[\Omega_+]$ and $\mathbb{F}_2[\Omega_-]$ differ only by an irreducible module of dimension 2^d; this is one of the main results in [30]. Therefore it suffices to show that the principal module appears at least six times in $\mathbb{F}_2[\Omega_+ \cup \Omega_-]$.

Let ϕ be a non-degenerate symplectic form on the vector space $V = \mathbb{F}_2^{2d}$. Let Ω be the set of quadratic forms $\theta : V \to \mathbb{F}_2$ which polarize to ϕ, that is $\phi(u, v) = \theta(u + v) + \theta(u) + \theta(v)$ for all $u, v \in V$. Clearly $G = \mathrm{Sp}(2d, 2)$ acts on Ω via $\theta^g(v) = \theta(v^{g^{-1}})$. This action has two orbits, which can be identified with Ω_+ and Ω_- from above, see e.g. [26], [30], or [9] Chapter 7.7. Fix any $\theta_0 \in \Omega$. Then Ω consists of the elements θ_a, $a \in V$, where $\theta_a(v) = \theta_0(v) + \phi(v, a)$. (For if $\theta \in \Omega$, then $\theta - \theta_0$ is a linear form on V, so $(\theta - \theta_0)(v) = \phi(v, a)$ for some $a \in V$.)

Let $f(g)$ be the number of fixed points of $g \in G$ on $\Omega_+ \cup \Omega_-$. From $\theta_a^g(v) = \theta_a(v^{g^{-1}}) = \theta_0(v^{g^{-1}}) + \phi(v^{g^{-1}}, a) = \theta_0(v^{g^{-1}}) + \phi(v, a^g)$ we get that θ_a is fixed under g if and only if $\theta_0(v^{g^{-1}}) - \theta_0(v) = \phi(v, a - a^g)$ for all $v \in V$. Choose b with $\theta_0(v^{g^{-1}}) = \theta_0^g(v) = \theta_0(v) + \phi(v, b)$, so $\theta_a^g = \theta_a$ if and only if $\phi(v, b - a - a^g) = 0$ for all v, which is equivalent to $b = a - a^g$. Thus either $f(g) = 0$, or $f(g) = |\{v \in V | v^g = v\}|$. We obtain $f(g) \leq \bar{f}(g)$, where $\bar{f}(g)$ is the number of fixed points of g on V. On the other hand, G is transitive on $V \setminus \{0\}$ (e.g. by Witt's Lemma), so $\sum_{g\in G} f(g) = 2|G| = \sum_{g\in G} \bar{f}(g)$, hence $f(g) = \bar{f}(g)$ for all $g \in G$. Also, g has as many fixed elements in V as in the dual space $V^\star$.

Therefore $\mathbb{F}_2[\Omega_+ \cup \Omega_-]$ and $\mathbb{F}_2[V^\star]$ are modulo 2 reductions of integral representations which are equivalent as complex representations. By a result of Brauer, $\mathbb{F}_2[\Omega_+ \cup \Omega_-]$ and $\mathbb{F}_2[V^\star]$ have the same multisets of composition factors, see [4] (14B).

We identify $\mathbb{F}_2[V^\star]$ with the G-module $V^{\star\mathbb{F}_2}$ of maps from $V^\star$ to $\mathbb{F}_2$, where $g \in G$ acts on $V^{\star\mathbb{F}_2}$ via $f^g(v) = f(v^{g^{-1}})$, for $f \in V^{\star\mathbb{F}_2}$ and $v \in V^\star$. Let $x_1, x_2, \ldots, x_{2d}$ be a basis of $(V^\star)^\star$, and $X_1, X_2, \ldots, X_{2d}$ be variables over $\mathbb{F}_2$. Each element in $V^{\star\mathbb{F}_2}$ is a polynomial in the x_i, so we get a G-equivariant surjection $\psi : \mathbb{F}_2[X_1, X_2, \ldots, X_{2d}] \to V^{\star\mathbb{F}_2}$ of G-modules. The kernel is generated by $X_i^2 - X_i$, $1 \leq i \leq 2d$. Let M_i be the G-submodule of $\mathbb{F}_2[X_1, X_2, \ldots, X_{2d}]$ of polynomials of degree $\leq i$. Thus $\psi(M_i)$, $i = 0, 1, \ldots, 2d$, is a filtration of $V^{\star\mathbb{F}_2}$. Set $\Lambda = \mathbb{F}_2[X_1, X_2, \ldots, X_{2d}]/(X_i^2 \mid 1 \leq i \leq 2d)$, and let $\psi' : \mathbb{F}_2[X_1, X_2, \ldots, X_{2d}] \to \Lambda$ be the natural map. It is easy to see that $\psi(M_{i+1})/\psi(M_i)$ and $\psi'(M_{i+1})/\psi'(M_i)$ are isomorphic G-modules (see e.g. [1] Lemma 2.2). Thus $\mathbb{F}_2[V^\star]$ has the same multiset of composition factors as Λ. Note that Λ is the exterior algebra of $(V^\star)^\star = V$. Thus $\mathbb{F}_2[\Omega_+ \cup \Omega_-]$ has the same multiset of composition factors as the exterior algebra $\bigwedge V = \bigoplus_{i=0}^{2d} \bigwedge^i V$.

We claim that the principal module is a submodule of $\bigwedge^{2k} V$ for $0 \leq k \leq d$; this proves the lemma for $d \geq 5$.

Let $b_1, b_2, \ldots, b_d, b'_1, b'_2, \ldots, b'_d$ be a symplectic basis of V; thus $\phi(b_i, b'_i) = 1 = \phi(b'_i, b_i)$, and $\phi(b_i, b_j) = 0 = \phi(b_i, b'_j)$ in all other cases. For $I, J \subseteq D := \{1, 2, \ldots, d\}$ we write $b_{I,J}$ for the product $\prod_{i\in I} b_i \prod_{j\in J} b'_j$ in $\bigwedge V$ in the natural order. Actually, the order does not matter, as $\bigwedge V$ is commutative in characteristic 2. For $1 \le k \le d$ we set

$$w_k = \textstyle\sum_{I \in \binom{D}{k}} b_{I,I} \in \bigwedge^{2k} V,$$

where $\binom{X}{k} := \{Y \mid y \subseteq X, |Y| = k\}$. Then $w_k \neq 0$, since the elements $b_{I,J}$ with $|I| + |J| = 2k$ form a basis of $\bigwedge^{2k} V$. We are going to show that w_k is fixed by G, so w_k spans a principal submodule of $\bigwedge^{2k} V$.

The group G is generated by the symplectic transvections $t(a)$ with $a \in V$, where $v^{t(a)} = v + \phi(v,a)a$; see e.g. [32] Theorem 8.5. Thus it suffices to check that $w_k^{t(a)} = w_k$ for all $a \in V$. Write $a = \sum_{i \in D}(\alpha_i b_i + \alpha_i' b_i')$ with field elements α_i, α_i'. For $I \subseteq D$ we have

$$\begin{aligned} {b_{I,I}}^{t(a)} &= \prod_{i \in I}(b_i + \phi(b_i,a)a) \prod_{i \in I}(b_i' + \phi(b_i',a)a) = \prod_{i \in I}(b_i + \alpha_i' a) \prod_{i \in I}(b_i' + \alpha_i a) \\ &= b_{I,I} + \sum_{i \in I} \alpha_i' a b_{I \setminus \{i\}, I} + \sum_{i \in I} \alpha_i a b_{I, I \setminus \{i\}} \\ &= b_{I,I} + \sum_{i \in I} (\alpha_i b_i + \alpha_i' b_i') a b_{I \setminus \{i\}, I \setminus \{i\}} \end{aligned}$$

and therefore

$$\begin{aligned} w_k^{t(a)} - w_k &= \sum_{I \in \binom{D}{k}} \sum_{i \in I} (\alpha_i b_i + \alpha_i' b_i') a b_{I \setminus \{i\}, I \setminus \{i\}} \\ &= \sum_{i \in D} (\alpha_i b_i + \alpha_i' b_i') a \sum_{J \in \binom{D \setminus \{i\}}{k-1}} b_{J,J}. \end{aligned}$$

In the last sum, we may as well admit all $J \in \binom{D}{k-1}$, since for $i \in J \subseteq D$ we have $b_i b_{J,J} = 0 = b_i' b_{J,J}$. Thus

$$w_k^{t(a)} - w_k = \sum_{i \in D} (\alpha_i b_i + \alpha_i' b_i') a \sum_{J \in \binom{D}{k-1}} b_{J,J} = aa \sum_{J \in \binom{D}{k-1}} b_{J,J} = 0.$$

This proves the lemma for $d \ge 5$. For $d = 4$, the lemma follows from the above considerations and the fact that the principal module appears twice as a composition factor of $\bigwedge^2 V$. This can be seen as follows: Let W be the subspace of codimension 1 of $\bigwedge^2 V$ consisting of the elements $\sum_{|I|+|J|=2} \alpha_{I,J} b_{I,J}$ with $\sum_{|I|+|J|=2} \alpha_{I,J} = 0$. An easy calculation shows that W is invariant under all $t(a)$ and therefore also under G. Now $w_1 \in W$, so $(\bigwedge^2 V)/W$ is another principal composition factor. $\square$

We also need the following application of Proposition 1.3.

Theorem 1.7. *Let G be the Conway group* Co_3 *in its doubly transitive action of degree* 276. *Then the stabilizer G_ω of degree* 275 *has no sharply transitive subset. Therefore,* Co_3 *has no sharply* 2*-transitive subset.*

Proof. The stabilizer G_ω contains the McLaughlin group McL as a subgroup of index 2; see [7]. Let Ω' be the set of size 275 on which $G_\omega = \mathrm{McL}.2$ acts, and assume that G_ω has a sharply transitive subset. Then by Proposition 1.3, the principal module appears at most once among the composition factors of the permutation module $\mathbb{F}_3[\Omega']$ for the action of G_ω on Ω'.

In the following we use the atlas [7], together with the atlas of Brauer characters [18], in order to obtain information about the permutation module $\mathbb{F}_3[\Omega']$. The degrees below 275 of the irreducible Brauer characters of McL.2 modulo 3 are 1, 21, 104, and 210.

Let a, b, c, and d be the numbers of composition factors of $\mathbb{F}_3[\Omega']$ with degree 1, 21, 104 and 210, respectively. Thus $275 = a + 21b + 104c + 210d$. Note that $a \geq 1$ as $L := \mathbb{F}_3 \sum_{\omega\in\Omega'} \omega \subset \mathbb{F}_3[\Omega']$. In the following all congruences are modulo 3. We have $2 \equiv a + 2c$, hence $a \equiv c + 2$.

We will work with elements in the conjugacy classes $2A$, $4A$, and $8C$ of G_ω. Let $\pi(C)$ be the number of fixed points of an element in C. The atlas [7] gives $\pi(2A) = 35 \equiv 2$, $\pi(4A) = 7 \equiv 1$, and $\pi(8C) = 5 \equiv 2$.

First suppose that $d \geq 1$. Then $d = 1$ and $c = 0$, so $a \equiv 2$. The Brauer character values of $2A$ for the Brauer characters of degrees 1, 21, and 210 are 1, 5, and 2, respectively. Thus $2 \equiv \pi(2A) \equiv a + 5b + 2d \equiv 2 + 2b + 2$, so $b \equiv 2$. Analogously, the class $4A$ yields $1 \equiv a + b - 2d \equiv 2$, which is a contradiction.

Thus $d = 0$. Again using the element $2A$ and $a \equiv c+2$ we get $2 \equiv a+2b+8c \equiv a + 2b + 8(a + 2) \equiv 2 + 2b$, so $b \equiv 0$. Using $4A$ we obtain $1 \equiv a + b$, hence $a \equiv 1$.

Degree	$2A$	$4A$	$8C$
1	1	1	1
1	1	1	-1
21	5	1	3
21	5	1	-3
104	8	0	0
210	2	-2	-4
210	2	-2	4

Table 1. Part of the Brauer character table of McL.2 mod 3

Recall that the principal module appears only once in $\mathbb{F}_3[\Omega']$. Let α be the multiplicity of the one-dimensional non-principal module in $\mathbb{F}_3[\Omega']$. Then $a = 1 + \alpha$, hence $\alpha \equiv 0$. Each Brauer character of degree 21 and 104 is $\equiv 0$ on $8C$, and -1 on $8C$ for the non-principal one-dimensional character. Thus we obtain the contradiction $2 \equiv \pi(8C) = 1 - \alpha \equiv 1$. □

Theorem 1.8. *The Mathieu group $G = M_{12}$ of degree 12 contains no sharply 2-transitive subset.*

Proof. The sharply transitive subsets in M_{11} of degree 11 that contain the identity are just the Sylow subgroups of order 11. This can be shown by an exhaustive computer search; M_{11} is small enough such that a straightforward program in GAP or Magma quickly yields the result.

Let $S \subset M_{12}$ be sharply 2-transitive with $1 \in S$, and let $\alpha, \beta \in \Omega$ be distinct. Then $S_\alpha = C$ and $S_\beta = C^g$ for some $g \in M_{12}$, where C is a subgroup of order 11. However, another short program shows that $C^g C$ contains non-trivial elements with at least two fixed points, which is a contradiction. □

Theorem 1.9. *If an almost simple permutation group $G \leq S_n$ contains a sharply 2-transitive subset, then $G = A_n$ or $G = S_n$, or $n = 23$ and $G = M_{23}$, or $n = 24$ and $G = M_{24}$.*

This result is essential for proving Theorem 2.3 on affine planes. It would be desirable to exclude the two Mathieu groups in 1.9.

Proof. The doubly transitive group G is contained in the automorphism group of some non-abelian simple group. The following list covers all possibilities for G; see [6] 7.4 or [5] p. 8 (here we rely on the classification of finite simple groups).

(1) $G = A_n$ or $G = S_n$.

(2) $G \leq \mathrm{P\Gamma L}(d, q)$ of degree $n = (q^d - 1)/(q - 1)$ with $d \geq 2$; this includes the case where $G = A_7$ of degree $n = 15$. By O'Nan [27], such a permutation group $\mathrm{P\Gamma L}(d, q)$ contains a sharply 2-transitive subset only if $d = 2$ and $q \leq 4$ (compare also [22] for $q \geq 5$). For $q \leq 3$ the group $\mathrm{P\Gamma L}(2, q)$ is solvable, and $\mathrm{PSL}(2, 4)$ of degree 5 coincides with the alternating group A_5.

(3) G is contained in the automorphism group of $\mathrm{PSU}(3, q)$, $\mathrm{Sz}(q)$ or ${}^2\mathrm{G}_2(q)$ in its natural action (always $q > 2$). By Lorimer [23], [24], [21] and [25], p. 433, these groups do not contain any sharply 2-transitive subset; we remark that the information from (2) makes it possible to apply Lorimer's argument for ${}^2\mathrm{G}_2(q)$ in [21] (i.e., the consideration of the global stabilizer of a block in the Ree unital) to the automorphism group of ${}^2\mathrm{G}_2(q)$. Note that (3) includes the group $G = \mathrm{P\Gamma L}(2, 8) \cong {}^2\mathrm{G}_2(3)$ of degree 28.

(4) G is contained in the symplectic group $\mathrm{Sp}(2d, 2)$ with degrees $n = 2^{2d-1} \pm 2^{d-1}$, $d \geq 3$; this includes the subgroup $G = \mathrm{P\Gamma L}(2, 8)$ of $\mathrm{Sp}(6, 2)$ of degree 28 (see [7]). This case is ruled out by Theorems 1.4 and 1.5.

(5) G is the Higman-Sims group HS in one of its two doubly transitive actions of degree $n = 176$. Then the stabilizer $G_\alpha \cong \mathrm{P\Sigma U}(3, 5)$ is the automorphism group of the Hofmann-Singleton graph, acting on the 175 edges. It is easy to show that G_α has no sharply transitive subset, using O'Nan's technique [27] of contradicting subgroups; see [14] p. 271. Alternatively, one can use 1.10 below.

(6) G is the Conway group Co_3 of degree $n = 276$. This case is ruled out by Theorem 1.7.

(7) G is contained in the Mathieu group M_{11} with degree $n = 11$; this includes the group $G = \mathrm{PSL}(2, 11)$ of degree 11. Then the stabilizer G_α is a subgroup of $M_{10} = \mathrm{PSL}(2, 9).2$, which has index 2 in $\mathrm{P\Gamma L}(2, 9)$. However, M_{10} of degree 10

has no sharply transitive subset. This can be easily verified for instance using GAP or Magma; the paper [10] mentions an early verification.

Alternatively, one can show directly that M_{11} of degree 11 has no sharply 2-transitive subset, using the information from Corollary 1.2 for $F = \mathbb{F}_3$ and $F = \mathbb{F}_{11}$ together: The fixed-point-free elements from M_{11} have order 6 or 11. Let $S \subset M_{11}$ be sharply 2-transitive on 11 points with $1 \in S$, and let x be the number of elements in S of order 6. Clearly $x \leq 10$. Corollary 1.2 for $F = \mathbb{F}_{11}$ yields $x \equiv 0 \pmod{11}$, together with $0 \leq x \leq 10$ this gives $x = 0$. However, Corollary 1.2 for $F = \mathbb{F}_3$ shows that $x \equiv 1 \pmod 3$, a contradiction.

(8) G is contained in the Mathieu group M_{12} with degree $n = 12$; this includes the group $G = M_{11}$ of degree 12. This case is ruled out by Theorem 1.8.

(9) G contains one of the large Mathieu groups M_n with degree $n = 22, 23, 24$ as a normal subgroup. By the Bruck-Ryser theorem, the symmetric group S_{22} has no sharply 2-transitive subset; see [8] 3.2.13 and 3.2.6. Hence $n \neq 22$. The groups M_{23} and M_{24} have no outer automorphisms (see [7]), thus $G = M_n$ with $n \in \{23, 24\}$. □

If G is doubly transitive, then Corollary 1.2 yields no restriction at all if F has characteristic 0. Similarly, if G is 4-transitive and the characteristic of F is 0, then 1.2 yields only the obvious relation $\sum b_i a_i = |\Omega|(|\Omega|-1)/2$, where b_i denotes the number of transpositions appearing in the cycle decomposition of $g \in C_i$. Also, it is not hard to see that O'Nan's technique [27] of contradicting subgroups cannot work for sharply transitive sets in doubly transitive groups, and neither for sharply 2-transitive sets in 4-transitive groups.

However, the picture changes if we consider fields F of positive characteristic. For example, one can show that a sharply transitive subset S of the Mathieu group M_{22} of degree 22 with $1 \in S$ has to contain an odd number of elements of order 8 (by considering the 10-dimensional 2-modular representation arising from the binary Golay code).

The non-existence of sharply 2-transitive subsets of $\mathrm{P\Gamma L}(m, q)$ for $m \geq 3$ and of the Higman-Sims group HS can be obtained from the fact that these groups are automorphism groups of symmetric block designs. Also, the 2-transitive affine group $\mathrm{ASp}(2m, 2)$ does not contain a sharply 2-transitive set, because it is the automorphism group of a symmetric block design, see e.g. [20], Chapter 3.2. Indeed, the following holds.

Theorem 1.10. *Let G be an automorphism group of a symmetric block design. Then the stabilizer in G of a point does not contain a subset which is sharply transitive on the remaining points.*

In particular, G does not contain a subset which is sharply 2-transitive on the points of the design.

Proof. Let P be the set of points of the design, $n := |P|$, and let $H = G_p$ be the stabilizer of a point $p \in P$. Suppose that H contains a set which is sharply transitive on $P \setminus \{p\}$. Then H has the orbits $\{p\}$ and $P \setminus \{p\}$ on P. Let $1 + \chi$ be the permutation character of H on P, where χ is the permutation character of H on $P \setminus \{p\}$. It is well known that $1+\chi$ is also the permutation character of the action of H on the set of blocks; see e.g. [20], Chapter 3, Theorem 3.1 (or problem 1), or [8] 2.3.12. In particular, H has two orbits Λ_1 and Λ_2 on the blocks. These orbits therefore consist of those blocks which are incident with p and those which are not, respectively. Let m and $n-m$ be the lengths of these orbits, and let ψ_i be the permutation character for the action of H on Λ_i. Then $1+\chi = \psi_1 + \psi_2$. So each irreducible constituent of each ψ_i appears in χ. The stabilizer in H of an element of Λ_i cannot be a contradicting subgroup in the sense of [27], so both m and $n-m$ divide $n-1$. But m and $n-m$ are proper divisors of $n-1$, for otherwise p were incident with only one block or with all blocks but one. Hence $n = m + (n-m) \leq \frac{n-1}{2} + \frac{n-1}{2} = n-1$, which is a contradiction. □

2. Projective and affine planes

Let $\mathcal{P}$ be a projective plane and let L, M be lines of $\mathcal{P}$ (considered as sets of points). Any point p not on L or M gives rise to the bijection $[L,p,M] : L \to M : x \mapsto (xp) \cap M$. Projectivities are concatenations of bijections of this type. The projectivities of L onto itself form a triply transitive group of permutations of L; choosing another line in $\mathcal{P}$ leads to an isomorphic permutation group (compare [8] p. 160). We denote this permutation group by $\Pi(\mathcal{P})$ and call it the *group of projectivities* of $\mathcal{P}$.

Lemma 2.1. *Let L be a line of a projective plane $\mathcal{P}$, and let $u \neq v$ be points with $u, v \notin L$. Then $S := \{ [L,u,M][M,v,L] \mid M$ is a line with $u, v \notin M \}$ is a set of projectivities which fix the point $\infty := L \cap uv$, and S is sharply 2-transitive on $L \setminus \{\infty\}$.*

This well-known lemma is a direct consequence of the axioms for projective planes. (In fact, if $\mathcal{P}$ is coordinatized by a ternary operation $x \cdot a \circ b$ with respect to a quadrangle o, u, v, e such that $L = oe$, then S consists essentially of the bijections $x \mapsto x \cdot a \circ b$ with $a \neq 0$; see [8] p. 127, p. 140 or [13] p. 438).

If $\mathcal{P}$ is the desarguesian projective plane coordinatized by a skew field F, then $\Pi(\mathcal{P}) = \mathrm{PGL}(2,F)$ in its natural action on the projective line $F \cup \{\infty\}$. The problem to determine the groups of projectivities of non-desarguesian planes has been addressed for the first time by Barlotti [2], [3]; he showed that $\Pi(\mathcal{P}) = S_{10}$ if $\mathcal{P}$ is the Hughes plane of order 9 or the nearfield plane of order 9, and that $\Pi(\mathcal{P}) = A_{17}$ for the Hall plane $\mathcal{P}$ of order 16.

The proofs of 2.2 and 2.3 below depend on the classification of all finite almost simple permutation groups that are triply (or doubly) transitive, hence on the classification of all finite simple groups. This appears to be unavoidable, because an explicit classification of all finite non-classical planes is neither available nor expected.

Theorem 2.2. *Let $\mathcal{P}$ be a finite non-desarguesian projective plane of order n. Then the group $\Pi(\mathcal{P})$ of all projectivities of $\mathcal{P}$ is the alternating group A_{n+1} or the symmetric group S_{n+1} of degree $n+1$, or $n = 23$ and $\Pi(\mathcal{P}) = M_{24}$.*

This result was proved in [14]. It is also a consequence of 2.3 below, because a finite non-desarguesian projective plane has at most one translation line, compare [17], Theorem 6.18, p. 151 (note that M_{24} is maximal in A_{24} and has no transitive extension).

If $n = 23$ and $\Pi(\mathcal{P}) = M_{24}$, then the Mathieu group M_{22} of degree 22 contains a sharply transitive subset. By excluding this possibility, one would prove the conjecture on the groups $\Pi(\mathcal{P})$ in [8], p. 160.

In the situation of Theorem 2.2, one might expect that $\Pi(\mathcal{P}) = A_{n+1}$ if n is even, and $\Pi(\mathcal{P}) = S_{n+1}$ if n is odd. This is true for André planes ([16] and [12]), for planes over commutative semifields ([19] and[15]), and for semifield planes and nearfield planes of odd order ([15], [12]). However, Kilmer [19] describes semifield planes $\mathcal{P}$ with orders $n = 16$, 32 and 64 such that $\Pi(\mathcal{P}) = S_{n+1}$. We do not know if there exists a finite projective plane $\mathcal{P}$ of odd order n such that $\Pi(\mathcal{P}) = A_{n+1}$.

Let $\mathcal{A}$ be an *affine plane* and let L, M be lines of $\mathcal{A}$ (considered as sets of points). Any point p at infinity defines a parallel projection $L \to M$ in the direction of p. The affine projectivities are the concatenations of bijections of this type. The affine projectivities of L onto itself form a doubly transitive group of permutations of L; choosing another line in $\mathcal{A}$ leads to an isomorphic permutation group. We denote this permutation group by $\Pi^{\mathrm{aff}}(\mathcal{A})$ and call it the *group of affine projectivities* of $\mathcal{A}$. (This is the group Π^W in [8] p. 161.)

In the projective closure $\mathcal{P}$ of $\mathcal{A}$, the parallel projection $L \to M$ in the direction of p is just the projectivity $[L, p, M]$ considered above (if we ignore the points at infinity of L and M). This observation implies that $\Pi^{\mathrm{aff}}(\mathcal{A})$ is a subgroup of the stabilizer $\Pi(\mathcal{P})_\infty$, where ∞ is a point at infinity.

If $\mathcal{A}$ is the desarguesian affine plane coordinatized by a skew field F, then $\Pi^{\mathrm{aff}}(\mathcal{A}) = \mathrm{AGL}(1, F)$ in its natural action on F. If $\mathcal{A}$ is a finite translation plane of order q^d with kernel $\mathbb{F}_q$, then $\mathrm{ASL}(d, q) \leq \Pi^{\mathrm{aff}}(\mathcal{A}) \leq \mathrm{AGL}(d, q)$ as permutation groups; see [11] (often one has $\Pi^{\mathrm{aff}}(\mathcal{A}) = \mathrm{AGL}(d, q)$, see [12], [15]).

Now we consider finite affine planes that are not translation planes.

Theorem 2.3. *Let $\mathcal{A}$ be a finite affine plane of order n that is not a translation plane. Then the group $\Pi^{\mathrm{aff}}(\mathcal{A})$ of all affine projectivities of $\mathcal{A}$ is the alternating group A_n or the symmetric group S_n of degree n, or $n = 23$ and $\Pi^{\mathrm{aff}}(\mathcal{A}) = M_{23}$, or $n = 24$ and $\Pi^{\mathrm{aff}}(\mathcal{A}) = M_{24}$.*

Proof. The group $G := \Pi^{\mathrm{aff}}(\mathcal{A}) \leq S_n$ has no sharply transitive normal subgroup; otherwise $\mathcal{A}$ would be a translation plane by Schleiermacher [31] Satz 2 (compare also [28] Proposition 7). Hence by a result of Burnside, the doubly transitive group G is almost simple; see [6] p. 110 or [5] 5.2. Moreover, G contains a sharply 2-transitive subset by Lemma 2.1 (just take u and v to be points at infinity). Therefore the assertion is a consequence of Theorem 1.9. □

Concerning the exceptions in Theorem 2.3, we remark that presently no non-desarguesian plane of order 23 or 24 is known. The exception $n = 23$ and $\Pi^{\mathrm{aff}}(\mathcal{A}) = M_{23}$ (and the exception in 2.2) would be ruled out by showing that

M_{22} of degree 22 has no sharply transitive subset. Using algorithms of Ulrich Dempwolff and his team, it might be feasible to prove this computationally. The exception $n = 24$ and $\Pi^{\mathrm{aff}}(\mathcal{A}) = M_{24}$ would disappear if one could prove that M_{24} contains no sharply 2-transitive subset (here, a direct computational approach seems to be hopeless).

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References

[1] Bardoe, M.; Sin, P.: *The permutation modules for* $\mathrm{GL}(n+1, \mathbf{F}_q)$ *acting on* $\mathbf{P}^n(\mathbf{F}_q)$ *and* $\mathbf{F}_q^{n-1}$. J. Lond. Math. Soc., II. Ser. **61** (2000), 58–80. Zbl 0961.20010

[2] Barlotti, A.: *La determinazione del gruppo delle proiettività di una retta in sè in alcuni particolari piani grafici finiti non desarguesiani.* Boll. Unione Mat. Ital., III. Ser. **14** (1959), 543–547. Zbl 0091.32601

[3] Barlotti, A.: *Il gruppo delle proiettività di una retta in sè in un particolare piano non desarguesiano di ordine sedici.* Matematiche **19** (1964), 89–95. Zbl 0192.57402

[4] Brauer, R.: *Zur Darstellungstheorie der Gruppen endlicher Ordnung. I.* Math. Z. **63** (1956), 406–444. Zbl 0093.03101

[5] Cameron, P. J.: *Finite permutation groups and finite simple groups.* Bull. Lond. Math. Soc. **13** (1981), 1–22. Zbl 0463.20003

[6] Cameron, P. J.: *Permutation groups.* Cambridge University Press, Cambridge 1999. Zbl 0922.20003

[7] Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A.: *Atlas of finite groups.* Oxford University Press, Eynsham 1985. Zbl 0568.20001

[8] Dembowski, P.: *Finite geometries.* Ergebnisse der Mathematik und ihrer Grenzgebiete **44**, Springer-Verlag, Berlin 1968. Zbl 0159.50001

[9] Dixon, J. D.; Mortimer, B.: *Permutation Groups.* Springer-Verlag, New York 1996. Zbl 0951.20001

[10] Fiori, C.; Zironi, F.: *Esistenza e costruzione di sottoinsiemi di permutazioni strettamente* 1*-transitivi.* Atti Semin. Mat. Fis. Univ. Modena **34** (1986), 317–335. Zbl 0644.20006

[11] Grundhöfer, T.: *Die Projektivitätengruppen der endlichen Translationsebenen.* J. Geom. **20** (1983), 74–85. Zbl 0514.51013

[12] Grundhöfer, T.: *Projektivitätengruppen von Translationsebenen.* Result. Math. **6** (1983), 163–182. Zbl 0532.51002

[13] Grundhöfer, T.: *Über Projektivitätengruppen affiner und projektiver Ebenen unter besonderer Berücksichtigung von Moufangebenen.* Geom. Dedicata **13** (1983), 435–458. Zbl 0523.51006

[14] Grundhöfer, T.: *The groups of projectivities of finite projective and affine planes*. Ars Comb. **25**(A) (1988), 269–275. Zbl 0643.51008

[15] Grundhöfer, T.: *Projektivitätengruppen von Ebenen über endlichen Semikörpern*. J. Geom. **40** (1991), 74–76. Zbl 0726.51005

[16] Herzer, A.: *Die Gruppe* $\Pi(g)$ *in den endlichen André-Ebenen gerader Ordnung*. Geom. Dedicata **3** (1974), 241–249. Zbl 0241.50023

[17] Hughes, D. R.; Piper, F. C.: *Projective Planes*. Springer, New York 1973. Zbl 0267.50018

[18] Jansen, C.; Lux, K.; Parker, R.; Wilson, R.: *An atlas of Brauer characters*. Oxford University Press, New York 1995. Zbl 0831.20001

[19] Kilmer, D.: *Über die Projektivitätengruppen von Semikörperebenen gerader Ordnung*. J. Geom. **35** (1989), 108–119. Zbl 0678.51006

[20] Lander, E. S.: *Symmetric Designs: An Algebraic Approach*. Lond. Math. Soc. Lect. Note Ser. **74**, Cambridge University Press, 1983. Zbl 0502.05010

[21] Lorimer, P.: *A note on groups of Ree type*. Can. Math. Bull. **16** (1973), 451–452. Zbl 0285.20002

[22] Lorimer, P.: *A property of the groups* $\mathrm{P\Gamma L}(m, q)$, $q \geq 5$. Proc. Am. Math. Soc. **37** (1973), 393–396. Zbl 0254.50009

[23] Lorimer, P.: *A property of the groups* Aut $\mathrm{PU}(3, q^2)$. Pac. J. Math. **46** (1973), 225–230. Zbl 0267.20003

[24] Lorimer, P.: *A property of the Suzuki groups*. Nanta Math. **6** (1973), 71–74. Zbl 0291.50022

[25] Lorimer, P.: *Finite projective planes and sharply* 2*-transitive subsets of finite groups*. Lect. Notes Math. **372**, Springer, Berlin 1974, 432–436. Zbl 0291.50021

[26] Mortimer, B.: *The modular permutation representations of the known doubly transitive groups*. Proc. Lond. Math. Soc., III. Ser. **41** (1980), 1–20. Zbl 0393.20002

[27] O'Nan, M. E.: *Sharply* 2*-transitive sets of permutations*. Proc. Rutgers group theory year, 1983–1984 (1984), 63–67. Zbl 0639.20002

[28] Pickert, G.: *Projectivities in projective planes*. In: Plaumann, P.; Strambach, K. (eds.): Geometry – von Staudt's point of view. (Bad Windsheim 1980), 1–49, Reidel, Dordrecht 1981. Zbl 0496.51002

[29] Plaumann, P.; Strambach, K. (eds.): *Geometry - von Staudt's point of view*. (Bad Windsheim 1980). Reidel, Dordrecht 1981. Zbl 0451.00016

[30] Sastry, N. S. N.; Sin, P.: *On the doubly transitive permutation representations of* $\mathrm{Sp}(2n, \mathbb{F}_2)$. J. Algebra **257** (2002), 509–527. Zbl 1050.20032

[31] Schleiermacher, A.: *Reguläre Normalteiler in der Gruppe der Projektivitäten bei projektiven und affinen Ebenen*. Math. Z. **114** (1970), 313–320. Zbl 0186.54401

[32] Taylor, D. E.: *The Geometry of the Classical Groups*. Heldermann-Verlag, Berlin 1992. Zbl 0767.20001

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