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On the Arithmetical Rank of a Special Class of Minimal Varieties

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Abstract. We study the arithmetical ranks and the cohomological dimensions of an infinite class of Cohen-Macaulay varieties of minimal degree. Among these we find, on the one hand, infinitely many set-theoretic complete intersections, on the other hand examples where the arithmetical rank is arbitrarily greater than the codimension.

Keywords: minimal variety, arithmetical rank, set-theoretic complete intersection, cohomological dimension

Introduction

Let K be an algebraically closed field, and let R be a polynomial ring in N indeterminates over K. Let I be a proper reduced ideal of R and consider the variety $V(I)$ defined in the affine space K^N (or in the projective space $\mathbf{P}_K^{N-1}$, if I is homogeneous and different from the maximal irrelevant ideal) by the vanishing of all polynomials in I. By Hilbert's Basissatz there are finitely many polynomials $F_1, \ldots, F_r \in R$ such that $V(I)$ is defined by the equations $F_1 = \cdots = F_r = 0$. By Hilbert's Nullstellensatz this is equivalent to the ideal-theoretic condition

$$I = \sqrt{(F_1, \ldots, F_r)}.$$

Suppose r is minimal with respect to this property. It is well known that $\operatorname{codim} V(I) \leq r$. If equality holds, $V(I)$ is called a *set-theoretic complete intersection* on $F_1, \ldots, F_r$.

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Exhibiting significant examples of set-theoretic complete intersections (or, more generally, determining the minimum number of equations defining given varieties, the so-called *arithmetical rank*, denoted ara, of their defining ideals) is one of the hardest problems in algebraic geometry. In [2] we already determined infinitely many set-theoretic complete intersections among the Cohen-Macaulay varieties of minimal degree which were classified geometrically by Bertini [6], Del Pezzo [11], Harris [14] and Xambó [27], and whose defining ideals were determined in an explicit algorithmic way in [5]. In this paper we present a new class of minimal varieties, where the gap between the arithmetical rank and the codimension can be arbitrarily high. It includes an infinite set of set-theoretic complete intersections. For the arithmetical ranks of the complementary set of varieties we determine a lower bound (given by étale cohomology) and an upper bound (resulting from the computation of an explicit set of defining equations) that only differ by one: the equality between the lower bound and the actual value of the arithmetical rank is shown in few special cases. We also determine the cohomological dimensions of the defining ideals of each of these varieties. This invariant, in general, also provides a lower bound for the arithmetical rank, and the cases where it is known to be smaller are rare. Those which were found so far are the determinantal and Pfaffian ideals considered in [9] and in [1]: there the strict inequality holds in all positive characteristics. We prove that the same is true for the minimal varieties investigated in the present paper that are not set-theoretic complete intersections.

Some crucial results on arithmetical ranks and cohomological dimensions are due to Bruns et al. and are quoted from [9] and [10].

1. Preliminaries

For all integers $s \geq 2$ and $t \geq 1$ consider the two-row matrix

$$A_{s,t} = \left(\begin{array}{cccc||c||c||c||c} x_1 & x_2 & \cdots & x_s & y_0 & y_1 & \cdots & y_{t-1} \\ x_{s+1} & x_{s+2} & \cdots & x_{2s} & z_1 & z_2 & \cdots & z_t \end{array} \right),$$

where $x_1, x_2, \ldots, x_{2s}, y_0, y_1, \ldots, y_{t-1}, z_1, z_2, \ldots, z_t$ are N indeterminates over K. We assume that they are pairwise distinct, possibly with the following exception: we can have $x_{2s} = y_0$ or $z_i = y_j$ for some indices i and j such that $1 \leq i \leq j \leq t-1$, but no entry appears more than twice in $A_{s,t}$. We have the least possible number of indeterminates if $x_{2s} = y_0$ and $z_i = y_i$ for $1 = 1, \ldots, t-1$, in which case $N = 2s + t$, and the matrix takes the following form:

$$\bar{A}_{s,t} = \left(\begin{array}{cccc||c||c||c||c} x_1 & x_2 & \cdots & x_s & x_{2s} & y_1 & \cdots & y_{t-1} \\ x_{s+1} & x_{s+2} & \cdots & x_{2s} & y_1 & y_2 & \cdots & y_t \end{array} \right).$$

If the indeterminates are pairwise distinct, then $N = 2s + 2t$. The matrix $A_{s,t}$ belongs to the class of so-called *barred matrices* introduced in [4] and can be associated with the ideal $J_{s,t}$ of $R = K[x_1, x_2, \ldots, x_{2s}, y_0, y_1, \ldots, y_{t-1}, z_1, z_2, \ldots, z_t]$ generated by the union of

(I) the set $\mathcal{M}$ of two-minors of the submatrix of $A_{s,t}$ formed by the first s columns (the so-called first *big block*);

(II) the set of products x_iz_j, with $1 \leq i \leq s$ and $1 \leq j \leq t$;

(III) the set of products y_iz_j, with $0 \leq i \leq j-2 \leq t-2$.

We will denote by $\bar{J}_{s,t}$ the ideal associated with the matrix $\bar{A}_{s,t}$.

As shown in [4], Section 1, $J_{s,t}$ it is the defining ideal of a Cohen-Macaulay variety of minimal degree and it admits the prime decomposition

$$J_{s,t} = J_0 \cap J_1 \cap \cdots \cap J_t,$$

where

$$J_0 = (\mathcal{M}, \mathcal{D}_0), \quad \text{and } J_i = (\mathcal{P}_i, \mathcal{D}_i) \quad \text{for } i = 1, \ldots, t,$$

with

$$\mathcal{P}_i = \{x_1, \ldots, x_s, y_0, \ldots, y_{i-2}\} \quad \text{for } i = 1, \ldots, t,$$
$$\mathcal{D}_i = \{z_{i+1}, \ldots, z_t\} \quad \text{for } i = 0, \ldots, t.$$

Thus the sequence of ideals $J_0, J_1, \ldots, J_t$ fulfils condition 2 of Theorem 1 in [21], which implies that it is *linearly joined*; this notion was introduced by Eisenbud, Green, Hulek and Popescu [12], and was later intensively investigated by Morales [21]. We also have

$$\text{height } J_{s,t} = s + t - 1. \tag{1}$$

In the sequel, we will set $V_{s,t} = V(J_{s,t})$, and also $\bar{V}_{s,t} = V(\bar{J}_{s,t})$. Note that $J_{s,1}$ has the same generators as $\bar{J}_{s,1}$, because the indeterminate y_0 does not appear in these generators. Consequently, we can identify $V_{s,1}$ with $\bar{V}_{s,1}$. One should observe that, apart from this special case, for any integers s and t, $J_{s,t}$ does not denote a single ideal, but a class of ideals, namely the ideals attached to a matrix $A_{s,t}$ for some choice of the (identification between) the indeterminates $x_1, x_2, \ldots, x_{2s}, y_0, y_1, \ldots, y_{t-1}, z_1, z_2, \ldots, z_t$. The same remark applies to the variety $V_{s,t}$.

For the proofs of the theorems on arithmetical ranks contained in the next section we will need the following two technical results, which are valid in any commutative unit ring R.

Lemma 1. ([3], Corollary 3.2) *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in R$. Then*

$$\sqrt{(\alpha_1\beta_1 - \alpha_2\beta_2,\ \beta_1\gamma,\ \beta_2\gamma)} = \sqrt{(\alpha_1(\alpha_1\beta_1 - \alpha_2\beta_2) + \beta_2\gamma,\ \alpha_2(\alpha_1\beta_1 - \alpha_2\beta_2) + \beta_1\gamma)}.$$

The next claim is a slightly generalized version of [3], Lemma 2.1 (which, in turn, extends [25], Lemma, p. 249). The proof is the same as the one given in [3], and will therefore be omitted here.

Lemma 2. *Let P be a finite subset of elements of R, and I an ideal of R. Let $P_1, \ldots, P_r$ be subsets of P such that*

(i) $\bigcup_{\ell=1}^{r} P_\ell = P$;

(ii) *if* p *and* p' *are different elements of* P_ℓ $(1 \leq \ell \leq r)$ *then* $(pp')^m \in I + \left(\bigcup_{i=1}^{\ell-1} P_i\right)$ *for some positive integer* m.

Let $1 \leq \ell \leq r$, *and, for any* $p \in P_\ell$, *let* $e(p) \geq 1$ *be an integer. We set* $q_\ell = \sum_{p \in P_\ell} p^{e(p)}$. *Then we get*

$$\sqrt{I + (P)} = \sqrt{I + (q_1, \ldots, q_r)},$$

where (P) *denotes the ideal of* R *generated by* P.

2. The arithmetical rank for $s = 2$: set-theoretic complete intersections

In this section we will show that, for all $t \geq 1$, the variety $V_{2,t}$ is a set-theoretic complete intersection. Recall that its defining ideal is the ideal $J_{2,t}$ of $R = K[x_1, x_2, x_3, x_4, y_0, y_2, \ldots, y_{t-1}, z_1, z_2, \ldots, z_t]$, which is associated with the matrix

$$A_{2,t} = \left(\begin{array}{cc||c||c||c||c} x_1 & x_2 & y_0 & y_1 & \cdots & y_{t-1} \\ x_3 & x_4 & z_1 & z_2 & \cdots & z_t \end{array} \right),$$

and is generated by the elements

$$\begin{array}{cccccc} x_1x_4 - x_2x_3, & x_1z_1, & x_1z_2, & \ldots, & x_1z_t, \\ & x_2z_1, & x_2z_2, & \ldots, & x_2z_t, \\ & y_0z_2, & \ldots, & \ldots, & y_0z_t, \\ & & y_1z_3, & \ldots, & y_1z_t, \\ & \ldots, & \ldots, & \ldots, & y_{t-2}z_t. \end{array}$$

The next result generalizes Example 5 in [2].

Theorem 1. *For all integers* $t \geq 1$, $\operatorname{ara} J_{2,t} = t + 1$, *i.e.,* $V_{2,t}$ *is a set-theoretic complete intersection.*

Proof. We proceed by induction on t, by showing that there are $F_1, \ldots, F_{t+1} \in R = K[x_1, x_2, x_3, x_4, y_0, \ldots, y_{t-1}, z_1, z_2, \ldots, z_t]$ such that

(a) $\sqrt{(F_1, \ldots, F_{t+1})} = J_{2,t}$,

(b) $F_1, F_2 \in (x_1, x_2)$,

(c) $F_i \in (x_1, x_2, y_0, \ldots, y_{i-3})$ for all $i = 3, \ldots, t+1$.

For the induction basis consider the case where $t = 1$. We have $J_{2,1} = (x_1x_4 - x_2x_3, x_1z_1, x_2z_1)$. Set

$$F_1 = x_4(x_1x_4 - x_2x_3) + x_2z_1, \qquad F_2 = x_3(x_1x_4 - x_2x_3) + x_1z_1. \tag{2}$$

Then F_1 and F_2 fulfil condition (b) and, by virtue of Lemma 1, they also fulfil condition (a). Now assume that $t \geq 2$ and suppose that $G_1, \ldots, G_t$ are polynomials fulfilling the claim for $t-1$. By condition (b) we have $G_1 = Px_1 - Qx_2$ for some $P, Q \in R$. Set

$$\begin{aligned}
F_1 &= QG_1 + x_1z_t\\
F_2 &= PG_1 + x_2z_t\\
F_3 &= G_2 + y_0z_t\\
&\vdots\\
F_i &= G_{i-1} + y_{i-3}z_t\\
&\vdots\\
F_{t+1} &= G_t + y_{t-2}z_t.
\end{aligned}$$

Then $F_1, F_2 \in (G_1, x_1, x_2) \subset (x_1, x_2)$. Moreover, for all $i = 3, \ldots, t+1$,

$$F_i \in (G_{i-1}, y_{i-3}) \subset (x_1, x_2, y_0, \ldots, y_{i-4}, y_{i-3}),$$

because G_{i-1} fulfils condition (c). Hence $F_1, \ldots, F_{t+1}$ fulfil conditions (b) and (c). Furthermore, by Lemma 1,

$$\sqrt{(F_1, F_2)} = \sqrt{(G_1, x_1z_t, x_2z_t)}, \tag{3}$$

and, for all $i = 2, \ldots, t$, the product of the two summands of F_{i+1} is

$$\begin{aligned}
G_i \cdot y_{i-2}z_t \in (x_1, x_2, y_0, \ldots, y_{i-3}) \cdot (z_t) &= (x_1z_t, x_2z_t) + (y_0z_t, \ldots, y_{i-3}z_t),\\
&\subset \sqrt{(F_1, F_2)} + (y_0z_t, \ldots, y_{i-3}z_t),
\end{aligned}$$

where the first membership relation is true because G_i fulfils condition (c). It follows that $(G_i \cdot y_{i-2}z_t)^m$ belongs to $(F_1, F_2) + (y_0z_t, \ldots, y_{i-3}z_t)$ for some positive integer m. Hence the assumption of Lemma 2 is fulfilled for $I = (F_1, F_2)$ and $P_i = \{G_{i+1}, y_{i-1}z_t\}$ $(i = 1, \ldots, t-1)$. Consequently,

$$\begin{aligned}
\sqrt{(F_1, F_2, F_3, \ldots, F_{t+1})} &= \sqrt{(F_1, F_2, G_2, \ldots, G_t, y_0z_t, \ldots, y_{t-2}z_t)}\\
&= \sqrt{(G_1, G_2, \ldots, G_t, x_1z_t, x_2z_t, y_0z_t, \ldots, y_{t-2}z_t)}\\
&= J_{2,t-1} + (x_1z_t, x_2z_t, y_0z_t, \ldots, y_{t-2}z_t) = J_{2,t},
\end{aligned}$$

where the second and the third equality follow from (3) and induction, respectively. Thus $F_1, \ldots, F_{t+1}$ fulfil condition (a) as well. This completes the proof.

Remark 1. The polynomials $F_1, \ldots, F_{t+1}$ defined in the proof of Theorem 1 still fulfil the required properties if in all monomial summands x_1z_t, x_2z_t, $y_0z_t, \ldots,$ $y_{t-2}z_t$ the factors z_t are raised to the same arbitrary positive power. This allows us, e.g., to replace the polynomials in (2) by

$$F_1 = x_4(x_1x_4 - x_2x_3) + x_2z_1^2, \qquad F_2 = x_3(x_1x_4 - x_2x_3) + x_1z_1^2,$$

which are homogeneous. Then, by a suitable adjustment of exponents, one can recursively construct a sequence of homogeneous polynomials $F_1, \ldots, F_{t+1}$ for any $t \geq 2$.

Example 1. Equalities (2) explicitly provide the defining polynomials for $V_{2,1}$. They are the starting point of the recursive procedure, described in the proof of Theorem 1, which allows us to construct $t+1$ polynomials defining $V_{2,t}$, for any $t \geq 2$. We perform the construction for $t = 2, 3$. First take $t = 2$. We have

$$A_{2,2} = \left(\begin{array}{cc||c||c} x_1 & x_2 & y_0 & y_1 \\ x_3 & x_4 & z_1 & z_2 \end{array} \right),$$

and

$$J_{2,2} = (x_1x_4 - x_2x_3, x_1z_1, x_1z_2, x_2z_1, x_2z_2, y_0z_2).$$

Let us rewrite the polynomials given in (2):

$$G_1 = x_1x_4^2 - x_2x_3x_4 + x_2z_1, \qquad G_2 = x_1x_3x_4 - x_2x_3^2 + x_1z_1.$$

Then, with the notation of the proof of Theorem 1, $P = x_4^2$ and $Q = x_3x_4 - z_1$. Thus

$$\begin{aligned} F_1 &= (x_3x_4 - z_1)G_1 + x_1z_2 \\ &= x_1x_3x_4^3 - x_1x_4^2z_1 - x_2x_3^2x_4^2 + 2x_2x_3x_4z_1 - x_2z_1^2 + x_1z_2, \\ F_2 &= x_4^2G_1 + x_2z_2 = x_1x_4^4 - x_2x_3x_4^3 + x_2x_4^2z_1 + x_2z_2, \\ F_3 &= G_2 + y_0z_2 = x_1x_4x_3 - x_2x_3^2 + x_1z_1 + y_0z_2 \end{aligned}$$

are three defining polynomials for $V_{2,2}$. Now let $t = 3$. We have

$$A_{2,3} = \left(\begin{array}{cc||c||c||c} x_1 & x_2 & y_0 & y_1 & y_2 \\ x_3 & x_4 & z_1 & z_2 & z_3 \end{array} \right),$$

$$J_{2,3} = (x_1x_4 - x_2x_3, x_1z_1, x_1z_2, x_1z_3, x_2z_1, x_2z_2, x_2z_3, y_0z_2, y_0z_3, y_1z_3).$$

In order to obtain four defining polynomials for $V_{2,3}$ we take the above polynomials F_1, F_2, F_3 as G_1, G_2, G_3. Thus $P = x_3x_4^3 - x_4^2z_1 + z_2$ and $Q = x_3^2x_4^2 - 2x_3x_4z_1 + z_1^2$. Hence, the four sought polynomials are

$$\begin{aligned} F_1 &= (x_3^2x_4^2 - 2x_3x_4z_1 + z_1^2)G_1 + x_1z_3 = x_1x_3^3x_4^5 - 3x_1x_3^2x_4^4z_1 + 3x_1x_3x_4^3z_1^2 \\ &\quad -x_1x_4^2z_1^3 - x_2x_3^4x_4^4 + 4x_2x_3^3x_4^3z_1 \\ &\quad -6x_2x_3^2x_4^2z_1^2 + 4x_2x_3x_4z_1^3 - x_2z_1^4 \\ &\quad +x_1x_3^2x_4^2z_2 - 2x_1x_3x_4z_1z_2 + x_1z_1^2z_2 + x_1z_3, \\ F_2 &= (x_3x_4^3 - x_4^2z_1 + z_2)G_1 + x_2z_3 = x_1x_3^2x_4^6 - 2x_1x_3x_4^5z_1 + 2x_1x_3x_4^3z_2 \\ &\quad +x_1x_4^4z_1^2 - 2x_1x_4^2z_1z_2 - x_2x_3^3x_4^5 - x_2x_3^2x_4^2z_2 \\ &\quad +3x_2x_3^2x_4^4z_1 - 3x_2x_3x_4^3z_1^2 + 2x_2x_3x_4z_1z_2 \\ &\quad +x_2x_4^2z_1^3 - x_2z_1^2z_2 + x_1z_2^2 + x_2z_3, \\ F_3 &= G_2 + y_0z_3 = x_1x_4^4 - x_2x_3x_4^3 + x_2x_4^2z_1 + x_2z_2 + y_0z_3, \\ F_4 &= G_3 + y_1z_3 = x_1x_3x_4 - x_2x_3^2 + x_1z_1 + y_0z_2 + y_1z_3. \end{aligned}$$

3. The arithmetical rank for $s \geq 3$: upper and lower bounds

The aim of this section is to show that, for $s \geq 3$, the ideal $J_{s,t}$ is never a set-theoretic complete intersection. We will determine a lower bound for $\operatorname{ara} J_{s,t}$, which shows that the difference between the arithmetical rank and the height strictly increases with s. For our purpose we will need the following cohomological criterion by Newstead [22].

Lemma 3. ([9], Lemma 3′) *Let $W \subset \tilde{W}$ be affine varieties. Let $d = \dim \tilde{W} \setminus W$. If there are r polynomials $F_1, \ldots, F_r$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_r)$, then*

$$H^{d+i}_{\mathrm{et}}(\tilde{W} \setminus W, \mathbb{Z}/m\mathbb{Z}) = 0 \qquad \textit{for all } i \geq r$$

and for all $m \in \mathbb{Z}$ which are prime to $\operatorname{char} K$.

We refer to [19] or [20] for the basic notions on étale cohomology. We are now ready to prove the first of the two main results of this section.

Theorem 2. *For all integers $s \geq 2$ and $t \geq 1$*

$$\operatorname{ara} J_{s,t} \geq 2s + t - 3.$$

Proof. For $s = 2$ the claim is a trivial consequence of Theorem 1. So let $s \geq 3$. It suffices to prove the claim for $\bar{J}_{s,t}$, because $\operatorname{ara} J_{s,t} \geq \operatorname{ara} \bar{J}_{s,t}$: in fact, given r defining polynomials for $V_{s,t}$, they can be transformed in r defining polynomials for $\bar{V}_{s,t}$ by performing on them the suitable identifications between the indeterminates. Let p be a prime different from $\operatorname{char} K$. According to Lemma 3, it suffices to show that

$$H^{4s+2t-4}_{\mathrm{et}}(K^{2s+t} \setminus \bar{V}_{s,t}, \mathbb{Z}/p\mathbb{Z}) \neq 0, \tag{4}$$

since this will imply that $\bar{V}_{s,t}$ cannot be defined by $2s + t - 4$ equations. By Poincaré Duality (see [20], Theorem 14.7, p. 83) we have

$$\operatorname{Hom}_{\mathbb{Z}/p\mathbb{Z}}(H^{4s+2t-4}_{\mathrm{et}}(K^{2s+t} \setminus \bar{V}_{s,t}, \mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \simeq H^4_{\mathrm{c}}(K^{2s+t} \setminus \bar{V}_{s,t}, \mathbb{Z}/p\mathbb{Z}), \tag{5}$$

where H_{c} denotes étale cohomology with compact support. For the sake of simplicity, we will omit the coefficient group $\mathbb{Z}/p\mathbb{Z}$ henceforth. In view of (5), it suffices to show that

$$H^4_{\mathrm{c}}(K^{2s+t} \setminus \bar{V}_{s,t}) \neq 0. \tag{6}$$

Let W be the subvariety of K^{2s+t} defined by the vanishing of y_t and of all generators of $\bar{J}_{s,t}$ listed in Section 1 under (I) and (II), and those listed in (III) for which $j \leq t - 1$. Then $W \subset \bar{V}_{s,t}$, and

$$\begin{aligned} &\bar{V}_{s,t} \setminus W = \\ &\{(x_1, \ldots x_{2s}, y_1, \ldots, y_t) | x_1 = \cdots = x_s = x_{2s} = y_1 = \cdots = y_{t-2} = 0,\ y_t \neq 0\} \\ &\qquad \simeq K^s \times (K \setminus \{0\}). \end{aligned} \tag{7}$$

It is well known that

$$H_c^i(K^r) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 2r \\ 0 & \text{else,} \end{cases} \tag{8}$$

and

$$H_c^i(K^r \setminus \{0\}) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 1, 2r \\ 0 & \text{else.} \end{cases} \tag{9}$$

Moreover, in view of (7), by the Künneth formula for étale cohomology ([20], Theorem 22.1),

$$H_c^i(\bar{V}_{s,t} \setminus W) \simeq \bigoplus_{h+k=i} H_c^h(K^s) \otimes H_c^k(K \setminus \{0\}),$$

so that, by (8) and (9), we have $H_c^i(\bar{V}_{s,t} \setminus W) \neq 0$ if and only if $i = 2s+1, 2s+2$. But $4 < 2s \leq 2s+1$, so that, in particular

$$H_c^3(\bar{V}_{s,t} \setminus W) = H_c^4(\bar{V}_{s,t} \setminus W) = 0. \tag{10}$$

We have a long exact sequence of étale cohomology with compact support (see [19], Remark 1.30, p. 94):

$$\cdots \to H_c^3(\bar{V}_{s,t} \setminus W) \to H_c^4(K^{2s+t} \setminus \bar{V}_{s,t}) \to H_c^4(K^{2s+t} \setminus W) \to H_c^4(\bar{V}_{s,t} \setminus W) \to \cdots.$$

By (10) it follows that

$$H_c^4(K^{2s+t} \setminus \bar{V}_{s,t}) \simeq H_c^4(K^{2s+t} \setminus W). \tag{11}$$

Note that W can be described as the variety of K^{2s+t} defined by the vanishing of y_t and of all polynomials defining $\bar{V}_{s,t-1}$ in K^{2s+t-1}. Note that a point of K^{2s+t} belongs to $K^{2s+t} \setminus W$ if and only if it fulfils one of the two following complementary cases:

- either its y_t-coordinate is zero, and it does not annihilate all polynomials of $\bar{J}_{s,t-1}$, or
- its y_t-coordinate is non zero.

Therefore we have

$$K^{2s+t} \setminus W = (K^{2s+t-1} \setminus \bar{V}_{s,t-1}) \cup Z, \tag{12}$$

where the union is disjoint, and Z is the open subset given by

$$Z = K^{2s+t-1} \times (K \setminus \{0\}). \tag{13}$$

We thus have a long exact sequence of étale cohomology with compact support:

$$\cdots \to H_c^4(Z) \to H_c^4(K^{2s+t} \setminus W) \to H_c^4(K^{2s+t-1} \setminus \bar{V}_{s,t-1}) \to H_c^5(Z) \to \cdots. \tag{14}$$

By the Künneth formula for étale cohomology, (8), (9) and (13), we have $H_c^i(Z) \neq 0$ if and only if $i = 4s+2t-1, 4s+2t$. But $4s+2t-1 > 5$, whence, in particular,

$$H_c^4(Z) = H_c^5(Z) = 0.$$

It follows that (14) gives rise to an isomorphism:

$$H^4_{\rm c}(K^{2s+t} \setminus W) \simeq H^4_{\rm c}(K^{2s+t-1} \setminus \bar{V}_{s,t-1}).$$

Hence, in view of (11), claim (6) follows by induction on t if it is true that

$$H^4_{\rm c}(K^{2s} \setminus V_{s,0}) \neq 0, \tag{15}$$

where $V_{s,0} \subset K^{2s}$ denotes the variety defined by the vanishing of the two-minors of the first big block of $A_{s,t}$. But according to [9], Lemma 2′, $H^{4s-4}_{\rm et}(K^{2s} \setminus V_{s,0}) \neq 0$, from which (15) can be deduced by Poincaré Duality. This completes the proof of the theorem.

Remark 2. According to (1) and Theorem 2, the difference between the arithmetical rank and the height of $J_{s,t}$ is at least $2s+t-3-(s+t-1) = s-2$. Thus it strictly increases with s. In view of Theorem 1, it is zero if and only if $s = 2$.

Corollary 1. *The variety $V_{s,t}$ is a set-theoretic complete intersection if and only if $s = 2$.*

Next we give an upper bound for ara $J_{s,t}$. In the sequel, for the sake of simplicity, we will denote by $[ij]$ $(1 \leq i < j \leq s)$ the minor formed by the ith and the jth column of $A_{s,t}$. We will call I_s the ideal generated by these minors (it is the defining ideal of the variety $V_{s,0}$ mentioned in the proof of Theorem 2). Moreover, for all $k = 1, \ldots, 2s-3$, we set

$$S_k = \sum_{i+j=k+2} [ij].$$

We preliminarily recall an important result by Bruns et al.

Theorem 3. ([9], Theorem 2 and [10], Corollary 5.21) *With the notation just introduced,*

$$\operatorname{ara} I_s = 2s-3,$$

and

$$I_s = \sqrt{(S_1, \ldots, S_{2s-3})}.$$

We can now prove the second result of this section.

Theorem 4. *For all integers $s \geq 2$ and $t \geq 1$,*

$$\operatorname{ara} J_{s,t} \leq 2s+t-2.$$

Proof. Again, in view of Theorem 1, it suffices to prove the claim for $s \geq 3$. Let $L_{s,t}$ be the ideal generated by the products listed in Section 1 under (II) and (III). For convenience of notation we set

$$\begin{array}{rclr} \xi_i & = & x_i & (1 \leq i \leq s) \\ \xi_i & = & y_{i-s-1} & (s+1 \leq i \leq s+t). \end{array}$$

In other words, the entries of the first row of $A_{s,t}$ are denoted by $\xi_1, \ldots, \xi_{s+t}$, and the monomial generators of $L_{s,t}$ are

$$\xi_i z_j, \qquad \text{where } 1 \leq i \leq s+t-1,\ i-s+1 \leq j \leq t. \tag{16}$$

Let

$$T_h = \sum_{i=1}^{s+t-1} \xi_i z_{i+t-h} \qquad (1 \leq h \leq s+t-1),$$

where we have set $z_j = 0$ for $j \notin \{1, \ldots, t\}$. Then the set of non zero monomial summands in $T_1, \ldots, T_{s+t-1}$ coincides with the set of monomial generators of $L_{s,t}$, as the following elementary argument shows. On the one hand, given a non zero monomial summand $\xi_i z_{i+t-h}$ of some T_h, it holds

$$i - s + 1 = i + t - s - t + 1 \leq i + t - h,$$

so that $\xi_i z_{i+t-h}$ is of the form (16). On the other hand, given a monomial $\xi_i z_j$ as in (16), we have $j = i + t - h$ for $h = i + t - j$, where $j \leq t$ and $i - s + 1 \leq j$. Therefore,

$$1 \leq i \leq h \leq i + t - (i - s + 1) = s + t - 1,$$

which implies that $\xi_i z_j$ is a monomial summand of T_h.

Moreover, $T_1 = \xi_1 z_t$. Now consider, for any h such that $1 \leq h \leq s+t-1$, the product of two non zero distinct monomial summands of T_h: it is of the form $\xi_p z_{p+t-h} \xi_q z_{q+t-h}$ with $1 \leq p < q \leq s+t-1$. Hence it is divisible by $\xi_p z_{q+t-h} = \xi_p z_{p+t-(h+p-q)}$, which is one of the non zero monomial summands of T_{h+p-q}. Since $q + t - h \leq t$, we have $h - q \geq 0$, whence it follows that $1 \leq p \leq h + p - q < h$. Thus the assumption of Lemma 2 is fulfilled if we take $I = (T_1)$, P_h equal to the set of all non zero monomial summands of T_h and $q_h = T_h$ for $h = 2, \ldots, s+t-1$. Therefore

$$L_{s,t} = \sqrt{(T_1, \ldots, T_{s+t-1})}. \tag{17}$$

For some arbitrarily fixed ℓ with $1 \leq \ell \leq 2s-3$, let $[ij]$ be a summand of S_ℓ. Then the monomial terms of $[ij]$ are of the form

$$\xi_u x_v, \qquad \text{where } 1 \leq u \leq \ell + 1. \tag{18}$$

For some fixed h such that $1 \leq h \leq s+t-1$, let $\xi_i z_{i+t-h}$ be a non zero monomial summand of T_h. Then $i + t - h \geq 1$ implies that

$$h - i \leq t - 1. \tag{19}$$

For all $\ell = 1, \ldots, s-2$ let

$$U_\ell = S_\ell + T_{\ell+t+1}. \tag{20}$$

Then, if $\xi_u x_v$ is a monomial term in S_ℓ and $\xi_i z_{i+t-(\ell+t+1)}$ a non zero monomial summand in $T_{\ell+t+1}$, their product is divisible by

$$\xi_u z_{i+t-(\ell+t+1)} = \xi_u z_{u+t-(\ell+t+1+u-i)}. \tag{21}$$

Set $h' = \ell + t + 1 + u - i$. Now, according to (18), $u \leq \ell + 1$, so that, applying (19) for $h = \ell + t + 1$, we obtain $h' = \ell + t + 1 - i + u \leq t - 1 + \ell + 1 = \ell + t$. On the other hand, since $z_{i+t-(\ell+t+1)} \neq 0$, we have $i + t - (\ell + t + 1) \leq t$, i.e., $\ell + t + 1 \geq i$. This implies that $h' = \ell + t + 1 + u - i \geq u \geq 1$. Thus (21) shows that the product of each two-minor appearing as a summand in S_ℓ and each non zero monomial summand of $T_{\ell+t+1}$ is divisible by a monomial summand of $T_{h'}$, for some h' such that $1 \leq h' < \ell+t+1$. Thus Lemma 1 applies to $I = (T_1, \ldots, T_{t+1})$, $P'_\ell = \{S_\ell, T_{\ell+t+1}\}$ and $q'_\ell = U_\ell$ for $\ell = 1, \ldots, s-2$, whence, in view of (20), we conclude that

$$\begin{aligned} \sqrt{(T_1, \ldots, T_{t+1}, U_1, \ldots, U_{s-2})} &= \\ \sqrt{(T_1, \ldots, T_{t+1}, T_{t+2}, \ldots, T_{s+t-1}, S_1, \ldots, S_{s-2})} &= \sqrt{L_{s,t} + (S_1, \ldots, S_{s-2})}, \end{aligned}$$

where the last equality is a consequence of (17). Thus we have

$$\begin{aligned} \sqrt{(T_1, \ldots, T_{t+1}, U_1, \ldots, U_{s-2}, S_{s-1}, \ldots, S_{2s-3})} &= \\ \sqrt{L_{s,t} + (S_1, \ldots, S_{2s-3})} &= \sqrt{L_{s,t} + I_s} = J_{s,t}, \end{aligned} \tag{22}$$

where the second equality follows from Theorem 3. Since the number of generators of the ideal in (22) is $t + 1 + 2s - 3 = 2s + t - 2$, this completes the proof.

The gap between the lower bound given in Theorem 2 and the upper bound given in Theorem 4 is equal to 1. Theorem 1 also shows that the lower bound is sharp.

Corollary 2. *For all integers $s \geq 2$ and $t \geq 1$,*

$$2s + t - 3 \leq \operatorname{ara} J_{s,t} \leq 2s + t - 2.$$

If $s = 2$, then the first inequality is an equality.

There are other cases where the lower bound is sharp. In fact it is the exact value of $\operatorname{ara} J_{s,1}$ for $s = 3, 4, 5$, i.e., we have $\operatorname{ara} J_{3,1} = 4$, $\operatorname{ara} J_{4,1} = 6$, $\operatorname{ara} J_{5,1} = 8$. This is what we are going to show in the next example: it will suffice to produce, in the three aforementioned cases 4, 5 and 6 defining polynomials, respectively.

Example 2. With the notation introduced above, we have

$$A_{3,1} = \left(\begin{array}{ccc||c} x_1 & x_2 & x_3 & y_0 \\ x_4 & x_5 & x_6 & z_1 \end{array} \right),$$

and

$$J_{3,1} = ([12], [13], [23], x_1z_1, x_2z_1, x_3z_1),$$

where

$$[12] = x_1x_5 - x_2x_4, \qquad [23] = x_2x_6 - x_3x_5, \qquad [13] = x_1x_6 - x_3x_4.$$

We show that four defining polynomials are:

$$\begin{aligned} F_1 &= [23] \\ F_2 &= \quad x_1z_1 + x_4[12] \\ F_3 &= [13] + x_2z_1 + x_5[12] \\ F_4 &= \quad x_3z_1 + x_6[12]. \end{aligned}$$

Since $F_1, F_2, F_3, F_4 \in J_{3,1}$, by virtue of Hilbert's Nullstellensatz it suffices to prove that every $\mathbf{v} = (x_1, \dots, x_6, z_1) \in K^7$ which annihilates all four polynomials annihilates all generators of $J_{3,1}$. In the sequel, we will use, when this does not cause any confusion, the same notation for the polynomials and for their values at $\mathbf{v}$. From $F_1 = 0$ we immediately get $[23] = 0$. Moreover, since $\mathbf{v}$ annihilates F_2, F_3, F_4, we have that the triple $([13], z_1, [12])$ is a solution of the 3×3 system of homogeneous linear equations associated with the matrix

$$\begin{pmatrix} 0 & x_1 & x_4 \\ 1 & x_2 & x_5 \\ 0 & x_3 & x_6 \end{pmatrix},$$

whose determinant is

$$\Delta = -x_1x_6 + x_3x_4 = -[13].$$

By Cramer's Rule, whenever $\Delta \neq 0$, the only solution is the trivial one, so that, in particular, $[13] = 0$, a contradiction. Thus we always have $\Delta = 0$, i.e., $[13] = 0$. Hence, in view of Lemma 1, $F_2 = F_3 = 0$ implies that $[12] = x_1z_1 = x_2z_1 = 0$. Consequently, $F_4 = 0$ implies that $x_3z_1 = 0$. Thus $\mathbf{v}$ annihilates all generators of $J_{3,1}$, as required. This shows that $\operatorname{ara} J_{3,1} = 4$.

Now consider

$$A_{4,1} = \left(\begin{array}{cccc||c} x_1 & x_2 & x_3 & x_4 & y_0 \\ x_5 & x_6 & x_7 & x_8 & z_1 \end{array} \right).$$

By Theorem 3 we have

$$\begin{aligned} J_{4,1} &= ([12], [13], [14], [23], [24], [34], x_1z_1, x_2z_1, x_3z_1, x_4z_1), \\ &= \sqrt{([12], [13], [14] + [23], [24], [34], x_1z_1, x_2z_1, x_3z_1, x_4z_1)}, \end{aligned} \tag{23}$$

where

$$[12] = x_1x_6 - x_2x_5, \qquad [13] = x_1x_7 - x_3x_5, \qquad [14] = x_1x_8 - x_4x_5,$$

$$[23] = x_2x_7 - x_3x_6, \qquad [24] = x_2x_8 - x_4x_6, \qquad [34] = x_3x_8 - x_4x_7.$$

Six defining polynomials are:

$$\begin{aligned}
F_1 &= [24]\\
F_2 &= [14]+[23]\\
F_3 &= [34]+ \qquad\quad x_1z_1+x_5[12]\\
F_4 &= \qquad\quad [13]+x_2z_1+x_6[12]\\
F_5 &= \qquad\qquad\qquad x_3z_1+x_7[12]\\
F_6 &= \qquad\qquad\qquad x_4z_1+x_8[12].
\end{aligned}$$

Suppose that all these polynomials vanish at $\mathbf{v}=(x_1,\dots,x_8,z_1)\in K^9$. We show that then $\mathbf{v}$ annihilates all generators of the ideal appearing under the radical sign in (23). From $F_1=F_2=0$ we get that $[24]=[14]+[23]=0$. Moreover, since $\mathbf{v}$ annihilates $F_3,\dots,F_6$, we have that the quadruple $([34],[13],z_1,[12])$ is a solution of the 4×4 system of homogeneous linear equations associated with the matrix

$$\begin{pmatrix} 1 & 0 & x_1 & x_5\\ 0 & 1 & x_2 & x_6\\ 0 & 0 & x_3 & x_7\\ 0 & 0 & x_4 & x_8 \end{pmatrix},$$

whose determinant is

$$\Delta = x_3x_8-x_4x_7=[34].$$

By Cramer's Rule, if $\Delta\neq 0$, the only solution is the trivial one, so that, in particular, $[34]=0$, a contradiction. Hence we always have $\Delta=0$, i.e., $[34]=0$. Hence, in analogy to what has been shown for $J_{3,1}$, $F_3=F_4=F_5=0$ implies that $[13]=x_1z_1=x_2z_1=x_3z_1=[12]=0$. Consequently, $F_6=0$ implies that $x_4z_1=0$. Thus $\mathbf{v}$ annihilates all generators of the ideal in (23), as required. This shows that $\operatorname{ara} J_{4,1}=6$.

Finally consider

$$A_{5,1}=\left(\begin{array}{ccccc||c} x_1 & x_2 & x_3 & x_4 & x_5 & y_0\\ x_6 & x_7 & x_8 & x_9 & x_{10} & z_1 \end{array}\right).$$

By Theorem 3 we have

$$\begin{aligned}
&J_{5,1}=\\
&([12],[13],[14],[15],[23],[24],[25],[34],[35],[45],x_1z_1,x_2z_1,x_3z_1,x_4z_1,x_5z_1)=\\
&\sqrt{([12],[13],[14]+[23],[15]+[24],[25]+[34],[35],[45],x_1z_1,x_2z_1,x_3z_1,x_4z_1,x_5z_1)}.
\end{aligned}$$

where

$$[12]=x_1x_7-x_2x_6,\ [13]=x_1x_8-x_3x_6,\ [14]=x_1x_9-x_4x_6,\ [15]=x_1x_{10}-x_5x_6,$$

$$[23]=x_2x_8-x_3x_7,\ [24]=x_2x_9-x_4x_7,\ [25]=x_2x_{10}-x_5x_7,\ [34]=x_3x_9-x_4x_8,$$

$$[35]=x_3x_{10}-x_5x_8,\ [45]=x_4x_{10}-x_5x_9.$$

Eight defining polynomials are:

$$\begin{array}{rcl} F_1 & = & [14]+[23] \\ F_2 & = & [15]+[24] \\ F_3 & = & [25]+[34] \\ F_4 & = & [35] + x_1z_1 + x_6[12] \\ F_5 & = & [13] + x_2z_1 + x_7[12] \\ F_6 & = & [45] + x_3z_1 + x_8[12] \\ F_7 & = & x_4z_1 + x_9[12] \\ F_8 & = & x_5z_1 + x_{10}[12]. \end{array}$$

Suppose that all these polynomials vanish at $\mathbf{v} = (x_1, \ldots, x_{10}, z_1) \in K^{11}$. We show that then $\mathbf{v}$ annihilates all generators of the ideal appearing under the radical sign in (24). From $F_1 = F_2 = F_3 = 0$ we get that $[14] + [23] = [15] + [24] = [25] + [34] = 0$. Moreover, since $\mathbf{v}$ annihilates $F_4, \ldots, F_8$, we have that the 5-uple $([45], [35], [13], z_1, [12])$ is a solution of the 5×5 system of homogeneous linear equations associated with the matrix

$$\begin{pmatrix} 0 & 1 & 0 & x_1 & x_6 \\ 0 & 0 & 1 & x_2 & x_7 \\ 1 & 0 & 0 & x_3 & x_8 \\ 0 & 0 & 0 & x_4 & x_9 \\ 0 & 0 & 0 & x_5 & x_{10} \end{pmatrix},$$

whose determinant is

$$\Delta = x_4x_{10} - x_5x_9 = [45].$$

By Cramer's Rule, if $\Delta \neq 0$, the only solution is the trivial one, so that, in particular, $[45] = 0$, a contradiction. Hence we always have $\Delta = 0$, i.e., $[45] = 0$. Hence, in analogy to what has been shown for $J_{4,1}$, $F_4 = F_5 = F_6 = F_8 = 0$ implies that $[13] = [35] = x_1z_1 = x_2z_1 = x_3z_1 = x_5z_1 = [12] = 0$. Consequently, $F_7 = 0$ implies that $x_4z_1 = 0$. Thus $\mathbf{v}$ annihilates all generators of the ideal in (24), as required. This shows that $\operatorname{ara} J_{5,1} = 8$.

4. On cohomological dimensions

Recall that, for any proper ideal I of R, the *(local) cohomological dimension* of I is defined as the number

$$\begin{array}{rcl} \operatorname{cd} I & = & \max\{i | H_I^i(R) \neq 0\}, \\ & = & \min\{i | H_I^j(M) = 0 \text{ for all } j > i \text{ and all } R\text{-modules } M\}, \end{array}$$

where H_I^i denotes the ith right derived functor of the local cohomology functor Γ_I; we refer to Brodmann and Sharp [7] or to Huneke and Taylor [17] for an extensive exposition of this subject. In this section we will determine $\operatorname{cd} J_{s,t}$ for all integers $s \geq 2$ and $t \geq 1$. We will use the following technical results on De Rham (H_{DR})

and singular cohomology (H) with respect to the coefficient field $\mathbb{C}$. The first involves sheaf cohomology (see [7], Chapter 20, or [17], Section 2.3) with respect to the structure sheaf $\tilde{R}$ of K^N. The second result is analogous to Lemma 3.

Lemma 4. ([16], Proposition 7.2) *Let $V \subset K^N$ be a non singular complex variety of dimension d such that $H^i(V, \tilde{R}) = 0$ for all $i \geq r$. Then $H^i_{\mathrm{DR}}(V, \mathbb{C}) = 0$ for all $i \geq d + r$.*

Lemma 5. ([9], Lemma 3) *Let $W \subset \tilde{W}$ be affine complex varieties such that $\tilde{W} \setminus W$ is non singular of pure dimension d. If there are r polynomials $F_1, \ldots, F_r$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_r)$, then*

$$H^{d+i}(\tilde{W} \setminus W, \mathbb{C}) = 0$$

for all $i \geq r$.

We also recall that, for every proper ideal I of R,

$$\operatorname{cd} I \leq \operatorname{ara} I, \tag{24}$$

which is shown in [15], Example 2, p. 414 (and also in [7], Corollary 3.3.3, and in [17], Theorem 4.4). Equality holds if I is generated by a regular sequence, in which case the arithmetical rank is equal to the length of that sequence.

In the proof of the next result we will use the well known characterization of local cohomology in terms of Koszul (or Čech) cohomology (see [7], Section 5.2, or [17], Section 2.1). Let $u_1, \ldots, u_h \in R$ be non zero generators of the proper ideal I of R. For all $S \subset \{1, \ldots, h\}$ let R_S denote the localization of R with respect to the multiplicative set of R generated by $\{u_i | i \in S\}$; set $R_\emptyset = R$. Then, according to [17], Theorem 2.10, or [7], Theorem 5.1.19, for all $i \geq 0$, $H^i_I(R)$ is isomorphic to the ith cohomology module of a cochain complex $(C^\cdot, \phi.)$ of R-modules constructed as follows (see [7], Proposition 5.1.5). For all $i \geq 0$, set

$$C^i = \bigoplus_{\substack{S \subset \{1,\ldots,h\} \\ |S| = i}} R_S.$$

Given any $\alpha \in C^i$, for all $i \geq 1$ and all $S \subset \{1, \ldots, h\}$ such that $|S| = i$, α_S will denote the component of α in R_S. The map $\phi_{i-1} : C^{i-1} \to C^i$ is defined in such a way that, for every $\alpha \in C^{i-1}$, and for all $S \subset \{1, \ldots, h\}$ for which $|S| = i$,

$$\phi_{i-1}(\alpha)_S = \sum_{k \in S} c_{S,k} \frac{\alpha_{S \setminus \{k\}}}{1},$$

where $c_{S,k} \in \{-1, 1\}$ and $\frac{\alpha_{S \setminus \{k\}}}{1}$ is the image of $\alpha_{S \setminus \{k\}}$ under the localization map $R_{S \setminus \{k\}} \to R_S$.

Lemma 6. *Let z be one of the indeterminates of R and let I be an ideal of R generated by polynomials in which z does not occur. Then, for all $i \geq 0$,*

(i) *z is regular on $H^i_I(R)$;*
(ii) *if $H^i_I(R) \neq 0$, then $H^i_I(R) \neq zH^i_I(R)$.*

Proof. Let $u_1, \ldots, u_h$ be non zero generators of I not containing the indeterminate z. Let $S \subset \{1, \ldots, h\}$. In this proof, we will say that an element $a \in R_S$ does not contain the indeterminate z if

$$a = \frac{f}{\prod_{k \in S} u_k^{s_k}},$$

where $f \in R$ is a polynomial not containing the indeterminate z. This definition is of course independent of the choice of f and of the exponents s_k. Moreover, there is a unique decomposition

$$a = \bar{a} + z\tilde{a}$$

such that $\bar{a}, \tilde{a} \in R_S$ and $\bar{a}$ does not contain z. Given $\alpha \in C_i$, for some $i \geq 0$, we will set $\bar{\alpha} = (\bar{\alpha}_S)_S$ and $\tilde{\alpha} = (\tilde{\alpha}_S)_S$, so that we have

$$\alpha = \bar{\alpha} + z\tilde{\alpha}. \tag{25}$$

We will say that α is z-free whenever $\alpha = \bar{\alpha}$. The decomposition (25) is unique, and will be called the z-decomposition of α. From the definition of ϕ_i it immediately follows that if α is z-free, so is $\phi_i(\alpha)$. Hence

$$\phi_i(\alpha) = \phi_i(\bar{\alpha}) + z\phi_i(\tilde{\alpha}) \tag{26}$$

is the z-decomposition of $\phi_i(\alpha)$. We thus have, for all $\alpha \in C_i$,

$$\alpha \in \operatorname{Ker} \phi_i \Longleftrightarrow \bar{\alpha}, \tilde{\alpha} \in \operatorname{Ker} \phi_i, \tag{27}$$

$$\alpha \in \operatorname{Im} \phi_{i-1} \Longleftrightarrow \bar{\alpha}, \tilde{\alpha} \in \operatorname{Im} \phi_{i-1}. \tag{28}$$

Let $\alpha \in C_i$. First suppose that $z\alpha \in \operatorname{Im} \phi_{i-1}$. Then, for some $\beta \in C_{i-1}$, $z\alpha = \phi_{i-1}(\beta) = \phi_{i-1}(\bar{\beta}) + z\phi_{i-1}(\tilde{\beta})$, whence $\phi_{i-1}(\bar{\beta}) = 0$ and $\alpha = \phi_{i-1}(\tilde{\beta})$. Thus $\alpha \in \operatorname{Im} \phi_{i-1}$. This proves part (i) of the claim. Now suppose that $H^i_I(R) \neq 0$. Then there is $\alpha \in \operatorname{Ker} \phi_i$ such that $\alpha \notin \operatorname{Im} \phi_{i-1}$. From (27) and (28) we can easily deduce that one can choose α to be z-free. Suppose that $\alpha \in \operatorname{Im} \phi_{i-1} + z \operatorname{Ker} \phi_i$, i.e., $\alpha = \phi_{i-1}(\beta) + z\alpha'$ for some $\beta \in C_{i-1}$, $\alpha \in \operatorname{Ker} \phi_i$. By the uniqueness of the z-decomposition of α it follows that $\alpha = \phi_{i-1}(\bar{\beta})$, a contradiction. This shows that $\operatorname{Ker} \phi_i \neq \operatorname{Im} \phi_{i-1} + z \operatorname{Ker} \phi_i$, so that $H^i_I(R) \neq zH^i_I(R)$. This shows part (ii) of the claim and completes the proof.

Lemma 7. *Let I be a proper ideal of R generated by polynomials in which the indeterminate z does not occur. Then*

$$\operatorname{cd}(I + (z)) = \operatorname{cd} I + 1.$$

Proof. The claim for $I = (0)$ is true because, by the observation following (24), we have that $\operatorname{cd}(z) = 1$. So assume that $I \neq (0)$. Set $d = \operatorname{cd} I$. We prove the claim by showing the two inequalities separately. We have the following exact sequence, the so-called Brodmann sequence (see [17], Theorem 3.2):

$$\cdots \to H_I^{i-1}(R_z) \to H^i_{I+(x)}(R) \to H^i_I(R) \to H^i_I(R_z) \to \cdots .$$

We deduce that $H^i_{I+(x)}(R) = 0$ whenever $H_I^{i-1}(R_z) = H^i_I(R) = 0$, which is certainly the case if $i > d+1$. It follows that $\operatorname{cd}(I+(z)) \leq d+1$. By virtue of Lemma 6, part (i), multiplication by z on $H^d_I(R)$ gives rise to a short exact sequence

$$0 \to H^d_I(R) \to H^d_I(R) \to H^i_I(R)/zH^d_I(R) \to 0$$

from which, in turn, we obtain the long exact sequence of local cohomology:

$$\cdots \to H^0_{(z)}(H^d_I(R)) \to H^0_{(z)}(H^d_I(R)/zH^d_I(R)) \to H^1_{(z)}(H^d_I(R)) \to \cdots . \tag{29}$$

Now $H^0_{(z)}(H^d_I(R)) \simeq \Gamma_{(z)}(H^d_I(R)) = 0$, because z is regular on $H^d_I(R)$ by Lemma 6, part (i). Moreover, $H^0_{(z)}(H^d_I(R)/zH^d_I(R)) \simeq \Gamma_{(z)}(H^d_I(R)/zH^d_I(R)) = H^d_I(R)/zH^d_I(R)$, since $H^d_I(R)/zH^d_I(R)$ is annihilated by z. Hence, by Lemma 6, part (ii), we deduce that $H^0_{(z)}(H^d_I(R)/zH^d_I(R)) \neq 0$. Therefore, from (29) it follows that

$$H^1_{(z)}(H^d_I(R)) \neq 0,$$

whereas from (24) we know that

$$H^i_{(z)}(H^d_I(R)) = 0 \qquad \text{for all } i > 1.$$

We have a Grothendieck spectral sequence for local cohomology (see [24], Theorem 11.38, or [18], Theorem 12.10),

$$E_2^{pq} = H^p_{(z)}(H^q_I(R)) \Rightarrow H^{p+q}_{I+(z)}(R).$$

The maximum value of $p+q$ for which $E_2^{pq} \neq 0$ is $d+1$ and is obtained only for $p = 1$ and $q = d$. Thus we get

$$H^{d+1}_{I+(z)}(R) \neq 0,$$

which yields $\operatorname{cd}(I+(z)) \geq d+1$. This completes the proof.

Before coming to the main result of this section, we first show one special case of its claim. This case deserves to be considered separately, because it is the only one where the cohomological dimension is independent of the characteristic of the ground field. The next proposition is an application of a recent result by Morales [21].

Proposition 1. *Let $t \geq 1$ be an integer. Then*

$$\operatorname{cd} J_{2,t} = t+1.$$

Proof. We refer to the prime decomposition of $J_{2,t}$ given in Section 1. Since $\mathcal{M} = \{x_1x_4 - x_2x_3\}$, all ideals $J_0, J_1, \ldots, J_t$ are complete intersections. According to [21], Theorem 4, this implies that

$$\operatorname{cd} J_{2,t} = \max_{j=1,\ldots,t} \dim_K(\langle \mathcal{P}_i \rangle + \langle \mathcal{D}_{i-1} \rangle) - 1. \tag{30}$$

Here the angle brackets denote linear spaces. It is evident from their definition that, for all $i = 1, \ldots, t$, $\mathcal{P}_i$ and $\mathcal{D}_{i-1}$ are disjoint sets of $i+1$ and $t-i+1$ indeterminates, respectively. Hence $\dim_K(\langle \mathcal{P}_i \rangle + \langle \mathcal{D}_{i-1} \rangle) = |\mathcal{P}_i| + |\mathcal{D}_{i-1}| = t+2$ for all $i = 1, \ldots, t$, whence, in view of (30), the claim follows.

Theorem 5. *Let $s \geq 2$ and $t \geq 1$ be integers. Then*
(a) *if* $\operatorname{char} K > 0$, $\operatorname{cd} J_{s,t} = s + t - 1$,
(b) *if* $\operatorname{char} K = 0$, $\operatorname{cd} J_{s,t} = 2s + t - 3$.

Proof. Claim (a) follows from (1) and [23], Proposition 4.1, p. 110, since $J_{s,t}$ is Cohen-Macaulay. We prove claim (b) by induction on t. Suppose that $\operatorname{char} K = 0$. The claim for $s = 2$ and any integer $t \geq 1$ is given by Proposition 1. Next we consider the case where $s = 3$ and $t = 1$. We have $\operatorname{cd} J_{3,1} \leq 4$: this follows from (24), since we have seen in Example 1 that $\operatorname{ara} J_{3,1} = 4$. The same inequality has also been proven, by other means, in [2], Example 6. In order to prove the opposite inequality, we have to show that

$$H^4_{J_{3,1}}(R) \neq 0. \tag{31}$$

By virtue of the flat basis change property of local cohomology (see [7], Theorem 4.3.2, or [17], Proposition 2.11 (1)), if this is true for $K = \mathbb{C}$, it remains true if K is replaced by $\mathbb{Z}$; then the same property allows us to conclude that it also holds for any algebraically closed field K of characteristic zero.

So let us prove the claim (31) for $K = \mathbb{C}$. As a consequence of Deligne's Correspondence Theorem (see [7], Theorem 20.3.11) for all indices i we have

$$H^i_{J_{3,1}}(R) \simeq H^{i-1}(\mathbb{C}^7 \setminus V_{3,1}, \tilde{R}).$$

Hence our claim can be restated equivalently as

$$H^3(\mathbb{C}^7 \setminus V_{3,1}, \tilde{R}) \neq 0.$$

Therefore, in view of Lemma 4, it suffices to show that

$$H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus V_{3,1}, \mathbb{C}) \neq 0, \tag{32}$$

a statement that is the De Rham analogue to (4) for $s = 3$, $t = 1$. For the sake of simplicity, we will omit the coefficient group $\mathbb{C}$ in the rest of the proof. Let $W \subset K^7$ be the variety defined as in the proof of Theorem 2, which in our present case is contained in $V_{3,1}$ and can be identified with the subvariety $V_{3,0}$ of K^6. By (7) we also have

$$V_{3,1} \setminus W \simeq \mathbb{C}^3 \times (\mathbb{C} \setminus \{0\}), \tag{33}$$

which is obviously non singular and pure-dimensional. It is well known that

$$H^i(\mathbb{C}^r) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{else,} \end{cases} \tag{34}$$

and

$$H^i(\mathbb{C}^r \setminus \{0\}) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0, 2r-1 \\ 0 & \text{else.} \end{cases} \tag{35}$$

Now, by (33) and the Künneth formula for singular cohomology (see [26], Theorem 3.6.1),

$$H^i(V_{3,1} \setminus W) \simeq \bigoplus_{h+k=i} H^h(\mathbb{C}^3) \otimes H^k(\mathbb{C} \setminus \{0\}),$$

so that, by (34) and (35), $H^i(V_{3,1} \setminus W) \neq 0$ if and only if $i = 0, 1$. In particular

$$H^4(V_{3,1} \setminus W) = H^5(V_{3,1} \setminus W) = 0. \tag{36}$$

Since, by (33), the set $V_{3,1} \setminus W$ is a closed non singular subvariety of $\mathbb{C}^7 \setminus W$ of codimension 3, by [16], Theorem 8.3, we have the following long exact sequence, which is the Gysin sequence for De Rham cohomology:

$$\cdots \to H^4_{\mathrm{DR}}(V_{3,1} \setminus W) \to H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W) \to H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus V_{3,1}) \to H^5_{\mathrm{DR}}(V_{3,1} \setminus W) \to \cdots \tag{37}$$

Now De Rham cohomology coincides with singular cohomology on non singular varieties, by virtue of Grothendieck's Comparison Theorem (see [13], Theorem 1′, or [16], Theorem, p. 147). Therefore, from (36) it follows that the leftmost and the rightmost terms in (37) vanish. Consequently,

$$H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W) \simeq H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus V_{3,1}). \tag{38}$$

In view of (38), our claim (32) will follow once we have proven that

$$H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W) \neq 0. \tag{39}$$

This is what we are going to show next. Recall from (12) and (13) that $\mathbb{C}^7 \setminus W = (\mathbb{C}^6 \setminus V_{3,0}) \cup Z$, where the union is disjoint and

$$Z = \mathbb{C}^6 \times (\mathbb{C} \setminus \{0\}) \tag{40}$$

is a open subset of $\mathbb{C}^7 \setminus W$. We thus have the following Gysin sequence of De Rham cohomology:

$$\cdots \to H^9_{\mathrm{DR}}(Z) \to H^8_{\mathrm{DR}}(\mathbb{C}^6 \setminus V_{3,0}) \to H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W) \to H^{10}_{\mathrm{DR}}(Z) \to \cdots, \tag{41}$$

where by the Künneth formula for singular cohomology, (34), (35) and (40),

$$H^9_{\mathrm{DR}}(Z) = H^{10}_{\mathrm{DR}}(Z) = 0.$$

It follows that (41) gives rise to an isomorphism:

$$H^8_{\mathrm{DR}}(\mathbb{C}^6 \setminus V_{3,0}) \simeq H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W).$$

But from [9], Lemma 2, we know that $H^8(\mathbb{C}^6 \setminus V_{3,0}) \neq 0$, so that $H^{10}_{\mathrm{DR}}(\mathbb{C}^7 \setminus W) \neq 0$. This proves our claim (39), which implies (32) and shows claim (b) for $s = 3$, $t = 1$.

Now suppose that $s > 3$ and $t = 1$. We have

$$J_{s,1} = I_s + (x_1 z_1, \ldots, x_s z_1).$$

Hence

$$\begin{aligned} J_{s,1} + (z_1) &= I_s + (z_1), && (42)\\ (J_{s,1})_{z_1} &= (x_1, \ldots, x_s) R_{z_1}. && (43) \end{aligned}$$

We recall from [9], Corollary, that

$$\operatorname{cd} I_s = 2s - 3. \tag{44}$$

Note that the indeterminate z_1 does not occur in the minors generating I_s. Therefore, by virtue of Lemma 7 we have $\operatorname{cd}(I_s + (z_1)) = \operatorname{cd} I_s + 1$. In view of (42) and (44) it then follows that

$$\operatorname{cd}(J_{s,1} + (z_1)) = 2s - 2. \tag{45}$$

Moreover, since $x_1/1, \ldots, x_s/1$ form a regular sequence in R_{z_1}, they generate an ideal of cohomological dimension s in R_{z_1}. Thus, in view of (43), we have

$$\operatorname{cd}(J_{s,1})_{z_1} = s, \tag{46}$$

where this cohomological dimension refers to the ring R_{z_1}. We have the following Brodmann sequence:

$$\cdots \to H^{i-1}_{(J_{s,1})_{z_1}}(R_{z_1}) \to H^i_{J_{s,1}+(z_1)}(R) \to H^i_{J_{s,1}}(R) \to H^i_{(J_{s,1})_{z_1}}(R_{z_1}) \to \cdots, \tag{47}$$

where we have used the fact that, due to the independence of base property (see [7], Theorem 4.2.1, or [17], Proposition 2.11 (2)), $H^i_{J_{s,1}}(R_{z_1}) \simeq H^i_{(J_{s,1})_{z_1}}(R_{z_1})$. Moreover, by (45) and (46),

$$H^i_{J_{s,1}+(z_1)}(R) = H^i_{(J_{s,1})_{z_1}}(R_{z_1}) = 0 \qquad \text{for } i \geq 2s - 1,$$

because $s < 2s - 1$. Thus, in view of (47),

$$H^i_{J_{s,1}}(R) = 0 \qquad \text{for } i \geq 2s - 1.$$

We conclude that $\operatorname{cd} J_{s,1} \leq 2s - 2$. On the other hand, by (46),

$$H^{2s-3}_{(J_{s,1})_{z_1}}(R_{z_1}) = H^{2s-2}_{(J_{s,1})_{z_1}}(R) = 0,$$

because $s < 2s - 3$. Thus from (45) and (47) we deduce

$$0 \neq H^{2s-2}_{J_{s,1}+(z_1)}(R) \simeq H^{2s-2}_{J_{s,1}}(R),$$

which proves that $\mathrm{cd}\, J_{s,1} \geq 2s-2$, whence we obtain $\mathrm{cd}\, J_{s,1} = 2s-2$, as claimed. Up to know we have proven claim (b) for all $s \geq 3$ and $t = 1$, which settles the basis of our induction. Now we perform the induction step by assuming that $s \geq 3$, $t \geq 2$ and supposing that

$$\mathrm{cd}\, J_{s,t-1} = 2s+t-4. \tag{48}$$

We have

$$\begin{aligned} J_{s,t} + (z_t) &= J_{s,t-1} + (z_t), &(49)\\ (J_{s,t})_{z_t} &= (x_1, \dots, x_s, y_1, \dots, y_{t-2})R_{z_t}. &(50) \end{aligned}$$

Since the indeterminate z_t does not occur in the generators of $J_{s,t-1}$ and the elements $x_1/1, \dots, x_s/1, y_1/1, \dots, y_{t-2}/1$ form a regular sequence in R_{z_t}, in view of Lemma 7, the relations (48), (49) and (50) allow us to deduce that

$$\begin{aligned} \mathrm{cd}\,(J_{s,t} + (z_t)) &= 2s+t-3 &(51)\\ \mathrm{cd}\,(J_{s,t})_{z_t} &= s+t-2, &(52) \end{aligned}$$

where $s+t-2 \leq 2s+t-3$. We have the following Brodmann sequence:

$$\cdots \to H^{i-1}_{(J_{s,t})_{z_t}}(R_{z_t}) \to H^i_{J_{s,t}+(z_t)}(R) \to H^i_{J_{s,t}}(R) \to H^i_{(J_{s,t})_{z_t}}(R_{z_t}) \to \cdots. \tag{53}$$

In view of (51) and (52), in (53) we have

$$H^i_{J_{s,t}+(z_t)}(R) = H^i_{(J_{s,t})_{z_t}}(R_{z_t}) = 0 \qquad \text{for } i \geq 2s+t-2,$$

which implies that $\mathrm{cd}\, J_{s,t} \leq 2s+t-3$. Moreover, from (52) we obtain

$$H^{2s+t-4}_{(J_{s,t})_{z_t}}(R_{z_t}) = H^{2s+t-3}_{(J_{s,t})_{z_t}}(R_{z_t}) = 0,$$

since $s > 2$ implies that $2s+t-4 > s+t-2$. Therefore, in view of (51), in (53) we have

$$0 \neq H^{2s+t-3}_{J_{s,t}+(z_t)}(R) \simeq H^{2s+t-3}_{J_{s,t}}(R).$$

This yields $\mathrm{cd}\, J_{s,t} = 2s+t-3$, as claimed, and completes the proof.

Remark 3. Theorem 2 and Theorem 5 (a) show that the inequality (24) is strict for $J_{s,t}$ if $s \geq 3$ and $\mathrm{char}\, K > 0$. In fact, in this case we have

$$\mathrm{cd}\, J_{s,t} = s+t-1 < s+t+s-3 = 2s+t-3 \leq \mathrm{ara}\, J_{s,t}.$$

According to Theorem 1, however, equality always holds for $s = 2$; in turn, Theorem 5 (b) and Example 1 show that equality also holds for $s = 3, 4, 5$ and $t = 1$ provided that $\mathrm{char}\, K = 0$. The question in the remaining cases is open. In any case, Theorem 4 tells us that in characteristic zero the cohomological dimension and the arithmetical rank are close to each other, since

$$\mathrm{ara}\, J_{s,t} \leq \mathrm{cd}\, J_{s,t} + 1.$$

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