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Spatial Graphs, Knots and the Cyclic Polytope

J. L. Ramírez Alfonsín*

Université Pierre et Marie Curie, Paris 6
Combinatoire et Optimisation - Case 189
4 Place Jussieu Paris 75252 Cedex 05, France

Abstract. Let $r(K)$ be the smallest positive integer such that every linear spatial representation of the complete graph on $n \geq r(K)$ vertices contains a cycle isotopic to knot K. In this paper, we prove that $r(L) \leq 2^{8^c}$ where $c = 4^{18n-7}$ by using a special embedding of K in the cyclic polytope. We also show that $r(F_8)$, $r(T(5,2)) \geq 9$ where F_8 and $T(5,2)$ are the "Figure-eight" and the $(5,2)$-torus knots respectively.

1. Introduction

A *link* L with k components consists of k disjoint simple closed curves in $\mathbb{R}^3$. A *knot* K is a link with one component. A *link diagram* $D(L)$ is obtained from L by projecting it onto a plane in such a way that the projection of each component is smooth and at most two curves intersect at any point. At each crossing point of the link diagram the curve which goes over the other is specified. The fundamental theorem of Reidemeister [14] states:

Theorem 1.1. [14] *Two links L_1 and L_2 are isotopic if and only if any link diagram $D(L_1)$ can be transformed into any link diagram $D(L_2)$ by a finite sequence of moves I, II and II and their inverses (see Figure* 1*).*

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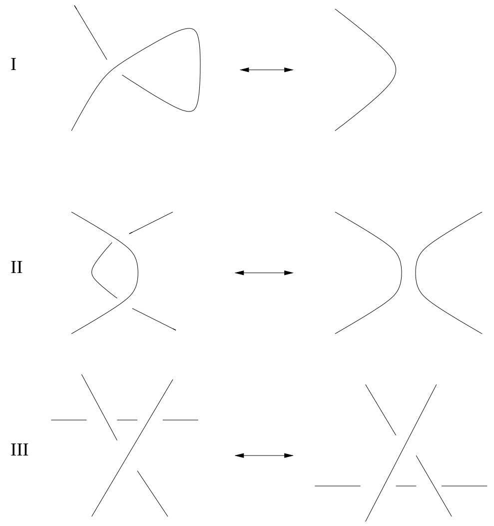


Figure 1. Reidemeister moves

A *spatial representation* $\mathcal{R}(\mathbf{K}_n)$, of the complete graph $\mathbf{K}_n$ (i.e., a graph in which any pair of vertices is joined by an edge) is the embedded image of $\mathbf{K}_n$ in $\mathbb{R}^3$, that is, the vertices of $\mathbf{K}_n$ are distinct points in $\mathbb{R}^3$ and the edges are simple Jordan curves between them in such a way that any two curves are either disjoint or meet at a common point. $\mathcal{R}(\mathbf{K}_n)$ is *linear* if each edge is represented by a straight line segment. A set of cycles of $\mathcal{R}(\mathbf{K}_n)$ can be thought as simple closed curves in $\mathbb{R}^3$ and they thus may be regarded as a link.

Let $r = r(L)$ be the smallest positive integer such that every linear $\mathcal{R}(\mathbf{K}_n)$, $n \geq r$, contains cycles isotopic to link L. S. Negami [9] showed that for any given knot, link or spatial graph $\mathcal{R}(G)$ there is a sufficiently large integer t such that every linear $\mathcal{R}(\mathbf{K}_t)$ always contains a subdivision of $\mathcal{R}(G)$. This implies the following result.

Theorem 1.2. [9] *Let K be a knot. Then, there exists a finite number $t = t(K)$ such that any linear $\mathcal{R}(\mathbf{K}_n)$, $n \geq t$, contain a cycle isotopic to K. In particular, $r(K)$ exists and it is finite.*

The idea of Negami's proof relies on a Ramsey-type result in connection to the cyclic polytope. The *cyclic polytope* of dimension d with n vertices $C_d(n) = C_d(t_1, \ldots, t_n)$ was discovered by C. Carathéodory [3, 4] and many times rediscovered; it is usually defined as the convex hull in $\mathbb{R}^d$, $d \geq 2$, of n, $n \geq d+1$, different points $x(t_1), l \ldots, x(t_n)$ of the *moment curve* $x : R \rightarrow R^d,\ t \rightarrow (t, t^2, \ldots, t^d)$. Cyclic polytopes, and simplicial neighbourly polytopes, in general, play an important role in the combinatorial convex geometry due to their connection with certain extremal problems. For example, the Upper Bound Theorem established by P. McMullen [8], says that the number of j-dimensional faces of a k-polytope with n vertices is maximal for $C_d(n)$. Throughout the rest of the paper, we will denote $C_3(n)$ by $C(n)$.

A classical Ramsey theorem states:

Theorem 1.3. [13] *Let r, k, n be given positive integers, $n \leq k$. Then, there exists a number $t = R_r(k; n)$ with the following property. If all k-subsets of $\{1, \ldots, t\}$ are colored with r colours, then there is an n-subset of $\{1, \ldots, t\}$ all of whose k-subsets have the same colour.*

The following proposition can be easily obtained from Theorem 1.3.

Proposition 1.4. [2, Proposition 9.4.7] *Let n, d be integers with $n \geq d+1 \geq 3$. Then, there exists an integer $t = t(n, d)$ such that every set of t points in general position in affine d-space contains the n vertices of a cyclic polytope of dimension d.*

Proof. Let S be a set of $t = R_2(d+1; n)$ points in $\mathbb{R}^3$ in general position and let $\mathcal{M}$ be the affine oriented matroid associated to S. Since the points are in general position then the bases of $\mathcal{M}$ are all the $(d+1)$-subset of S and they are signed with either $+$ or $-$ (which can be considered as a 2-colouring). Then, by Proposition 1.4, there is a n-subset of $\{1, \ldots, t\}$ all of whose $(d+1)$-subsets have the same color, and thus, the bases of the oriented matroid corresponding to this n-set points, have all the same sign, implying that these points are the vertices of a polytope combinatorial equivalent to $C_d(n)$. $\square$

Negami proved Theorem 1.2 by showing that there is a sufficiently large integer m such that $C(m)$ contains a cycle (that is, a sequence of line segments $l_1, \ldots, l_r, l_1$, $r \leq m$ where two consecutive segments have a common vertex belonging to $C(m)$) isotopic to a given knot and then Proposition 1.4 is applied. However, Negami's approach does not give an order of magnitude of m and thus an explicit value for $r(K)$ is not given. Let us see how Nagami proceeded for showing the existence of the desired integer m. First, a knot diagram $D(K)$ with n crossings is transformed into what Negami called an *n-plat representation* (better known as an *n-braid* representation of K – this always exists by Alexander's Theorem [1]). Then, such representation is oriented (creating *positive* and *negative* crossings). Since a *positive plat representation* (a plat representation with only positive crossings) is needed then each negative crossing is changed into $(2n+1)(n-1)$ crossings of the opposite sign. Finally, Negami constructed an embedding of a positive plat representation on $C(m)$ inductively. The induction is involved and it assumes 'certain conditions' that may increase the value of m which already depends on the number of crossings of the positive plat representation. So, the order of magnitude of m is difficult to estimate.

In the next section, we propose a completely new approach (avoiding the above mentioned drawbacks) to show the existence of the desired cycles in cyclic polytopes. Our method is constructive and yields to the following result.

Theorem 1.5. *Let L be a link and let $cr(L)$ be its crossing number (the minimum number of crossings in any diagram representing L). Then, there exist cycles in $C(9cr(L))$ isotopic to L.*

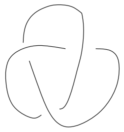
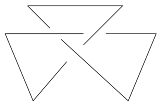
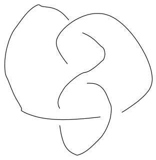
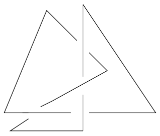
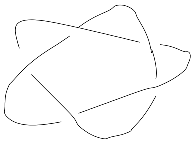
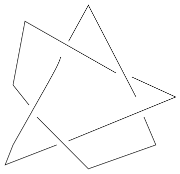
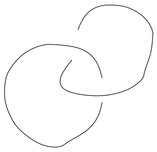
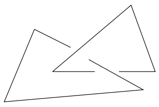
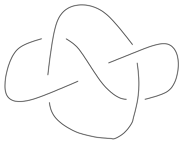
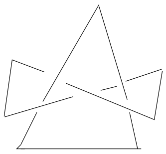
Link or Knot	Diagram	Stick representation	Sticky number
Trefoil T			6
Figure−eight F_8			7
T(5,2)			$\geqslant 7$
Hopf link 2^2_1			6
4^2_1			7

Figure 2. Basic knots and links

P. Erdös and R. Rado [5] gave an explicit upper bound for $R_r(k; n)$. By combining this upper bound (with $r = 2$ and $k = 4$), Proposition 1.4 and Theorem 1.5, the following upper bound for $r(L)$ can be obtained.

Theorem 1.6. *Let $D(L)$ be a diagram of link L with n crossings. Then, $r(L) \leq 2^{8^c}$ where $c = 4^{18n-7}$.*

Although, this is an exponentially huge upper bound, it can be stated explicitly which is not the case by using Negami's approach. Recently, A. V. Kostochka and V. Rödl [6] proved that for every $\epsilon > 0$, $R_r(k; s) \leq c(n, k, \epsilon)n^{1+\epsilon}$ where $c(n, k, \epsilon)$ depends on n, k and ϵ. This upper bound could be used to improve the upper bound for $r(L)$ however it is not easy to state it explicitly.

There is not much known about $r(L)$ for particular links L. It is clear that $r(L) \geq s(L)$ where $s(L)$ is the *sticky number* of L (that is, the smallest number of *sticks* needed to represent L in $\mathbb{R}^3$). H. Sacks [18] proved that $r(2^2_1) = 6$. In a series of three papers, N. Robertson, P. Seymour and R. Thomas [15], [16], [17] strengthened the latter by characterizing the *linklessly graphs*. In [10], we showed that $r(T \text{ or } T^*) = 7$ where T is the *Trefoil* and T^* its *mirror* (recall that the *mirror* of a knot K is obtained by a reflection of K in a plane). We also have proved [11] that $r(4_1^2) > 7$. We refer the reader to [12] for further details on knots and links in spatial graphs.

Here, we also investigate the value r for Figure-eight knot (denoted by F_8) and the $(5,2)$-torus knot (denoted by $T(5,2)$) which are the second and the third nontrivial knots respectively. As illustrated in Figure 2, we have that $s(F_8) = 7$ and $s(T(5,2)) \geq 7$. It will be proved that any cycle in $C(8)$ is isotopic to either the trivial knot (i.e., a loop) or T or T^*, implying the following theorem.

Theorem 1.7. *$r(F_8), r(T(5,2)) \geq 9$.*

2. Knots in the cyclic polytope

Let $D(C(n))$ be the projection of $C(n) = C(t_1, \ldots, t_n)$, $0 < t_1 < \cdots < t_n$ onto the xy-plane, see Figure 3 for the case $n = 8$.

Proposition 2.1. *Let $\{t_i, t_k\}$ and $\{t_j, t_l\}$ be two edges in $D(C(n))$. Then $\{t_j, t_l\}$ passes over $\{t_i, t_k\}$ if and only if $1 \leq i < j < k < l \leq n$.*

Proof. Let $\mathcal{M}$ be the affine oriented matroid of rank 4 on n elements arising from the vertices of $C(t_1, \ldots, t_n)$, $0 < t_1 < \cdots < t_n$. It is known [2] that the circuits of $\mathcal{M}$ are the signed subsets $B = \{t_i, \bar{t}_j, t_k, \bar{t}_l, t_m\}$, $1 \leq i < j < k < l < m \leq n$. It is also known that the signed circuits of affine oriented matroids are exactly the *minimal Radon partitions*, i.e., the convex hull of the positive elements intersects the convex hull of negative elements. So, circuit B of $\mathcal{M}$ implies that the triangle formed by t_i, t_k and t_m intersects the line segment formed by t_j and t_l and so the edge $\{t_j, t_l\}$ passes over $\{t_i, t_k\}$ in $D(C(n))$, see Figure 4. □

A pair of edges of $D(C(n))$ as in Proposition 2.1 are said to have the *crossing property* (we also may say the crossing verifies the crossing property). Throughout the rest of the paper, we will assume that the diagram $D(C(m))$ has all its vertices on the x-axis and all the arcs (representing the segments) lying above the x-axis and verifying the crossing property (this can easily be done by deforming continuously the plane), see Figure 5.

Lemma 2.2. *Let $D(K)$ be a diagram of knot K and let us mark with a segment the overcrossing piece of $D(K)$ at each crossing. Then, there exists a non self-intersecting curve Γ in the plane, starting and ending outside $D(K)$ (i.e., outside of a circle containing $D(K)$) such that it passes over each marked segment of $D(K)$ exactly once and it intersects $D(K)$ transversally elsewhere.*

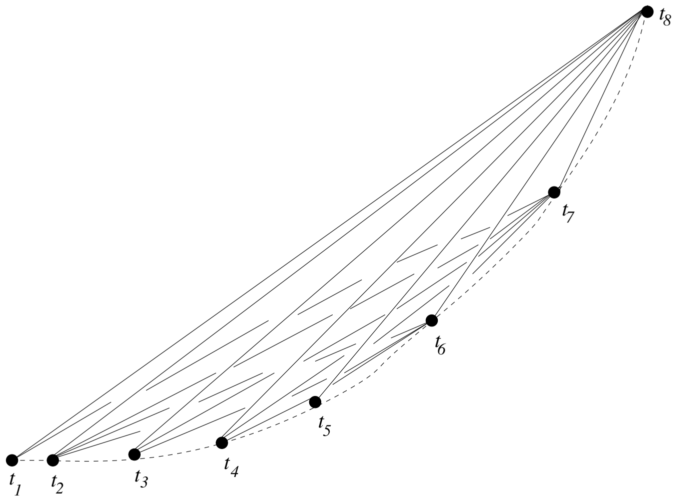


Figure 3. Projection of $C(8)$

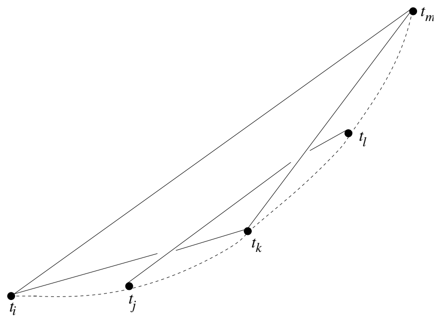


Figure 4. Edges with the crossing property

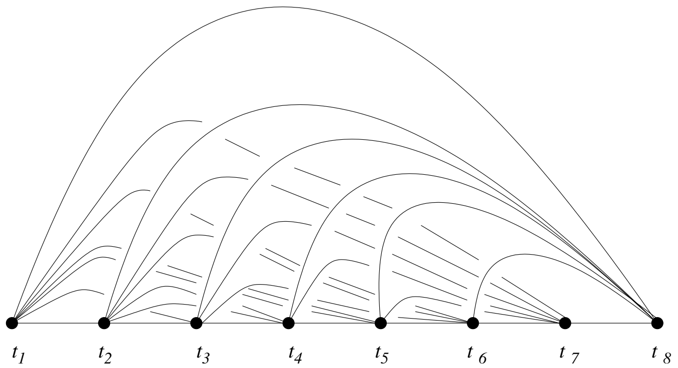


Figure 5. Representation of the diagram $D(C(8))$

Proof. Given any knot diagram $D(K)$ we note that if the over/under crossings are ignored it can be regarded as a 4-regular plane graph G_K. Accordingly, G_K is Eulerian and thus its dual plane graph consisting of faces of G_K can be 2-coloured, say with colours black and white. Since G_K is Eulerien then its set of edges can

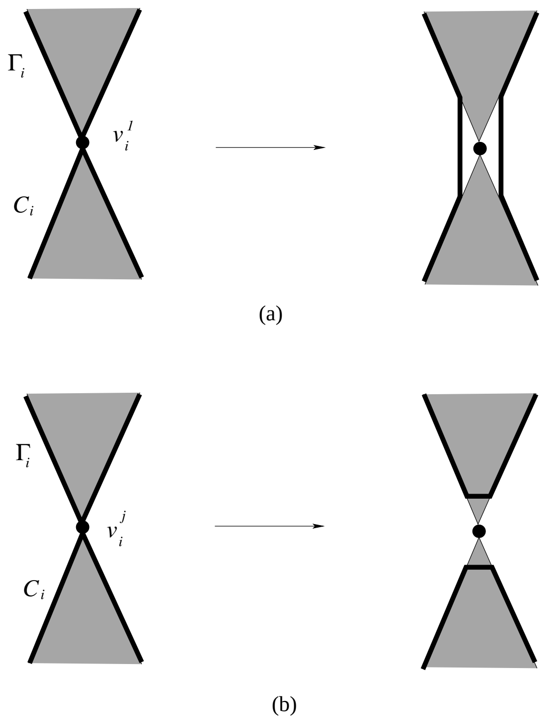


Figure 6. Modifications around a vertex for constructing Γ_h vertices.

be partitioned in edge-disjoint cycles, say $C_1, \ldots, C_h$ where each vertex of G_K belongs to exactly two different cycles (since G_K is 4-regular). We may construct the desired curve as follows. Let $C_1, \ldots, C_h$ be the set of cycles bounding the black faces of G_K. Since G_K is connected then we can suppose that the cycles $C_1, \ldots, C_h$ are ordered such that $\{V(C_1) \cup \cdots \cup V(C_{i-1})\} \cap V(C_i) \neq \emptyset$ for each $i = 2, \ldots, h$. Let $V(C_1) = \{v_1^1, \ldots, v_{f_1}^1\}$ and let $\{v_1^i, \ldots, v_{f_i}^i\} = \{V(C_1) \cup \cdots \cup V(C_{i-1})\} \cap V(C_i)$ for each $i = 2, \ldots, h$. Let Γ_{C_i} be the closed no self-intersecting curve formed by the set of edges and vertices of cycle C_i and we set $\Gamma_1 = \Gamma_{C_1}$. We shall construct curve Γ_2 with Γ_1 and Γ_{C_2} where Γ_1 and Γ_{C_2} are modified at vertex v_1^2 according to Figure 6 (a) and at vertex v_j^2, $2 \leq j \leq f_2$ according to Figure 6 (b). We shall now construct curve Γ_3 from Γ_2 and Γ_{C_3} where Γ_2 and Γ_{C_3} are modified at vertex v_1^3 according to Figure 6 (a) and at vertex v_j^3, $2 \leq j \leq f_3$ according to Figure 6(b). By carrying on this procedure we obtain a curve Γ_h.

Now, we shall make a local change to Γ_h around each crossing (according to the corresponding marked segment in the knot diagram), as it is shown in Figure 7. The desired curve Γ is obtained by breaking the modified Γ_h at a boundary face of G_K, and by moving Γ_h away from $D(K)$, say inside each black face, see Figure 8. □

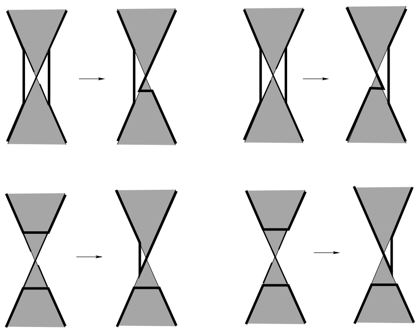


Figure 7. Local changes made to Γ_h

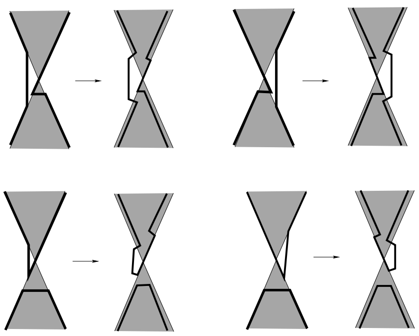


Figure 8. Moving Γ_h away from $D(K)$

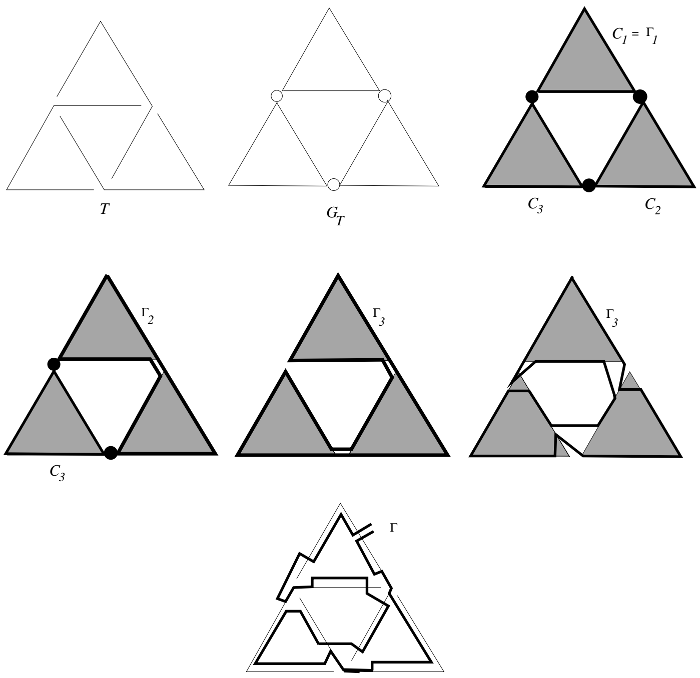


Figure 9. Construction of curve Γ for a Trefoil diagram

The construction of the curve Γ, given in Lemma 2.2, is illustrated in Figure 9 for the Trefoil.

We may now prove Theorem 1.5.

Proof of Theorem 1.5. Let $D(K)$ be a diagram of knot K with n crossings. We shall construct a cycle in $D(C(9n))$ isotopic to $D(K)$ in four steps (this construction is illustrated in Figure 11 for the Trefoil).

(1) Let Γ be the curve constructed as in the proof of Lemma 2.2 for diagram $D(K)$. Put a vertex at the extremes of each segment and each intersection between Γ and $D(K)$. Number the vertices in the order of appearance while tracing Γ in one direction. Notice that the number of vertices is at most $4n+2$ (we have at most 4 vertices for each crossing and two when breaking Γ). We deform the plane in such a way that Γ becomes a straight line, say the x-axis. We obtain a new diagram $D_1(K)$ (isotopic to $D(K)$) with all the vertices lying in the x-axis (and appearing in order from left to right), see Figure 11 (a).

(2) Put an extra vertex at each undercrossing (just below Γ) of $D_1(K)$. We then join the vertices (from left to right), with a curve Γ' and deform again the plane in such a way that Γ' becomes the x-axis. We obtain a new diagram $D_2(K)$ (isotopic to $D_1(K)$) with at most $5n+2$ vertices lying on the x-axis, see Figures 11 (b) and (c). Notice that $D_2(K)$ is composed by upper arcs (above Γ') and lower arcs (below Γ'). Also notice that $D_2(K)$ has n crossings (one for each segment) all induced by two upper arcs and the number of lower arcs is at most $3n$ (by construction of Γ, we have at most three lower arcs for each segment).

(3) Number the vertices (from left to right) of $D_2(K)$. Let $e=\{i,k\}$ and $e'=\{j,l\}$ be two upper arcs of $D_2(K)$ with $i<j<k<l$. If these arcs do not verify the crossing property (that is e crosses over e') then we may fix this as follows. Pull over arc e' (its vertices are fixed) to the left-hand side of the first point appearing in Γ' and stick it to a new vertex v on the x-axis (if while pulling e' another arc f is met then e' passes under f, and so the possible new created crossings will verify the crossing property), see Figure 10 (a). We repeat this procedure for each crossing not verifying the crossing property. We obtain a diagram $D_3(K)$ (isotopic to $D_2(K)$) with all its upper arcs verifying the crossing property and with at most $6n+2$ vertices. We notice that if we use the construction of Γ as in the proof of Lemma 2.2 then by the above numbering of vertices, all the crossings in $D_2(K)$ turned out to verify the crossing property, and thus, in this case $D_2(K)$ has at most $5n+2$ vertices.

(4) Finally, we get ride out of the lower arcs in $D_3(K)$ by doing a similar procedure as in step three. Pull over each lower arc e above Γ' and stick it to a new vertex w on the x-axis at the right-hand side of the last vertex appearing in Γ' (if while pulling e an upper arc f is met then e passes over f, and so the possible new created crossings will verify the crossing property), see Figure 10 (b).

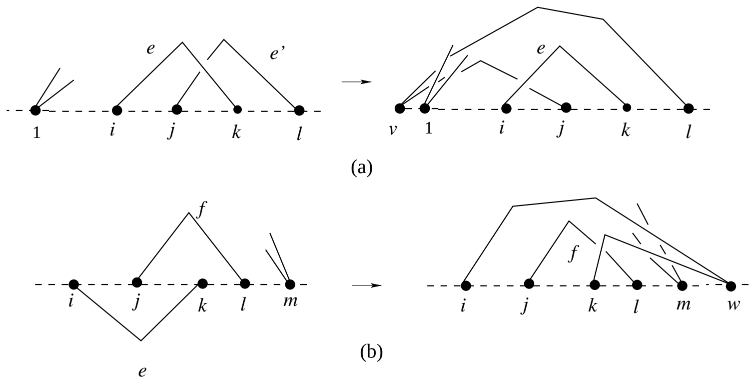


Figure 10. Fixing up upper and lower arcs

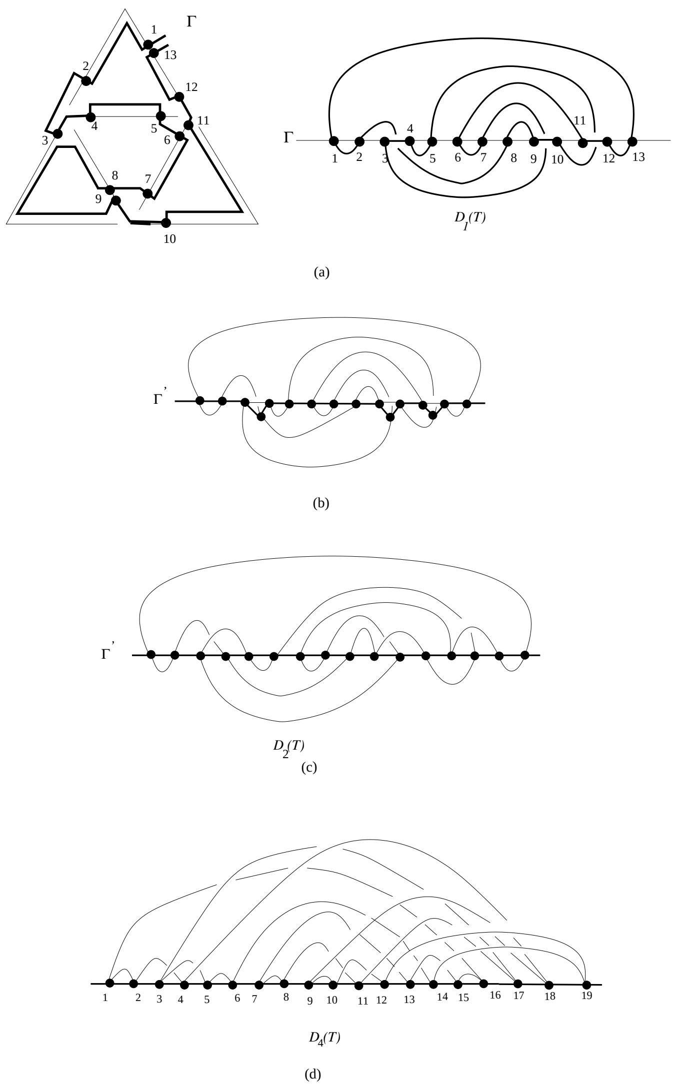


Figure 11. Construction of the Trefoil in $D(C(19))$

We repeat this procedure for each lower arc (if a lower edge joins two consecutive vertices then we may just put it above Γ' without creating an extra vertex). We obtain a diagram $D_4(K)$ (isotopic to $D_3(K)$) with at most $9n$ vertices, see Figure 11 (d). Since the diagram $D_4(K)$ is isotopic to K and it has only upper arcs (verifying the crossing property) then it corresponds to a cycle contained in the special representation of $D(C(9n))$, as desired. □

We notice that Theorem 1.5 implies that $s(K) \leq 9cr(K)$. However, the number of vertices of the cyclic polytope given in Theorem 1.5 is not optimal in general. For instance, Figure 12 shows a cycle in $C(7)$ isotopic to the Trefoil while the cycle obtained via Theorem 1.5 has 19 vertices.

Finally, we may prove Theorem 1.7.

Proof of Theorem 1.7. We shall show that the vertices of $C(8)$ (a particular linear spatial representation of $\mathbf{K}_8$) contain neither F_8 nor $T(5,2)$. We will show that any cycle in $C(8)$ is isotopic to either the trivial knot or T or T^*. Let B be a cycle in $D(C(8))$ of length $3 \leq m \leq 8$. It is easy to check that if $3 \leq m \leq 6$ then B is isotopic to the trivial knot. Let us suppose that $m = 7, 8$. We notice that, by Proposition 2.1, if B contains two consecutive vertices of the form $\ldots t_i, t_{i+1} \ldots$, $1 \leq i \leq 8$ (respectively, three consecutive vertices of the form $\ldots t_i, t_k, t_{i+1} \ldots$, $1 \leq i \leq 8$ and $k \neq i, i+1$) then these can be replaced by $\ldots t_i \ldots$ (respectively, by $\ldots t_i, t_{i+1} \ldots$) obtaining a smaller cycle isotopic to B. In this case we say that B is *reducible*. It can be checked that if B is not reducible then either

$$B \in \{(2,7,5,3,1,6,4), (1,3,7,5,2,6,4)\},$$

if $m = 7$ or

$$\begin{aligned} B \in \ & \{(1,8,3,5,7,2,4,6), (2,8,4,6,1,3,7,5), (2,8,5,3,7,1,4,6), \\ & (3,8,5,2,7,4,1,6), (3,8,6,2,4,7,1,5), (1,3,6,8,4,2,7,5), \\ & (1,3,5,7,2,4,6,8)\}. \end{aligned}$$

if $m = 8$. The result follows by verifying (via Theorem 1.1) that any of the above cycles is isotopic to either the trivial knot or T or T^*, see Figures 12 and 13. □

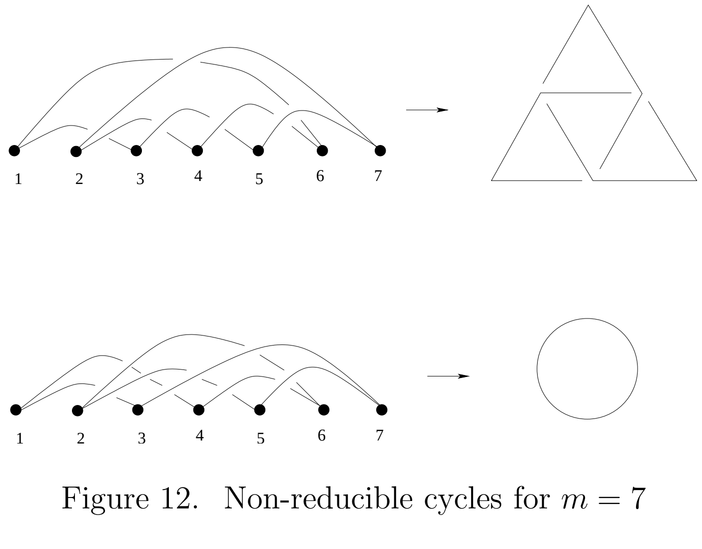


Figure 12. Non-reducible cycles for $m = 7$

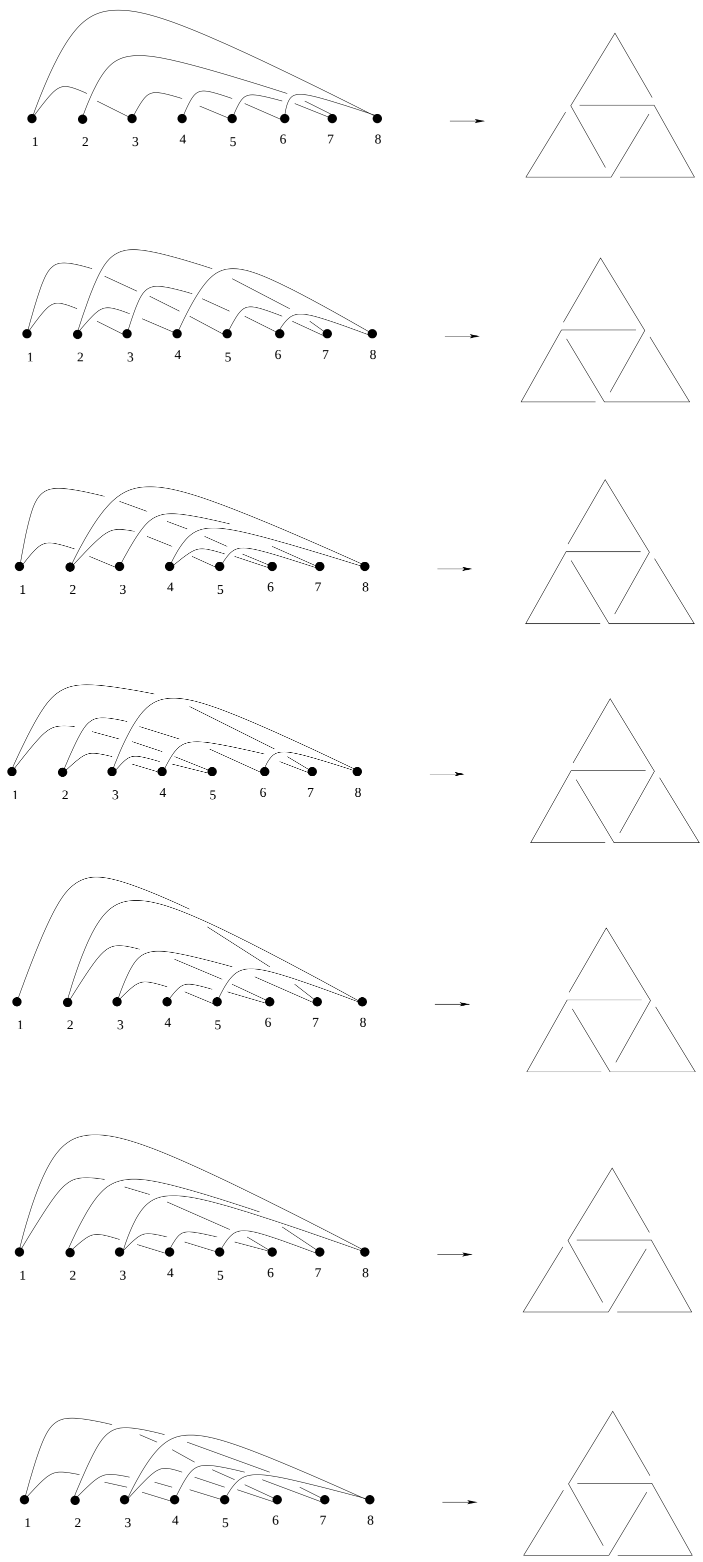


Figure 13. Non-reducible cycles for $m = 8$

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