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I-measures in Minkowski Planes

Andreas Fankhänel

Faculty of Mathematics, University of Technology Chemnitz
09107 Chemnitz, Germany

Abstract. Let μ be an angular measure in a 2-dimensional normed linear space (i.e., in a Minkowski plane). We consider certain measures μ, called I-measures, and show that their existence is sufficient for the plane to be Euclidean.

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1. Introduction

In [2], Brass considers angular measures in normed planes. He shows that in any Minkowski plane, except for the rectangular one, there exists an angular measure having the following property: Each equilateral triangle is equiangular. We want to investigate how this statement should be modified for considering angular measures that satisfy a stronger condition, namely: In each isosceles triangle, the two angles corresponding to the equal sides are equal.

We will see that this condition is strong enough to guarantee that the plane under consideration is Euclidean.

1.1. Background

Let $\mathbb{E}^2$ be the Euclidean plane with origin o, and $B \subset \mathbb{E}^2$ be a convex body (i.e., a compact, convex set) centred at o. Then B defines a norm on $\mathbb{E}^2$ by

$$||x||_B := \inf\{\lambda \in \mathbb{R}^+ \,:\, x \in \lambda \cdot B\}.$$

The plane $\mathbb{E}^2$ equipped with the norm $||\cdot||_B$ is called the *Minkowski plane with unit ball* B and denoted by $M^2(B)$. Obviously, we have $B = \{x : ||x||_B \leq 1\}$, and the set $C := \{x : ||x||_B = 1\}$ is called the *unit circle of* $M^2(B)$.

We say that $x, y \in M^2(B)$ are *James orthogonal* or *isosceles orthogonal*, denoted by $x\#y$, if

$$||x+y|| = ||x-y||,$$

and x, y are *Birkhoff orthogonal*, abbreviated by $x \dashv y$, if

$$||x|| \leq ||x + t \cdot y|| \quad \forall t \in \mathbb{R}.$$

If both $x \dashv y$ and $y \dashv x$ hold, we write $x \perp y$.

By $\widehat{uv}$ we denote the small arc from u to v, i.e., an interval on C with endpoints u and v lying in a half circle. (ab) is the line through the points $a \neq b$, $[ab]$ the respective (Euclidean) segment.

Following Brass [2], we introduce the notion of angular measure in Minkowski planes.

Definition 1. *Let μ be a measure on the unit circle C. μ is called an angular measure if it has the following properties:*

1) $\mu(C) = 360°$.
2) $\mu(S) = \mu(-S)$ *for all subsets S of C.*
3) $\mu(\{p\}) = 0$ *for an arbitrary point $p \in C$.*
4) μ *is translation invariant.*

For $u, v \in C$ we define the angle between u and v by

$$\sphericalangle(u, v) := \mu(\widehat{uv}).$$

Furthermore, for points $a, b, c \in M^2(B)$, $a \neq b, b \neq c$, we define the angle

$$\sphericalangle(abc) := \sphericalangle(\tilde{a}, \tilde{c}),$$

where $\tilde{a} = \frac{a-b}{||a-b||}$ and $\tilde{c} = \frac{c-b}{||c-b||}$.

The word "angle" may describe the corresponding geometrical figure as well, meaning that this figure $\sphericalangle(abc)$ is formed by two rays starting in b (the vertex of the angle) and passing through a and c, respectively.

1.2. Basic properties

The following properties hold for any angular measure.

Observation 1. For $u \in C$, $\sphericalangle(u, -u) = 180°$.

Observation 2. The sum of the interior angles of a triangle equals $180°$.

Both statements follow directly from the definition of an angular measure. They can be proved in the same way as in the Euclidean case; see [5].

Lemma 1. *The following statements are equivalent:*

i) $\sphericalangle(-uvu) = 90° \quad \forall u, v \in C, u \neq v.$

ii) $\sphericalangle(x, y) = 90° \quad \forall x, y$ *with* $x \# y$.

Proof. Assume that i) holds, and let $x \# y$. For $u = x - y, v = x + y$ we have

$$||u|| = ||-u|| = ||v||,$$

and thus $\sphericalangle(x, y) = \sphericalangle(-uvu) = 90°$.

Let now $u, v \in C, u \neq v$. Since $||(u-v)+(u+v)|| = ||(u-v)-(u+v)||$, we have $(u-v)\#(u+v)$, and thus $\sphericalangle(u-v, u+v) = \sphericalangle(-uvu) = 90°$. □

Remark 1. Property i) holds in the Euclidean plane and has been stated by Thales, see [4].

2. *I*-measures

In what follows, we look at special angular measures satisfying certain properties.

Proposition 1. *Let μ be an angular measure in a Minkowski plane. The following conditions are equivalent:*

i) *For any isosceles triangle $\triangle abc$ the corresponding angles are equal, i.e.,*

$$||a - c|| = ||b - c|| \Rightarrow \sphericalangle(cab) = \sphericalangle(cba).$$

ii) *Let $a, b, c \in C$ be pairwise distinct. If $c \notin \widehat{ab}$, then*

$$\sphericalangle(aob) = 2 \cdot \sphericalangle(acb),$$

else

$$360° - \sphericalangle(aob) = 2 \cdot \sphericalangle(acb).$$

iii) *Let $a, b \in C$ be distinct and $c := -a$. Then*

$$\sphericalangle(aob) = 2 \cdot \sphericalangle(acb).$$

Proof. i) $\Rightarrow$ ii): This direction can be proved in the same way as in the Euclidean case; see [3], p. 46.

ii) $\Rightarrow$ iii): This conclusion is obviously true.

iii) $\Rightarrow$i): Due to basic angle properties we have

$$\sphericalangle(aob) = 180° - \sphericalangle(boc) = \sphericalangle(obc) + \sphericalangle(ocb).$$

Since $\sphericalangle(aob) = 2 \cdot \sphericalangle(acb)$, it follows that $\sphericalangle(ocb) = \sphericalangle(obc)$ for any $a, b \in C$. □

Definition 2. *Let μ be an angular measure in a Minkowski plane. If μ satisfies one of the conditions in Proposition 1, then μ is said to be an I-measure.*

Theorem 1. *Let $M^2(B)$ be a Minkowski plane with unit circle C, and μ be an I-measure on C. Then $M^2(B)$ is Euclidean.*

Thus I-measures only exist in the Euclidean plane. We will prove this result in several steps.

In what follows, $M^2(B)$ is always a Minkowski plane with unit circle C, and μ denotes an I-measure on C.

For $u, v \in C$ we call $\sphericalangle(u,v)$ a *non-trivial zero angle* if it is an angle of measure $0°$, but $u \neq v$.

Theorem 2. *There exist no non-trivial zero angles, i.e.,*

$$\sphericalangle(u,v) > 0° \quad \forall u, v \in C, u \neq v.$$

Proof. Suppose $\sphericalangle(aob) = 0°$ for some $a, b \in C, a \neq b$. Without loss of generality, $\sphericalangle(a'ob') > 0°$ for all angles $\sphericalangle(a'ob')$ strictly containing $\sphericalangle(aob)$ (since if $\sphericalangle(a_n o b_n)$ is a sequence of angles of measure $0°$ that is increasing with respect to $\subseteq$, and $a_n \to \overline{a}, b_n \to \overline{b}$, then $\sphericalangle(\overline{a}o\overline{b}) = \sup \sphericalangle(aob) = 0°$).

Now we translate $\sphericalangle(aob)$ so that the vertex moves to the boundary of C and such that the translated angle $\sphericalangle(a'o'b')$ contains $\sphericalangle(aob)$; see Figure 1.

Then $\sphericalangle(a'ob') = 2 \cdot \angle(aob) = 0°$, but $\sphericalangle(a'ob')$ strictly contains $\sphericalangle(aob)$, a contradiction. □

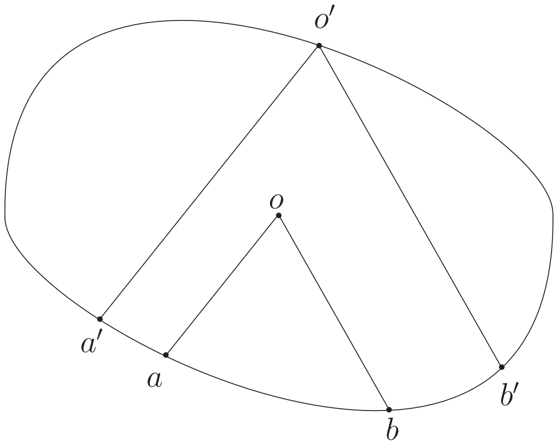

Figure 1. Notation in the proof of Theorem 2

The following statement is a direct consequence of Proposition 1.

Lemma 2. *Thales' theorem holds, i.e., for all $u, v \in C, u \neq v$, we have*

$$\sphericalangle(-uvu) = 90°.$$

Lemma 3. *Let t be a line supporting B at the point p. For an arbitrary point $q \in t$, $q \neq p$, we have that $\sphericalangle(opq) = 90°$. In other words: If $x, y \in M^2(B)$ are Birkhoff orthogonal, then $\sphericalangle(x,y) = 90°$.*

Proof. The statement follows from the previous lemma, taking into consideration the continuity of the measure. Let $p \in C$ be an arbitrary point on the unit circle, and let $\{s_k\} \subset C,\ k = 0, 1, \ldots$, be a sequence of points with $\lim_{k\to\infty} s_k = p$. Thus $\sphericalangle(-ps_kp) \to \sphericalangle(opq)$. By the previous lemma we have that $\sphericalangle(-ps_kp) = 90°$ for all k, which completes the proof. □

This lemma suffices to show that Theorem 1 is true.

Proof of Theorem 1. Let $x \in M^2(B)$ be arbitrary, and $y \in M^2(B)$ be such that $x\#y$. Then Lemma 1 and Lemma 2 yield that $\sphericalangle(x, y) = 90°$.

Now let $z \in M^2(B)$ be such that $||z|| = ||y||$ and $x \dashv z$. Then, by Lemma 3, we have that $\sphericalangle(x, z) = 90°$. Since there exist no non-trivial zero angles, it follows that $y = z$ or $y = -z$, and thus we have

$$x\#y \Rightarrow x \dashv y \quad \forall x, y \in M^2(B).$$

This relation is a well-known characterization of the Euclidean plane; see [8], p. 87, or [1], p. 33. □

References

[1] Amir, D.: *Characterizations of Inner Product Spaces.* Operator Theory: Advances and Applications **20**. Birkhäuser, Basel 1986. Zbl 0617.46030

[2] Brass, P.: *Erdős distance problems in normed spaces.* Comput. Geom. **6** (1996), 195–214. Zbl 0860.52008

[3] Euclid: *The Elements.* (translated by Sir Thomas L. Heath), Dover 1956. cf. Santa Fe 2002. Zbl 1026.01024

[4] Eves, H.: *An Introduction to the History of Mathematics.* CBS College Publishing, 1983. Zbl 0564.01002

[5] Jennings, G. A.: *Modern Geometry with Applications.* Springer, New York 1994. Zbl 0807.51001

[6] Martini, H.; Swanepoel, K.: *The geometry of Minkowski spaces – a survey. Part II.* Expo. Math. **22** (2004), 93–144. Zbl 1080.52005

[7] Martini, H.; Swanepoel, K.; Weiß, G.: *The geometry of Minkowski spaces – a survey. Part I.* Expo. Math. **19** (2001), 97–142. Zbl 0984.52004

[8] Thompson, A. C.: *Minkowski Geometry.* Encyclopedia of Mathematics and Its Applications **63**. Cambridge University Press, Cambridge 1996. Zbl 0868.52001

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