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Twisted Projective Spaces and Linear Completions of some Partial Steiner Triple Systems

Małgorzata Prażmowska

Institute of Mathematics, University of Białystok
ul. Akademicka 2, 15-267 Białystok, Poland
e-mail: malgpraz@math.uwb.edu.pl

Abstract. A class of Steiner triple systems is characterized. Each one is obtained as a linear completion of some $\left(15_4\,20_3\right)$ multiveblen configurations and it contains some Fano planes.

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Introduction

In the paper we discuss geometrical properties of the structures which appear as solutions of two different problems. The first problem is to find and classify linear completions of some partial Steiner triple systems. The second problem is to construct linear spaces – Steiner triple systems – which are not projective spaces but are in many points similar to Fano projective spaces; in particular, they contain many projective planes.

The partial Steiner triple systems discussed in the paper are $\left(15_4\,20_3\right)$-configurations, which generalize the combinatorial Grassmannian $\mathbf{G}_2(6)$ and the dual combinatorial Veronesian $(\mathbf{V}_3(4))^\circ$ (cf. [14], [16]). A detailed classification of the general construction of some $\left(\binom{n+2}{2}_n\,\binom{n+2}{3}_3\right)$-configurations called *multiveblen configurations* can be found in [15]. In particular, for $n = 4$ the construction produces our structures of this paper which we extend to some further examples. The obtained results can be found in Section 2.

Lindner's conjecture (cf. e.g. [18], [3], [4], and many others) states that every partial Steiner triple system $\mathfrak{A}$ can be embedded into a Steiner triple system, but following Lindner we can enlarge the point set of $\mathfrak{A}$. Given an arbitrary partial linear space $\mathfrak{A}$ it is natural to ask if $\mathfrak{A}$ is a reduct of some linear space. In a more constructive fashion we ask if for a given $\left(\boldsymbol{\nu}_{\boldsymbol{r}}\, \boldsymbol{b}_{\boldsymbol{\kappa}}\right)$-configuration $\mathfrak{A}$ there exists a $\left(\boldsymbol{\nu}_{\tilde{\boldsymbol{r}}}\, \widetilde{\boldsymbol{b}}_{\boldsymbol{\kappa}}\right)$-configuration $\widetilde{\mathfrak{A}}$ such that $\widetilde{\mathfrak{A}}$ is a linear space and every line of $\mathfrak{A}$ is a line of $\widetilde{\mathfrak{A}}$ as well. If so, we call $\widetilde{\mathfrak{A}}$ a *linear completion of* $\mathfrak{A}$. Thus for $\boldsymbol{\kappa} = 3$ constructing linear completions of $\mathfrak{A}$ is a problem to find a suitable embedding of a partial Steiner triple system into a Steiner triple system (comp. similar problems considered in [5]).

A simple though important result (see 3.1) states that every multiveblen configuration on 15 points has a unique linear completion.

In general, the question is not so trivial. Indeed, though there are Steiner triple systems on 21 points, the Grassmannian $\mathbf{G}_2(7)$ (which is a $\left(21_5\, 35_3\right)$-configuration) has not a linear completion (the result will be published in another paper).

The geometry of the obtained Steiner triple systems is studied in Section 4. It turns out that all these completions can be represented as twisted projective spaces – projective Fano 3-spaces in which some original lines are deleted and lacking connections are replaced by another family of 3-subsets so that a new linear space is obtained (cf. Construction 4.11 and Representation 4.14). At least three subplanes with a common line remain unchanged along this process.

The classical Fano 3-space appears among these completions, while none of the remaining completions can be embedded into a Fano projective space.

However, the most regular among our completions contain a point p such that any two lines through it span a Fano plane so the pencil with vertex p i.e. the lines and planes through p yield a Fano plane as well (cf. Proposition 4.8).

It is worth to point out that some of our twisted projective spaces appear as particular instances of the construction introduced in [8, Section 3].

Thus our structures, constructions and investigations touch also another problem – to find and characterize structures which contain a sufficiently regular family of Fano subplanes (see [2], [8], and [12]).

1. Generalities, definitions, and basic facts

Let X be a nonempty n-element set. For every nonnegative integer k the symbol $\wp_k(X)$ stands for the set of all k-element subsets of X. Three fundamental types of graphs (nonoriented, without loops, cf. [19]) defined on X will be used in the paper. We write K_n for the complete graph $\langle X, \wp_2(X)\rangle$, N_n for the empty graph $\langle X, \emptyset\rangle$, and L_n for the linear graph $\left\langle X, \left\{\{x_i, x_{i+1}\} \colon i = 1, \ldots, n-1\right\}\right\rangle$ for some ordering $x_1, \ldots, x_n$ of the set X. If $\mathcal{P}$ is a graph defined on a set X (i.e. $\mathcal{P} \subset \wp_2(X)$) and $A \subset X$, we write $\mathcal{P} \curlywedge A$ for the restriction $\mathcal{P} \cap \wp_2(A)$ of $\mathcal{P}$ to A.

First, let us recall some standard notations from the theory of partial linear spaces. If $\mathfrak{M}$ is a partial linear space with constant point degree and line size we write

$\boldsymbol{\nu}_{\mathfrak{M}}$ for the number of its points, $\boldsymbol{b}_{\mathfrak{M}}$ for the number of its lines, $\boldsymbol{r}_{\mathfrak{M}}$ for the degree (rank) of any of its points, and $\boldsymbol{\kappa}_{\mathfrak{M}}$ for the size (rank) of any of its lines. A partial linear space $\mathfrak{M}$ on $\boldsymbol{\nu}$ points and $\boldsymbol{b}$ lines with constant point degree $\boldsymbol{r} = \boldsymbol{r}_{\mathfrak{M}}$ and line size $\boldsymbol{\kappa} = \boldsymbol{\kappa}_{\mathfrak{M}}$ is also called a $\big(\boldsymbol{\nu}_{\boldsymbol{r}}\, \boldsymbol{b}_{\boldsymbol{\kappa}}\big)$-configuration. A partial Steiner triple system is a partial linear space whose lines have size 3; consequently, every $(\boldsymbol{\nu}_{\boldsymbol{r}} \boldsymbol{b}_3)$-configuration is a partial Steiner triple system.

Next, we recall definitions and constructions of some combinatorial structures, which will be used in the paper. Let X be an arbitrary (finite) set and $|X| = n$.

Construction 1.1. (Combinatorial Grassmannian $\mathbf{G}_k(X)$ (cf. [16], [14], also [6], [13])) For any positive integer k such that $1 \leq k < n$ we put

$$\mathbf{G}_k(X) := \langle \wp_k(X), \wp_{k+1}(X), \subset \rangle.$$

We write, shortly, $\mathbf{G}_k(n) \cong \mathbf{G}_k(X)$, where $|X| = n$.

A "dual" structure, isomorphic to $\mathbf{G}_k(X)$ under the map $\varkappa\colon a \longmapsto X \setminus a$ is the structure $\mathbf{G}^*_{n-k}(X) = \langle \wp_{n-k}(X), \wp_{n-k-1}(X), \supset \rangle$.

Let $\alpha \in S_X$ i.e. let α be a permutation of X; we write $\alpha^{(m)}$ for the natural action of α on $\wp_m(X)$. Clearly, $\alpha^{(k)} \in \mathrm{Aut}(\mathbf{G}_k(X))$ and $\alpha^{(k)} \in \mathrm{Aut}(\mathbf{G}^*_k(X))$.

Fact 1.2. The structure $\mathbf{G}_k(n)$ is an $\big(\binom{n}{k}_{n-k} \binom{n}{k+1}_{k+1}\big)$-configuration. Consequently, $\mathbf{G}_k(X)$ is a partial Steiner triple system iff $k = 2$.

Example 1.3. $\mathbf{G}_2(3)$ is a single 3-element line. $\mathbf{G}_2(4) \cong \mathbf{G}^*_2(4) \cong \mathbf{V}_2(3)$ is the Veblen configuration (cf. [16]). Moreover, $\mathfrak{D}^o := \mathbf{G}_2(5) \cong \mathbf{G}^*_3(5)$ is simply the Desargues configuration (a classical (10_3)-configuration, cf. also [7]).

Construction 1.4. Let $\mathfrak{H} = \langle \wp_2(X), \mathcal{L} \rangle$ be a partial Steiner triple system and $\mathcal{P}$ be a nonoriented graph without loops defined on X. We take any two distinct elements $p_1, p_2 \notin X$ and put $p = \{p_1, p_2\}$, $X' = X \cup p$. Consider the following families of blocks:

$$\begin{aligned}
\mathcal{L}_1 &= \Big\{ \big\{ \{p_1,p_2\}, \{p_1,i\}, \{p_2,i\} \big\} : i \in X \Big\}, \\
\mathcal{L}_2 &= \Big\{ \big\{ \{i,j\}, \{p_1,i\}, \{p_2,j\} \big\}, : i,j \in X,\ i \neq j,\ \{i,j\} \notin \mathcal{P} \Big\}, \\
\mathcal{L}_3 &= \Big\{ \big\{ \{i,j\}, \{p_1,i\}, \{p_1,j\} \big\}, \big\{ \{i,j\}, \{p_2,i\}, \{p_2,j\} \big\} : i,j \in X,\ \{i,j\} \in \mathcal{P} \Big\}.
\end{aligned}$$

Then the structure $\langle \wp_2(X'), \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \rangle$ will be denoted by $\mathbb{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$.

A particular role is played in the sequel by the structure

$$\mathbb{B}(X) = \mathbb{M}^p_X \triangleright_{\emptyset} \mathbf{G}_2(X).$$

We write $\mathbb{B}(n) := \mathbb{B}(X)$, where $|X| = n$, for short.

Example 1.5. The structure $\mathfrak{V}^o := \mathbb{B}(3)$ is the 10_3G-configuration of Kantor (cf. [10]); in the paper this one will be also called the Veronese configuration (as it is isomorphic with one of the combinatorial Veronese spaces, cf. [16]). It is known that $\mathfrak{D}^o \not\cong \mathfrak{V}^o$. In general, (see [15, Proposition 6]) the incidence structure $\mathbb{B}(n)$ is isomorphic to the dual of a suitable combinatorial Veronese space (cf. [16]).

As an example we present also Figure 1 which illustrates the structure of $\mathbb{B}(4)$.

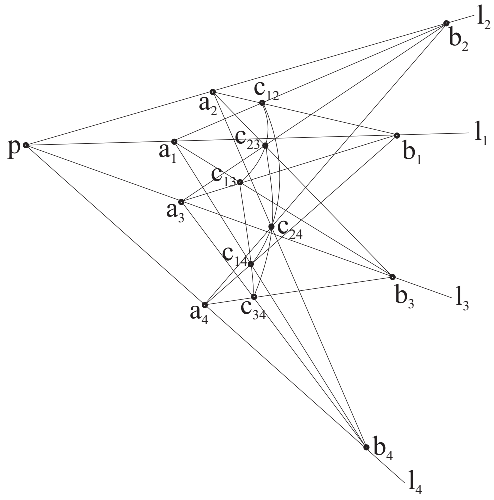


Figure 1. The configuration $\mathbb{B}(4)$

Adopt the notation of 1.4. Let $\alpha \in \mathrm{Aut}(\mathcal{P})$ be a permutation of X such that $\alpha^{(2)} \in \mathrm{Aut}(\mathfrak{H})$ and $\beta \in S_p$. Then, evidently, $\alpha \cup \beta \in \mathrm{Aut}(\mathbb{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H})$ (cf. [15, Lemma 2]).

Fact 1.6. If $\mathfrak{H}$ is an $\left(\binom{n}{2}_{n-2}\binom{n}{3}_3\right)$-configuration then $\mathbb{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$ is an $\left(\binom{n+2}{2}_n\binom{n+2}{3}_3\right)$-configuration.

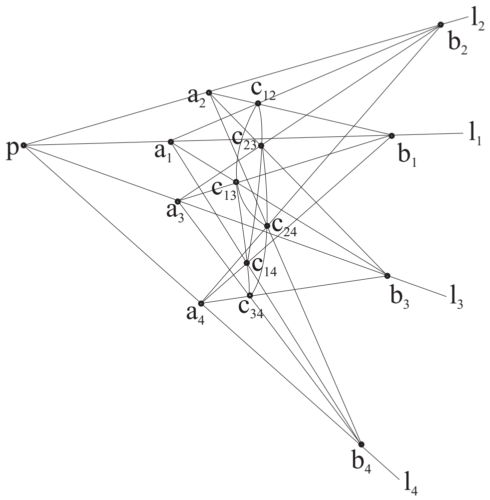


Figure 2. The configuration $\mathbb{M}^4 \triangleright_{\emptyset} \mathbb{B}(2)$

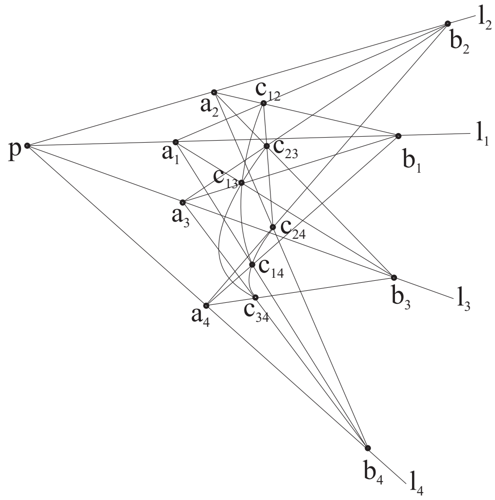


Figure 3. The configuration $\mathbb{M}^4 \triangleright_{\emptyset} \mathbf{G}_2^*(4)$

Construction 1.7. (Multi Veblen Configuration, [15]) The construction of the structure $\mathbb{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$ can be presented in a more geometrical version which will be frequently used in the following consideration of the obtained configurations. Let us adopt the notation of 1.4. Next, write

$$a_i = \{p_1, i\}, \quad b_i = \{p_2, i\} \quad \text{for } i \in X$$

and

$$c_z = z \quad \text{for } z \in \wp_2(X), \qquad \mathcal{C} = \big\{c_z \colon z \in \wp_2(X)\big\}.$$

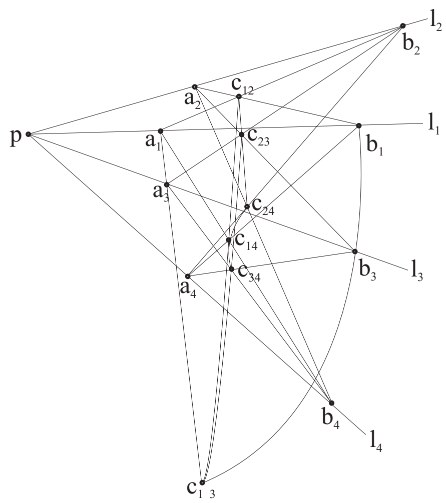


Figure 4. The configuration $\mathbb{M}^4 \triangleright_{L_2^4} \mathbf{G}_2^*(4)$

Step A. The set p is an arbitrary "abstract new point".

Step B. Through p we have the lines L_i, and the points a_i, b_i on L_i, for every $i \in X$.

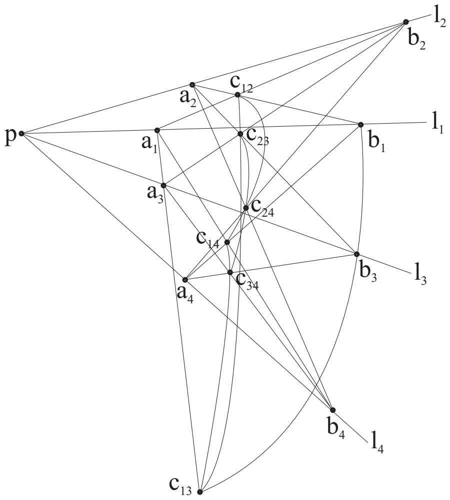


Figure 5. The configuration $\mathbb{M}^4 \rhd_{L_2^4} \mathbf{G}_2(4)$

Step C. We have a subset $\mathcal{P}$ of $\wp_2(X)$ distinguished, and after that

if $\{i,j\} \in \mathcal{P}$**:** we draw lines $A_{i,j} = \overline{a_i, a_j}$ and $B_{i,j} = \overline{b_i, b_j}$; the point $c_{\{i,j\}}$ is common for $A_{i,j}$ and $B_{i,j}$,

if $\{i,j\} \in \wp_2(X) \setminus \mathcal{P}$**:** we draw lines $G_{i,j} = \overline{a_i, b_j}$; the point $c_{\{i,j\}}$ is common for $G_{i,j}$ and $G_{j,i}$,

for every $\{i,j\} \in \wp_2(X)$. It is seen that the point p and the points a_i, b_i ($i \in X$) have degree n, while (up to now) c_z with $z \in \wp_2(X)$ has degree 2. Moreover, the number of the points c_z is $\binom{n}{2}$.

Step D. Let $\mathfrak{H}$ be any $\left(\binom{n}{2}_{n-2}, \binom{n}{3}_3\right)$-configuration. Finally, we identify the points c_z constructed above with points of $\mathfrak{H}$ (under some bijection γ) and, consequently, we group the points c_z into new $\binom{n}{3}$ lines obtained as coimages of the lines of $\mathfrak{H}$ under γ.

The resulting configuration will be written as $\mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H} \cong \mathbb{M}^p_X \rhd_{\mathcal{P}} \mathfrak{H}$. If this does not lead to a misunderstanding we write $\mathbb{M}^X \rhd_{\mathcal{P}} \mathfrak{H} = \mathbb{M}^{|X|} \rhd_{\mathcal{P}} \mathfrak{H}$.

If a bijection γ is fixed (or evident), we write simply $\mathbb{M}^X \rhd_{\mathcal{P}} \mathfrak{H}$. In particular, if $\mathfrak{H} = \mathbf{G}_2(X)$, it is natural to put $\gamma\colon c_{\{i,j\}} \longmapsto \{i,j\}$. It is immediate from 1.4 that

$$\mathbb{M}^n \rhd_{K_n} \mathbf{G}_2(n) \cong \mathbf{G}_2(n+2).$$

If $\delta \in S_{X'}$ (cf. 1.4) yields an automorphism of $\mathbb{M}^p_X \rhd_{\mathcal{P}} \mathfrak{H}$ then we frequently write F_δ instead of $\delta^{(2)}$ for the automorphism in question. In particular, if δ is the transposition (p_1, p_2) then $\sigma := F_\delta$ is the automorphism of $\mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H}$ which interchanges a_i and b_i for every $i \in X$. It is clear that $\sigma F_\alpha = F_\alpha \sigma$ for every $\alpha \in S_X$.

As a convenient tool for classification of the structures $\mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H}$ we have introduced in [15] the following classifications of graphs. Let $\mathcal{P}', \mathcal{P}''$ be two graphs on n vertices $x_1, \ldots, x_n$. We write $\mathcal{P}' \approx_0 \mathcal{P}''$ iff $\mathcal{P}', \mathcal{P}''$ are complementary in exactly one vertex x_t; this means x_i, x_t ($i \neq t$) are connected in $\mathcal{P}'$ iff they are not connected in $\mathcal{P}''$; while the remaining edges of the graphs are common. Let $\approx$ be the transitive closure of the relation $\approx_0$. Then (cf. [15, Proposition 9]) $\mathcal{P}' \approx \mathcal{P}''$ yields $\mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}'} \mathfrak{H} \cong \mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}''} \mathfrak{H}$. There are exactly 3 pairwise not $\approx$-equivalent graphs on 4 vertices (the case analyzed in detail in this paper): K_4, N_4, and L_4, and L_4 is equivalent to the graph L_2^4 on 4 vertices with a single one edge.

Another auxiliary notion that appears useful in our theory is the antineighborhood $\mathcal{N}^-(u)$ of a point u: $\mathcal{N}^-(u)$ consists of the points not collinear with u. Detailed technical lemmas characterizing when $\mathcal{N}^-(u)$ is a subspace of $\mathbb{M}^X \rhd^{\gamma}_{\mathcal{P}} \mathfrak{H}$ are given in [15, Lemmas 5, 7, 8, Remark 2]. We shall frequently use these characterizations without quoting their technical formulations.

Construction 1.8. (Convolution $\mathfrak{M} \bowtie_\theta \mathsf{G}$ (cf. [17])) Let $\mathfrak{M} = \langle X, \mathcal{L} \rangle$ be a partial Steiner triple system and G be an abelian group; let $\theta \in G$. The points of the structure $\mathfrak{M} \bowtie_\theta \mathsf{G}$ are the elements of $X \times G$, and its lines are all the sets

$$\{(x_1, g_1), (x_2, g_2), (x_3, g_3)\}, \text{ where } \{x_1, x_2, x_3\} \in \mathcal{L} \text{ and } g_1 + g_2 + g_3 = \theta.$$

In fact, only structures $\mathfrak{M} \bowtie_0 \mathsf{G}$ were analyzed in detail in [17], but the above construction yields a partial Steiner triple system for arbitrary θ. And also most of the results of [17] can be generalized for arbitrary θ.

The choice of $\theta \in G$ may be (sometimes) inessential.

Proposition 1.9. *Adopt the notation of* 1.8. *Let us fix* $g_0 \in G$ *and define the map*

$$F\colon X \times G \longrightarrow X \times G;\ (x, g) \longmapsto (x, g + g_0).$$

The map F *is an isomorphism of* $\mathfrak{M} \bowtie_\theta \mathsf{G}$ *on* $\mathfrak{M} \bowtie_{\theta + 3g_0} \mathsf{G}$*. In particular*

$$\mathfrak{M} \bowtie_0 C_2 \cong \mathfrak{M} \bowtie_1 C_2.$$

Construction 1.10. Let $\mathfrak{N} = \langle Z, \mathcal{L} \rangle$ be an arbitrary Steiner triple system and $\theta \in C_2$. Evidently, the pairs $((x, 0), (x, 1))$, where x is a point of $\mathfrak{N}$ are exactly all the pairs of noncollinear points of $\mathfrak{N} \bowtie_\theta C_2$. Let us write $\Lambda(\mathfrak{N}; \theta)$ for the structure obtained from $\mathfrak{N} \bowtie_\theta C_2$ by adding one new point p and the lines which join pairs of points noncollinear in $\mathfrak{N} \bowtie_\theta C_2$ and pass through the point p.

The construction of the structure $\Lambda(\mathfrak{N}; \theta)$ has also its own interest. Namely, let $\mathfrak{N}$ be a $(\boldsymbol{\nu}_{\boldsymbol{r}}\ \boldsymbol{b}_3)$-configuration. Then $\boldsymbol{\nu}\boldsymbol{r} = 3\boldsymbol{b}$ and, since $\mathfrak{N}$ is a linear space, $\binom{\boldsymbol{\nu}}{2} = 3\boldsymbol{b}$ which yields $\boldsymbol{\nu} - 1 = 2\boldsymbol{r}$. On the other hand, the degree of a point in $\mathfrak{N} \bowtie_\theta C_2$ is $2\boldsymbol{r}$, and thus its degree in $\Lambda(\mathfrak{N}; \theta)$ is $2\boldsymbol{r} + 1$. The degree of the added point p is $\boldsymbol{\nu}$, and $\boldsymbol{\nu} = 2\boldsymbol{r} + 1$ as well. Finally, the constructed structure $\Lambda(\mathfrak{N}; \theta)$ is again a Steiner triple system and a $\big((2\boldsymbol{\nu} + 1)_{2\boldsymbol{r}+1}\, (4\boldsymbol{b} + \boldsymbol{\nu})_3\big)$-configuration.

A considerable contribution to the theory on $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configurations can be found in the literature. Let us quote some more important results.

Proposition 1.11. (Kirkmann) *A Steiner triple system can be defined on a* $\boldsymbol{\nu}$*-element set if and only if* $\boldsymbol{\nu} \equiv 1 \mod 6$ *or* $\boldsymbol{\nu} \equiv 3 \mod 6$.

Proposition 1.12. ([1]) *If $\mathfrak{M}$ is a $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configuration, then $\boldsymbol{\nu r} = \boldsymbol{b \kappa}$. A $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_{\boldsymbol{\kappa}})$-configuration is a linear space if and only if $\binom{\boldsymbol{\nu}}{2} = \boldsymbol{b}\binom{\boldsymbol{\kappa}}{2}$.*

Proposition 1.13. ([9]) *There is a $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_3)$-configuration if and only if $\boldsymbol{\nu} \geq 2\boldsymbol{r} + 1$ and $\boldsymbol{\nu r} = 3\boldsymbol{b}$.*

2. Some more examples of multiveblen configurations

Let $X = \{1,2,3,4\}$ and $\mathfrak{H}$ be any representation of the Veblen configuration defined on the set $\wp_2(X)$. Two such representations were already discussed in [15]: $\mathfrak{H} = \mathbf{G}_2(4)$ (very widely) and $\mathfrak{H} = \mathbb{B}(2)$. A third one which is also interesting is the structure $\mathbf{G}_2^*(4)$, whose lines have form $\{a \in \wp_2(X) \colon i \in a\}$ with $i \in X$. First, since the case $\mathfrak{H} = \mathbf{G}_2^*(4)$ was not analyzed in [15] we shall establish here some basic features of the structures $\mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2^*(X)$.

Proposition 2.1. *Let $\mathcal{P}$ be any graph defined on X and $\mathfrak{B} = \mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2^*(X)$.*

(i) *The point p of $\mathfrak{B}$ is elementarily distinguishable in terms of the geometry of $\mathfrak{B}$ and, consequently, every automorphism of $\mathfrak{B}$ fixes p.*

(ii) *No three lines of $\mathfrak{B}$ which pass through p yield a Veronese nor a Desargues configuration.*

(iii) *$\mathfrak{B}$ is not isomorphic to any structure of the form $\mathbb{M}^4 \rhd_{\mathcal{P}'} \mathbf{G}_2(X)$ or $\mathbb{M}^4 \rhd_{\mathcal{P}'} \mathbb{B}(2)$ for any graph $\mathcal{P}'$ on X.*

(iv) $\mathrm{Aut}(\mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2^*(X)) \cong (\mathrm{Aut}(\mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2(X)))_{(p)}$.

Proof. From the results of [15] we get that the set of points q of $\mathfrak{B}$ such that $\mathcal{N}^-(q)$ is a subspace of $\mathfrak{B}$ coincides with $\{p\} \cup \mathcal{C}$, and p is the single isolated point in this set. This proves (i).

Consider three lines L_{i_i}, L_{i_2}, L_{i_3} through p. "Diagonal" points of the corresponding Veblen figures are $c_{\{i_1,i_2\}}$, $c_{\{i_1,i_3\}}$, and $c_{\{i_3,i_2\}}$. But these three points are never collinear in $\mathbf{G}_2^*(X)$; the third point on $\overline{c_{\{i_1,i_2\}}, c_{\{i_1,i_3\}}}$ is $c_{\{i_1,i_4\}}$, where $X = \{i_1, i_2, i_3, i_4\}$. This justifies (ii).

Finally, (iii) is an immediate consequence of (ii).

Let $F \in \mathrm{Aut}(\mathfrak{B})$ be arbitrary. From (i) $F(p) = p$ and thus F leaves the sets $\mathcal{C}$ and $\{a_i, b_i \colon i \in X\}$ invariant. Let us write $\mathbf{L}(c_{z_1}, c_{z_2}, c_{z_3})$ iff the c_{z_j} yield a triangle in $\mathbf{G}_2^*(X)$, $c_{z_1}, c_{z_2}, c_{z_3} \in \mathcal{C}$. Then F preserves the relation $\mathbf{L}$. But $\mathbf{L}$ defines (together with the lines through p and "old" lines which join points on lines through p and points in $\mathcal{C}$) the structure $\mathbb{M}^4 \rhd_{\mathcal{P}} \mathbf{G}_2(X)$, which closes the proof of (iv). $\square$

Similarly as it was done in [15] we get that every $\mathfrak{B}$ of the form considered in 2.1 is isomorphic to one of the following: $\mathbb{M}^4 \rhd_{N_4} \mathbf{G}_2^*(X)$, $\mathbb{M}^4 \rhd_{K_4} \mathbf{G}_2^*(X)$, $\mathbb{M}^4 \rhd_{L_4} \mathbf{G}_2^*(X)$. The arising structures are visualized in Figures 3, 4, and 7.
As a consequence of 2.1(iv) we get that

$$\mathrm{Aut}(\mathbb{M}^4 \rhd_{N_4} \mathbf{G}_2^*(X)) \cong C_2 \oplus S_4 \cong \mathrm{Aut}(\mathbb{M}^4 \rhd_{K_4} \mathbf{G}_2^*(X)),$$

$$\mathrm{Aut}(\mathbb{M}^4 \rhd_{L_4} \mathbf{G}_2^*(X)) \cong C_2^3.$$

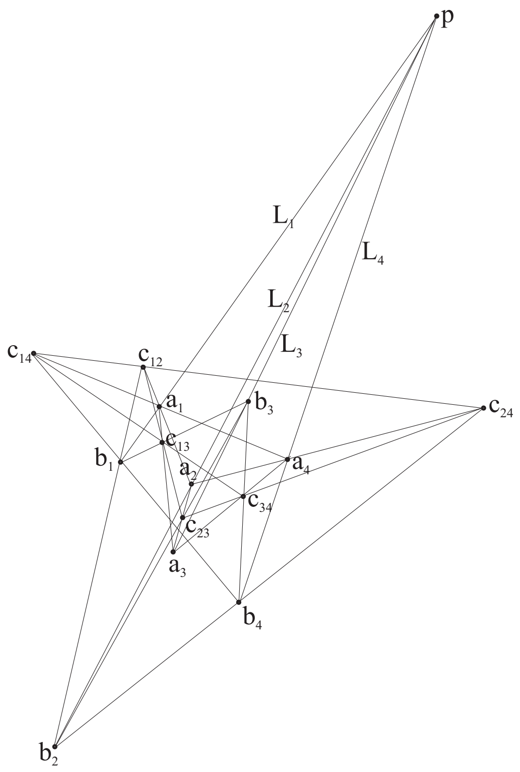


Figure 6. The configuration $\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)$

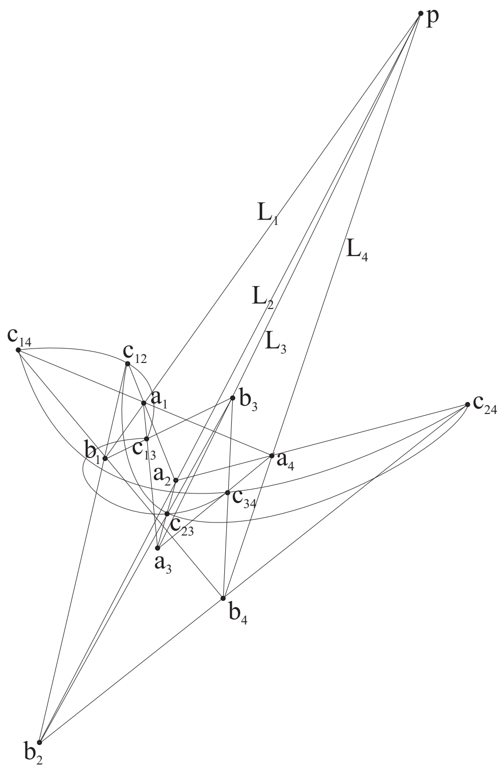


Figure 7. The configuration $\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2^*(4)$

Finally, the case where the Veblen configuration $\mathfrak{H}$ is represented in the form $\mathfrak{H} = \mathbb{B}(2) = \mathbb{M}^2 \triangleright_{N_2} \mathbf{G}_2(2)$ on a 4-element set X was only mentioned in [15]. Let $\mathfrak{B} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathbb{B}(2)$. From [15] we get that when $\mathcal{P} = K_4$, then $\mathfrak{B} \cong \mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$. When $\mathcal{P} = N_4$ then $\mathrm{Aut}(\mathfrak{B}) = C_2^3$, but $\mathfrak{B} \not\cong \mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$. The case when $\mathcal{P} \approx L_4$ was completely left.

Let us recall that if $\mathcal{P} \approx L_4$ then there is $y \in \wp_2(X)$ such that $\mathcal{P} \approx \{y\}$ and, consequently, $\mathbb{M}^X \triangleright_{\mathcal{P}} \mathfrak{H} \cong \mathbb{M}^X \triangleright_{\{y\}} \mathfrak{H}$ for every representation $\mathfrak{H}$ of the Veblen configuration.

Lemma 2.2. *Let $\mathcal{P}$ be a graph defined on the set $X = \{1, 2, 3, 4\}$ such that $\mathcal{P} \approx \{y'\}$ for some $y' \in \wp_2(X)$. Set $q = \{3, 4\}$, $r = \{1, 2\}$, $s = \{1, 3\}$, $\mathcal{Y} = \{q, r, s\}$, and $\mathfrak{H} = \mathbb{M}^q_r \triangleright_{\{\emptyset\}} \mathbf{G}_2(r) = \mathbb{B}(r)$. Then $\mathbb{M}^X \triangleright_{\mathcal{P}} \mathfrak{H} \cong \mathbb{M}^X \triangleright_{\{y\}} \mathfrak{H}$ for some $y \in \mathcal{Y}$.*

Proof. Let $\mathcal{P} \approx \{y\}$ for some $y \in \wp_2(X)$. If $y \neq q, r$ then there is a permutation $\alpha \in S_X$ which maps y onto s such that $\alpha^{(2)} \in \mathrm{Aut}(\mathfrak{H})$ and then α yields an isomorphism F_α of $\mathbb{M}^X \triangleright_{\{y\}} \mathfrak{H}$ onto $\mathbb{M}^X \triangleright_{\{s\}} \mathfrak{H}$. □

Proposition 2.3. *Let $\mathcal{P}$ be a graph defined on the set $X = \{1, \dots, 4\}$ such that $\mathcal{P} \approx \emptyset$ or $\mathcal{P} \approx L_4$. We adopt notation of 2.2 and then, without loss of generality we can assume that*

$$\mathcal{P} = \emptyset \text{ or } \mathcal{P} = \{y\} \text{ with } y \in \mathcal{Y}. \tag{1}$$

Set $c_p = p = \{5, 6\}$, $X' = X \cup p$, and $\mathfrak{B} = \mathbb{M}^X \triangleright_{\mathcal{P}} \mathbb{B}(2)$.

(i) *Let u be a point of $\mathfrak{B}$. Then $\mathcal{N}^-(u)$ is a subspace of $\mathfrak{B}$ iff $u = c_y$, where $y \in \{p, q, r\} =: \mathcal{Z}$.*

(ii) *Let $z \in \mathcal{Z}$.*

a) *Let $z = r$. Then there are two lines through c_z which do not yield in $\mathfrak{B}$ a Veblen figure.*

b) *Let $z = p$. Then every two lines through c_z yield in $\mathfrak{B}$ a Veblen figure.*

c) *Let $z = q$. If $q \notin \mathcal{P}$ then there are two lines through c_z which do not yield in $\mathfrak{B}$ a Veblen figure.*

(iii) *Let $\mathcal{P} = \{q\}$. There is an involutory automorphism ξ of $\mathfrak{B}$ which fixes c_r and interchanges c_p and c_q.*

(iv) *Three lines $L_{i_1}, L_{i_2}, L_{i_3}$ through p yield a Desargues configuration $\mathfrak{D}^o$ or a $\mathfrak{V}^o$-configuration iff $q \subset \{i_1, i_2, i_3\}$. Consequently, there are two such triples of lines, and the lines in one triple are numbered with the elements of the set $q \cup \{i\}$ where $i \in r$.*

(v) *Let $\mathcal{P} \neq \emptyset$, so, in view of* (1) $\mathcal{P} = \{y\}$ *where* $y \in \mathcal{Y}$.

d) *Assume that $y = r$, then the two triples of lines mentioned in* (iv) *yield two $\mathfrak{V}^o$-configurations and* $\mathrm{Aut}(\mathfrak{B}) = C_2^3$.

e) *If $y = q$ then these two triples of lines yield two Desargues configurations and* $\mathrm{Aut}(\mathfrak{B}) = C_2 \oplus D_4$.

f) *Assume that $y \neq q, r$. Then the two triples of lines mentioned in* (iv) *yield one Desargues configuration and one Veronese configuration and* $\mathrm{Aut}(\mathfrak{B}) = C_2^2$.

(vi) *The three structures* $\mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(2)$, $\mathbb{M}^4 \triangleright_{\{\{3,4\}\}} \mathbb{B}(2)$, *and* $\mathbb{M}^4 \triangleright_{\{\{1,3\}\}} \mathbb{B}(2)$ *are nonisomorphic, and not isomorphic to any of the structures of the form* $\mathbb{M}^4 \triangleright_{\mathcal{P}'} \mathbf{G}_2(4)$ *or* $\mathbb{M}^4 \triangleright_{\mathcal{P}'} \mathbf{G}_2^*(4)$.

(vii) *Moreover, none of them is isomorphic to* $\mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)$.

Proof. From [15, Lemma 7(ii)] it follows that $\mathcal{N}^-(c_z)$ is a subspace of $\mathfrak{B}$ iff $z \in \mathcal{Z}$. In every of the cases of (1) from [15, Lemma 8(ii)] we infer that $\mathcal{N}^-(a_i)$ or $\mathcal{N}^-(b_i)$ may be a subspace of $\mathfrak{B}$ only for $i \in r$, and from [15, Lemma 8(i)] it follows that neither $\mathcal{N}^-(a_i)$ nor $\mathcal{N}^-(b_i)$ is a subspace of $\mathfrak{B}$ for some $i \in r$. This closes the proof of (i).

Condition (ii) is justified by direct examples. Note, first, that in every case $\{2,3\}$, $\{2,4\}$, and $\{1,4\}$ are not in $\mathcal{P}$. If $r \in \mathcal{P}$ then required two lines through c_r are $\{c_{\{1,4\}}, c_{\{2,3\}}, c_{\{1,2\}}\}$ and $\{b_1, b_2, c_{\{1,2\}}\}$; if $r \notin \mathcal{P}$ then the lines $\{c_{\{1,4\}}, c_{\{2,3\}}, c_{\{1,2\}}\}$ and $\{b_1, b_2, c_{\{1,2\}}\}$ are as required.

If $q \notin \mathcal{P}$ then the two lines $\{c_{\{2,3\}}, c_{\{2,4\}}, c_{\{3,4\}}\}$ and $\{b_3, a_4, c_{\{3,4\}}\}$ through c_q do no yield a Veblen Figure.

To prove (iii) it suffices to note that the following involutory bijection ξ

$$\begin{array}{ccccccccc} c_{\{3,4\}} & c_{\{2,4\}} & c_{\{2,3\}} & c_{\{1,4\}} & c_{\{1,3\}} & a_3 & a_4 & b_4 & c_{\{1,2\}} \\ p & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & b_4 & c_{\{1,2\}} \end{array}$$

is a required automorphism of $\mathfrak{B}$.

Condition (iv) is evident.

(v): Recall that the map σ (interchanging every a_i with b_i) is an automorphism of $\mathfrak{B}$.

Let $F \in \mathrm{Aut}(\mathfrak{B})$. From (i) and (ii), we get that F leaves the set $\{c_z \colon z \in \mathcal{Z}\}$ invariant and either $F(c_p) = c_p$ or F interchanges c_p and c_q.

Assume that $F(p) = p$, so F determines a permutation α of X corresponding to the permutation of the lines through p. Clearly, α maps a triple of lines which yields a Desargues or Veronese configuration onto a triple with the same property. From (iv), $\alpha^{(2)}(q) = q$ and thus $\alpha^{(2)}(r) = r$. Let $\mathcal{P} = \{y\}$ and $y \in \mathcal{Y}$.

In the case (f) $y = s = \{1,3\}$ and then the triple $\{L_1, L_3, L_4\}$ yields a Desargues configuration, while $\{L_2, L_3, L_4\}$ yields a Veronese configuration so, corresponding triples of lines through q cannot be interchanged. Consequently, $F(c_y) = c_y$ or $F(c_y) = c_{\{1,4\}}$ and thus $\alpha^{(2)}(y) = y$ or $\alpha^{(2)}(y) = \{1,4\}$, which gives that either

$\alpha(3) = 3$ or $\alpha(3) = 4$, and α fixes 1 and 2. This yields $\alpha \in S_q$. Thus $F = F_\alpha \sigma$ or $F = F_\alpha$, where $\alpha \in S_q$.

In the cases (d) and (e) for arbitrary permutations $\alpha_1 \in S_q$, $\alpha_2 \in S_r$ the map $\alpha = \alpha_1 \cup \alpha_2$ determines the automorphism F_α of $\mathfrak{B}$, and $F = F_\alpha \sigma$ or $F = F_\alpha$, where $\alpha \in S_q \circ S_r$. If $y = r$ then $F(c_p) = c_p$ follows from (ii) so the proof of (d) is completed.

To close the proof in the case (e) we note that if F is an arbitrary automorphism of $\mathfrak{B}$ and $F(p) \neq p$ then $\xi(F(p)) = p$ for ξ defined in (iii). Direct computation proves that $\xi F_\beta = F_\beta \xi$ for $\beta \in S_r$. Let α be the transposition $(3, 4)$. Analogous computation gives $\xi \sigma \xi = F_\alpha$. Consequently, $\big\{\xi, F_\alpha, \sigma\big\}$ generates the D_4 group, which gives our claim.

Clearly, (vi) is an immediate consequence of (v), (iv), and 2.1(ii).

In view of (v), the only one suspected isomorphism is $\mathbb{M}^4 \rhd_{\{\{1,2\}\}} \mathbb{B}(2) \overset{?}{\cong} \mathbb{M}^4 \rhd_{N_4} \mathbb{B}(2)$, as these two structures have the same automorphism group. Suppose that F is such an isomorphism. From the above and [15, Example 2], $F(c_p) = c_p$, $F(\mathcal{C}) = \mathcal{C}$, $F(c_q) = c_q$, and $F(c_r) = c_r$. Let us replace the lines through c_r contained in $\mathcal{C}$ by the following triples:

$$\{c_r, c_u, c_v\}, \text{ where } c_r \text{ is collinear with } c_u, c_v \text{ and } c_u, c_v \text{ are not collinear;}$$

then the structure $\mathbf{G}_2(X)$ emerges on $\mathcal{C}$ and F appears to be an isomorphism of $\mathbb{M}^4 \rhd_{\{\{1,2\}\}} \mathbf{G}_2(4)$ and $\mathbb{M}^4 \rhd_{N_4} \mathbf{G}_2(4)$, which is impossible. $\square$

The structures discussed in 2.3 are drawn in Figures 2, 8, 9, and 10.

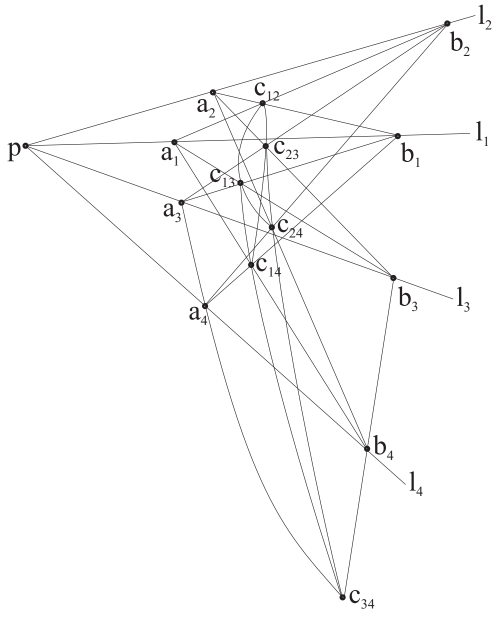


Figure 8. The configuration $\mathbb{M}^4 \rhd_{\{\{3,4\}\}} \mathbb{B}(2)$

Clearly, the three structures $\mathbf{G}_2(4)$, $\mathbf{G}_2^*(4)$, and $\mathbb{B}(2)$ do not exhaust all the possible labelings of the points of a Veblen configurations by elements of $\wp_2(X)$, where

$|X| = 4$. For the other ways, however, we could not find any natural "constructive" interpretation.

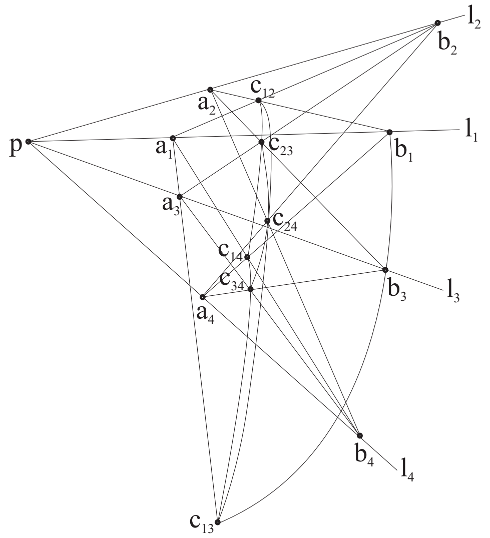


Figure 9. The configuration $\mathbb{M}^4 \triangleright_{\{\{1,3\}\}} \mathbb{B}(2)$

Example 2.4. Let us define on the set $\wp_2(X)$, where $X = \{1, 2, 3, 4\}$ the following system of lines:

$$\big\{\{1,2\},\{1,4\},\{2,3\}\big\}, \qquad \big\{\{1,2\},\{2,4\},\{3,4\}\big\},$$
$$\big\{\{1,3\},\{3,4\},\{2,3\}\big\}, \qquad \big\{\{1,3\},\{1,4\},\{2,4\}\big\}.$$

One can note that a Veblen configuration $\mathfrak{V}$ arises, in which for every pair u, y of noncollinear points we have $u \cap y \neq \emptyset$. Consequently, no point c_u of $\mathfrak{B} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{V}$ yields a subspace of the form $\mathcal{N}^-(c_u)$. Similarly, no point a_i nor b_i yields a subspace. No triple $a \in \wp_3(X)$ may yield a Desargues configuration or a Veronese configuration with center p. Slightly less irregular labeling of the Veblen configuration is given by the following:

$$\big\{\{1,2\},\{1,4\},\{2,4\}\big\}, \qquad \big\{\{1,2\},\{2,3\},\{3,4\}\big\},$$
$$\big\{\{1,3\},\{1,4\},\{2,3\}\big\}, \qquad \big\{\{1,3\},\{3,4\},\{2,4\}\big\}.$$

In this case $\mathfrak{B}$ has one Desargues or Veronese configuration on lines with numbers $1, 2, 4$.

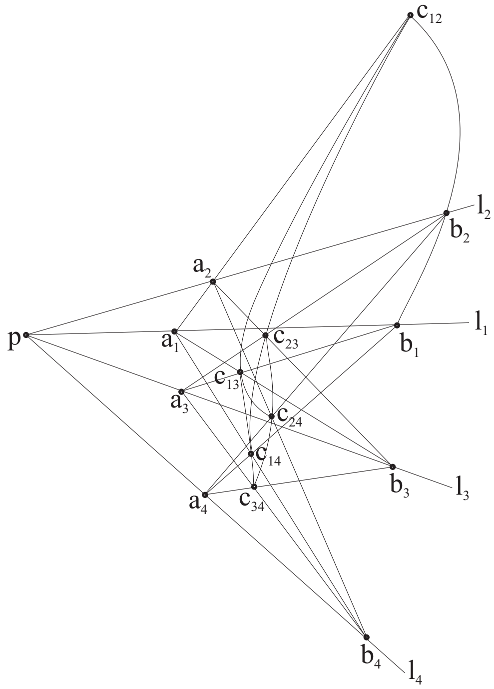


Figure 10. The configuration $\mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(2)$

3. Linear completions, problems on existence

3.1. General theory

Let $\mathfrak{A}$ be a partial Steiner triple system. *A linear completion* of $\mathfrak{A}$ is a Steiner triple system $\widetilde{\mathfrak{A}}$ defined on the point universe of $\mathfrak{A}$ such that every line of $\mathfrak{A}$ is a line of $\widetilde{\mathfrak{A}}$ as well. Moreover, if $\mathfrak{A}$ has a constant point degree we assume that $\widetilde{\mathfrak{A}}$ also has constant point degree.

Let $\mathfrak{A}$ be a $(\boldsymbol{\nu}_{\boldsymbol{r}}, \boldsymbol{b}_3)$-configuration and let $\widetilde{\mathfrak{A}}$ be a configuration defined on the point set of $\mathfrak{A}$ such that its line set extends the line set of $\mathfrak{A}$. Thus $\widetilde{\mathfrak{A}}$ is a $(\boldsymbol{\nu}_{\widetilde{\boldsymbol{r}}}, \widetilde{\boldsymbol{b}}_3)$-configuration whose parameters satisfy $\boldsymbol{\nu}\widetilde{\boldsymbol{r}} = 3\widetilde{\boldsymbol{b}}$.

$\widetilde{\mathfrak{A}}$ is a linear space if and only if $\binom{\boldsymbol{\nu}}{2} = \widetilde{\boldsymbol{b}}\binom{3}{2}$ (cf. [1]) i.e. iff $\boldsymbol{\nu}(\boldsymbol{\nu}-1) = 6\widetilde{\boldsymbol{b}}$. Since $\boldsymbol{\nu}\widetilde{\boldsymbol{r}} = 3\widetilde{\boldsymbol{b}}$, $\widetilde{\mathfrak{A}}$ is a linear space if and only if $\boldsymbol{\nu} = 2\widetilde{\boldsymbol{r}} + 1$ and thus $\boldsymbol{\nu}$ must be even. Given $\boldsymbol{\nu}$ this determines $\widetilde{\boldsymbol{r}} = \frac{\boldsymbol{\nu}-1}{2}$.

Let $\widetilde{\mathfrak{A}}$ be a linear completion of $\mathfrak{A}$. Then $\widetilde{\mathfrak{A}}$ is a Steiner triple system, so $\boldsymbol{\nu} = 6k + t$ for some k and $t \in \{1, 3\}$. Substituting $\widetilde{\boldsymbol{r}} = 3k$ and $\widetilde{\boldsymbol{r}} = 3k + 1$ resp. we determine $\widetilde{\boldsymbol{b}} = k(6k+1)$ and $\widetilde{\boldsymbol{b}} = (2k+1)(3k+1)$. In any case, if $\boldsymbol{\nu}$ is admissible (cf. 1.11) then there exists a Steiner triple system on the universe of points of $\mathfrak{A}$ with constant point degree. The question is whether there exists one which completes $\mathfrak{A}$.

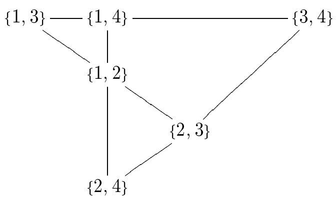
3.2. Completing the Veblen and related configurations

Let $\mathfrak{H} = \langle Z, \mathcal{L} \rangle$ be the Veblen configuration. For every point $a \in Z$ there is the unique point $\eta(a) \in Z$ such that $a, \eta(a)$ are not collinear in $\mathfrak{H}$. It is known that, if we add one "abstract" point $p \notin Z$ and define the family $\mathcal{L}_0$ consisting of three new lines $\{p, a, \eta(a)\}$ with $a \in Z$, then the structure $\mathfrak{P} = \langle Z \cup \{p\}, \mathcal{L} \cup \mathcal{L}_0 \rangle$ is the Fano plane. Recall (cf. [17]) that the Veblen configuration can be presented as $\mathfrak{T} \bowtie_0 C_2$, where $\mathfrak{T}$ is a single 3-element line. Thus the above remark and 1.10 yield that $\Lambda(\mathfrak{T}; 0) \cong \mathfrak{P}$ is simply the Fano plane.

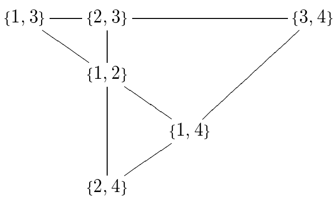
On the other hand recall that the Veblen configuration can be also presented as

- Veronese space $\mathbf{V}_2(X)$, where X is a 3-element set (cf. [16]),
- $\mathbf{G}_2(X)$ and $\mathbf{G}_2^*(X)$, where X is a 4-element set,
- $\mathbb{B}(q) = \mathbb{M}^{p}_{q} \triangleright_{\emptyset} \wp_2(q)$, where p and q are disjoint 2-element sets.

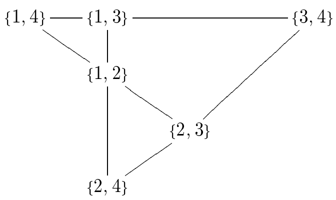
Some more frequently used labelings of the points of the Veblen configuration by the elements of $\wp_2(X)$, where $X = \{1, 2, 3, 4\}$ is a 4-element set are shown in Table 1.



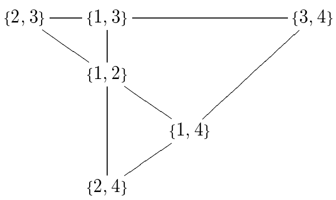
$\mathbf{G}_2(\{1, 2, 3, 4\})$



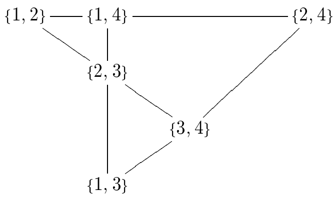
$\mathbf{G}_2^*(\{1, 2, 3, 4\})$



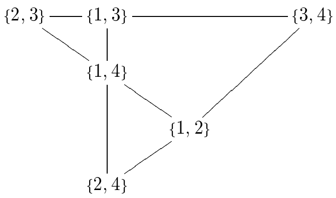
$\mathbb{B}(\{1, 2\})$



$\varkappa(\mathbb{B}(\{3, 4\}))$



cf. 2.4



cf. 2.4

Table 1. Various labeling of the points of the Veblen figure by the elements of $\wp_2(\{1, 2, 3, 4\})$

3.3. Completing $\left(15_4\ 20_3\right)$ multiveblen configurations

In [16, Proposition 4.13] we proved for some $\left(15_4\, 20_3\right)$-configuration that it has the unique completion to a Steiner triple system. In the sequel we shall construct such completions of multiveblen configurations.

Proposition 3.1. *Let $\mathcal{P}$ be an arbitrary graph on vertices $X = \{1,2,3,4\}$. The structure $\mathfrak{B} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{H}$ has the unique completion to a Steiner triple system.*

Proof. From assumptions, the structure $\mathfrak{H}$ must be a $\left(6_2\, 4_3\right)$-configuration and thus $\mathfrak{H}$ is simply the Veblen configuration defined on the set $\wp_2(X)$. We are going to construct the system of new lines on the universe of points of $\mathfrak{B}$ so as the resulting incidence structure $\mathfrak{B}^{\mathfrak{c}}$ will be a partial linear space. The first three lines of $\mathfrak{B}^{\mathfrak{c}}$ are given by the following formula:

$$\{p, c_{z_1}, c_{z_2}\}, \quad z_1, z_2 \in \wp_2(X),\ z_1 = \eta(z_2).$$

Indeed, if $z \in \wp_2(X)$ then the only point noncollinear with both p and c_z is $c_{\eta(z)}$. For any two distinct $i, j \in X$ we have the following pair of lines of $\mathfrak{B}^{\mathfrak{c}}$:

$$\begin{matrix}\{a_i, a_j, c_{X\setminus\{i,j\}}\},\\ \{b_i, b_j, c_{X\setminus\{i,j\}}\}\end{matrix} \ \text{if } \{i,j\} \notin \mathcal{P}, \ \text{ or } \ \begin{matrix}\{a_i, b_j, c_{X\setminus\{i,j\}}\},\\ \{b_i, a_j, c_{X\setminus\{i,j\}}\}\end{matrix} \ \text{if } \{i,j\} \in \mathcal{P}. \qquad (2)$$

With the above we have obtained 15 new lines which join points noncollinear in $\mathfrak{B}$, thus (cf. [16], [1]) the resulting system of lines is a linear space. It is seen that the above completion is unique. $\square$

4. Twisted projective spaces and the geometry of completions

One can directly verify that $\widetilde{\mathbf{G}_2(6)}$ is simply the Fano 3-space $PG(3,2)$. Let us analyze in some detail the remaining $(15_7, 35_3)$-configurations which arise as the completions of the structures of the form $\mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{H}$, where $\mathfrak{H} = \mathbf{G}_2(4)$ or $\mathfrak{H} = \mathbf{G}_2^*(4)$. In particular, let us examine the completions $\widetilde{\mathfrak{A}}$, where $\mathfrak{A} = \mathbb{B}(4)$ and $\mathfrak{A} = \mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$, as they are constructed in 3.1. Note the following simple observations, which give an insight into the geometry of our completions. Let $X = \{1, \ldots, 4\}$.

Analysis 4.1.

(I) The set $\mathcal{C} \cup \{p\}$ yields in $\widetilde{\mathfrak{A}}$ a subspace isomorphic to the the Fano plane $PG(2,2)$ (the natural completion of the Veblen figure, cf. Subsection 3.2).

(II) For every distinct $i, j \in X$ the set $\{p, a_i, a_j, b_i, b_j, c_{\{i,j\}}, c_{X\setminus\{i,j\}}\}$ yields in $\widetilde{\mathfrak{A}}$ a subspace $\mathcal{D}_{\{i,j\}}$ isomorphic to the Fano plane. Consequently, every two lines through p yield in $\widetilde{\mathfrak{A}}$ a Fano plane; the $\mathcal{D}_{\{i,j\}}$ together with the one defined in (I) gives us 7 Fano planes through p contained in $\widetilde{\mathfrak{A}}$, which pairwise intersect in a line and, clearly, cover the point set of $\widetilde{\mathfrak{A}}$.

(III) The structure $\mathfrak{A}$ contains the configuration $\mathfrak{V}^\circ$. Therefore, in accordance with [16], $\mathfrak{A}$ can not be embedded into a Fano projective space and thus $\widetilde{\mathfrak{A}}$ cannot be a projective space.

(IV) Let us consider the lines through p and the Fano planes through p, as they are constructed above. It is seen that we obtain a (7_3)-configuration i.e. a Fano plane. Old lines, those of $\mathfrak{A}$ are labeled as L_i with $i \in X$. Let us label the new three lines through p by elements of $\wp_2(\{1,2,3\})$ in such a way that $L_{\{i,j\}}$ lies in the plane spanned by L_i, L_j. Every line L_x contains two points distinct from p; to construct $\widetilde{\mathfrak{A}}$ we must characterize the way in which they are joined in corresponding planes.

(V) Let $\mathfrak{A} = \mathbb{B}(4)$. Let us take two lines $K_1 = \overline{a_i, b_{j_1}}$ and $K_2 = \overline{a_i, b_{j_2}}$, where i, j_1, j_2 are pairwise distinct; without loss of generality we can assume that $i = 1$, $j_1 = 2$, $j_2 = 3$. Consider the following "generating" sequence:

$$\begin{aligned}
&c_{\{1,2\}} \mathrel{|} \overline{a_1, b_2}, \quad c_{\{1,3\}} \mathrel{|} \overline{a_1, b_3}, \quad c_{\{2,3\}} \mathrel{|} \overline{c_{\{1,2\}}, c_{\{1,3\}}},\\
&\qquad c_{\{1,4\}} \mathrel{|} \overline{b_2, b_3}, \quad p \mathrel{|} \overline{c_{\{2,3\}}, c_{\{1,4\}}}, \quad a_4 \mathrel{|} \overline{a_1, c_{\{2,3\}}},\\
&\qquad b_1 \mathrel{|} \overline{p, a_1}, \quad a_2 \mathrel{|} \overline{p, b_2}, \quad a_3 \mathrel{|} \overline{p, b_3}, \quad b_4 \mathrel{|} \overline{p, a_4},\\
&\qquad\qquad c_{\{2,4\}} \mathrel{|} \overline{a_2, b_4}, \quad c_{\{3,4\}} \mathrel{|} \overline{a_3, b_4}.
\end{aligned}$$

Consequently, the smallest subspace of $\widetilde{\mathfrak{A}}$ which contains $K_1 \cup K_2$ is the whole point set of $\widetilde{\mathfrak{A}}$. Analogous reasoning shows that the following two pair of lines: $\overline{a_1, a_2}, \overline{a_1, a_3}$ and $\overline{a_1, a_2}, \overline{a_1, b_3}$ also span the space.
Let us write down all the lines through a_4; these are $L_0 = \overline{p, a_4}$, $M_i = \overline{a_i, a_4}$, $N_i = \overline{b_i, a_4}$ with $i = 1, 2, 3$. Every pair (L_0, M_i), (L_0, N_i), and (M_i, N_i) yields (the same) Fano plane – there are 3 such planes and 9 such pairs. Every pair (M_i, M_j), (M_i, N_j), (N_i, N_j) spans the whole space, then number of pairs of lines through a_4 which span the whole space is 12.
From the above we get also that for every $z \in \wp_2(X)$ there are two lines through c_z which span the whole space (e.g. the lines $\overline{a_1, b_2}$ and $\overline{a_4, a_3}$ through $c_{\{1,2\}}$).

(VI) Analogous computation can be provided for $\mathfrak{A} = \mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$ (which is a quite expected result, cf. 4.7).

Proposition 4.2. *Let $\mathfrak{F}$ be a Fano plane, $\theta \in C_2$, and $X = \{1, \ldots, 4\}$. Set $\mathfrak{M} = \Lambda(\mathfrak{F}; \theta)$. Then $\mathfrak{M} \cong \widetilde{\mathbf{G}_2(6)}$ for $\theta = 0$ and $\mathfrak{M} \cong \widetilde{\mathbb{M}^4 \triangleright_{\emptyset} \mathbf{G}_2^*}(X)$ for $\theta = 1$.*

Proof. The case $\theta = 0$ is evident; from [17] it follows that $\mathfrak{F} \bowtie_0 C_2$ is a suitable projective slit space, and its completion is the Fano 3-space. However, the construction presented below works well also in this case.
Let $\theta = 1$ and let us label points of $\mathfrak{F}$ in such a way that

- its points are: p_1, p_2, p_3, p_4, $q_{1,2}$, $q_{1,3}$, $q_{2,3}$;
- collinear triples are: $(p_i, p_j, q_{i,j})$ with $1 \le i < j \le 3$, $(p_i, p_4, q_{j,k})$, where $\{i, j, k\} = \{1, 2, 3\}$, and $(q_{1,2}, q_{1,3}, q_{2,3})$.

We embed $\mathbb{M}^4 \triangleright_{\emptyset} \mathbf{G}_2^*(X)$ into $\mathfrak{F} \bowtie_\theta C_2$ putting

$$\begin{array}{rcl} a_i & \longmapsto & (p_i, 0), \\ b_i & \longmapsto & (p_i, 1), \\ c_{\{i,j\}} & \longmapsto & \begin{cases} (q_{i,j}, 0) & \text{when } i, j \leq 3, \\ (q_{i',j'}, 1) & \text{where } \{i', j'\} = X \setminus \{i, j\} \text{ otherwise.} \end{cases} \end{array} \tag{3}$$

Direct verification shows that the above map (together with $p \longmapsto p$) is a required isomorphism. Besides, for the line $\mathfrak{T} = \{q_{1,2}, q_{1,3}, q_{2,3}\}$ we obtain $\mathfrak{T} \bowtie_1 C_2 \cong \mathbf{G}_2^*(X)$. □

In view of 4.2 and 1.9, the linear completion of $\mathbb{M}^4 \triangleright_{\emptyset} \mathbf{G}_2^*(4)$ is also the projective space $PG(3, 2)$.

To determine, which of the completing configurations are really distinct, we can apply the following lemma

Lemma 4.3. *Let $\mathcal{P}$ be any graph on the vertices $X = \{1, \ldots, 4\}$, and $\mathfrak{H}$ be a Veblen configuration represented on the set $\wp_2(X)$. Let η be a bijection of the points of $\mathfrak{H}$ such that $a, \eta(a)$ are not collinear in $\mathfrak{H}$ for every $a \in \wp_2(X)$. Finally, let $\mathfrak{B}$ be the unique linear completion of $\mathfrak{A} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{H}$ (cf.* 3.1 *and the construction presented there).*

Let $\mathfrak{H}'$ be the image of $\mathfrak{H}$ under the map $\varkappa$ (more precisely, let it be the representation of the Veblen configuration, with lines defined as images of lines of $\mathfrak{H}$ under the map $\varkappa$). Then $\mathfrak{A}' = \mathbb{M}^4 \triangleright_{\mu(\mathcal{P})} \mathfrak{H}'$ is contained in $\mathfrak{B}$ and therefore, $\widetilde{\mathfrak{A}} = \widetilde{\mathfrak{A}'}$.

Proof. Consider the points p, a_i, b_i of $\mathfrak{A}$. For every $a = \{i, j\} \in \wp_2(X)$, if $a \in \mathcal{P}$ then we choose in $\mathfrak{B}$ the lines $\overline{a_i, b_j}$ and $\overline{a_j, b_i}$, and if $a \notin \mathcal{P}$ we choose the lines $\overline{a_i, a_j}$ and $\overline{b_j, b_i}$. Their intersection point in $\mathfrak{B}$ is $c_{\varkappa(a)}$, which, in accordance with the construction of the $\mathfrak{A}'$ should be written as $c'_{\{i,j\}}$. To recover the original collinearity of points in $\mathfrak{H}$ we must put in $\mathfrak{H}'$: $c'_{a_1}, c'_{a_2}, c'_{a_3}$ form a line of $\mathfrak{H}'$ iff $c_{\varkappa(a_1)}, c_{\varkappa(a_2)}, c_{\varkappa(a_3)}$ form a line in $\mathfrak{H}$. This closes the proof. □

It is seen that the map $\varkappa$ transforms $\mathbf{G}_2(X)$ onto $\mathbf{G}^*_{|X|-2}(X)$; in particular the image of $\mathbf{G}_2(4)$ under $\varkappa$ is $\mathbf{G}_2^*(4)$. Thus, as an immediate consequence of 4.3 we infer, e.g.

Corollary 4.4. $\widetilde{\mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)} = \widetilde{\mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2^*(4)}$, $\widetilde{\mathbb{B}(4)} = \widetilde{\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2^*(4)}$, *and* $\widetilde{\mathbf{G}_2(6)} = \widetilde{\mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2^*(4)}$.

Lemma 4.5. *Let $\mathfrak{H} = \mathbf{G}_2(X)$, where $X = \{1, \ldots, 4\}$, let $\mathcal{P}$ be a graph defined on the set X, and let $\mathfrak{B}$ be the linear completion of the structure $\mathfrak{A} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{H}$. Let us adopt notation of* 4.1(IV) *and consider any three noncoplanar lines L', L'', L''' of $\mathfrak{B}$ which pass through p and the family $\mathcal{S}(L', L'', L''')$ of all the possible Desargues or Veronesian subconfigurations of $\mathfrak{B}$ for which the given lines are the rays.*

Let us number elements of X by i_1, i_2, i_3, i_4 and let $y = \{i_1, i_2, i_3\} \in \wp_3(X)$.

(i) *Consider the lines* $L_{i_1}, L_{i_2}, L_{i_3}$. *If* $\mathcal{P} \curlywedge y \approx K_3$, *then* $\mathcal{S}(L_{i_1}, L_{i_2}, L_{i_3})$ *consists of* 4 *Desargues configurations; otherwise it consists of* 4 *Veronese configurations.*

(ii) *Consider the lines* $L_{i_1}, L_{i_1,i_2}, L_{i_1,i_3}$. *If* $\mathcal{P} \curlywedge y \approx K_3$, *then the family* $\mathcal{S}(L_{i_1}, L_{i_1,i_2}, L_{i_1,i_3})$ *consists of* 4 *Desargues configurations; otherwise it consists of* 4 *Veronese configurations.*

(iii) *Consider the lines* $L_{i_1}, L_{i_2}, L_{i_2,i_3}$. *Let* $\mathcal{C}$ *be the graph* (i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_1). *If* $\mathcal{P}$ *contains* $\mathcal{C}$ *or odd number of edges of* $\mathcal{C}$ *then the family* $\mathcal{S}(L_{i_1}, L_{i_2}, L_{i_2,i_3})$ *consists of* 4 *Desargues configurations; otherwise it consists of* 4 *Veronese configurations.*

Proof. (i) Note that the pair of points on the Veblen figure inscribed into a pair of the given lines is $c_u, c_{\varkappa(u)}$, where $u \in \wp_2(y)$. The three points c_{u_i} with $u_i = y \setminus \{i\}$ are collinear in $\mathbf{G}_2(X)$. If $\varkappa$ is applied to one of the u_i or to three of the u_i, then the resulting points are not collinear in $\mathbf{G}_2(X)$ (though they are collinear in $\mathbf{G}^*_2(X)$). If $\varkappa$ is applied to two of the u_i, then we obtain a line of $\mathbf{G}_2(X)$. Directly we verify that the corresponding triples of collinear points c_{u_i} or $c_{\varkappa(u_i)}$ yield the same type of configuration as the original line $\wp_2(y)$.

(ii) In this case the pairs of points of corresponding Veblen figures inscribed into the given lines are: (a_{i_2}, b_{i_2}), (a_{i_3}, b_{i_3}), and $c_{\{i_2,i_3\}}, c_{\{i_1,i_4\}}$. If $(i_2, i_3) \in \mathcal{P}$ then the collinear triples are: $(a_{i_2}, a_{i_3}, c_{\{i_2,i_3\}})$, $(b_{i_2}, b_{i_3}, c_{\{i_2,i_3\}})$, $(a_{i_2}, b_{i_3}, c_{\{i_1,i_4\}})$, $(b_{i_2}, a_{i_3}, c_{\{i_1,i_4\}})$. The rest of the proof goes as in the case (i).

(iii) Now, pairs of points of suitable Veblen figures are: (b_{i_3}, a_{i_3}), (b_{i_4}, a_{i_4}), and $(c_{\{i_3,i_4\}}, c_{\{i_1,i_2\}})$. If $(i_3, i_4) \in \mathcal{P}$, then the collinear triples are: $(a_{i_3}, a_{i_4}, c_{\{i_3,i_4\}})$, $(b_{i_3}, b_{i_4}, c_{\{i_3,i_4\}})$, $(a_{i_3}, b_{i_4}, c_{\{i_1,i_2\}})$, and $(a_{i_4}, b_{i_3}, c_{\{i_1,i_2\}})$. □

Now we are in a position to determine the automorphism group of $\widetilde{\mathbb{B}(4)}$. Let us recall some elementary facts from group theory. Let $G \cong C_2^3$ be the subgroup of C_2^4 consisting of the elements $\sigma = (\sigma_1, \dots, \sigma_4) \in C_2^4$ such that $\sigma_1 + \dots + \sigma_4 = \theta$ (i.e. let G be the kernel of the homomorphism $\sigma \mapsto \sum_{i=1}^4 \sigma_i$). The group S_4 acts on C_2^4 via the map $S_4 \ni \alpha \longmapsto \alpha^*\colon \big((\sigma_1, \dots, \sigma_4) \mapsto (\sigma_{\alpha(1)}, \dots, \sigma_{\alpha(4)})\big)$ and G is invariant under this action (i.e. $\alpha^*(G) = G$ for every $\alpha \in S_4$). Consequently, S_4 acts via * on G, which means that S_4 acts on C_2^3 and this enables us to define $S_4 \ltimes C_2^3$.

Proposition 4.6. $\mathrm{Aut}(\widetilde{\mathbb{B}(4)}) \cong S_4 \ltimes C_2^3$.

Proof. Let $X = \{1, 2, 3, 4\}$, $P = \{a, b\}$, and $\mathfrak{B} = \widetilde{\mathbb{B}(4)}$. Finally, let $F \in \mathrm{Aut}(\mathfrak{B})$. From 4.1(V) we infer that $F(p) = p$, so F determines a permutation of the lines through p and, at the same time, an automorphism of the corresponding Fano plane $\mathfrak{F}$ (cf. 4.1(IV)). Let us adopt notation of 4.1(I)–(IV); from 4.5 we obtain that F maps the triples of lines of the form $L_{i_1}, L_{i_2}, L_{i_2,i_3}$ where $i_1, i_2, i_3 \in X$ onto triples of the same form ($\mathcal{P}$ contains 0 edges of $\mathcal{C}$ in notation of 4.5(iii)). From this we deduce that F permutes the lines L_i with $i \in X$ and leaves the family of the lines L_{i_1,i_2} invariant (this family is a line of $\mathfrak{F}$). Thus F determines the permutation $\alpha_F \in S_X$ such that $F(L_i) = L_{\alpha_F(i)}$. Evidently, every $\alpha \in S_X$ determines an automorphism $f_\alpha \in \mathrm{Aut}(\mathfrak{B})$ which extends the permutation $x_i \mapsto x_{\alpha(i)}$ ($x \in P$, $i \in X$) (cf. [15, Lemma 2]). To close the proof it remains to determine the subgroup of $\mathrm{Aut}(\mathfrak{B})$ consisting of the maps F such that $\alpha_F = \mathrm{id}$. Every such F is associated with a quadruple $\sigma = (\sigma_1, \dots, \sigma_4) \in (S_P)^4$ such that $F(x_i) = \sigma_i(x)_i$ for every symbol $x \in P$ and $i \in X$. Clearly, every σ as above determines the permutation F'_σ of the points a_i, b_j; direct verification shows that F'_σ can be extended to an automorphism F_σ of $\mathfrak{B}$ iff $\sigma_1 \circ \dots \circ \sigma_4 = \mathrm{id}$ so, the group of admissible quadruples is isomorphic to C_2^3. It is clear that $f_\alpha F_\sigma = F_{\alpha^*(\sigma)} f_\alpha$ for $\alpha \in S_X$. $\square$

Proposition 4.7. *The completions of* $\mathbb{B}(4)$ *and of* $\mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(4)$ *are isomorphic.*

Proof. Let $X = \{1,2,3,4\}$ and let $\mathfrak{B}$ be the linear completion of $\mathbb{B}(X)$, as defined in 3.1. Next, let $\mathcal{P} = L_4$ be the suitable linear graph on X. Direct verification shows that the following function

$$\begin{array}{ccccccccc} p & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & a_4 & b_4 \\ p & b_1 & a_1 & c_{\{2,4\}} & c_{\{1,3\}} & c_{\{1,2\}} & c_{\{3,4\}} & a_4 & b_4 \end{array}$$

$$\begin{array}{cccccc} c_{\{1,2\}} & c_{\{2,3\}} & c_{\{3,4\}} & c_{\{1,4\}} & c_{\{1,3\}} & c_{\{2,4\}} \\ b_3 & c_{\{1,4\}} & a_3 & c_{\{2,3\}} & b_2 & a_2 \end{array}$$

embeds $\mathbb{M}^4 \triangleright_{L_4} \mathbf{G}_2(X)$ into $\mathfrak{B}$, which yields our claim. $\square$

The two possible completions of the multiveblen configurations $\mathbb{M}^X \triangleright_{\mathcal{P}} \mathfrak{H}$ where $|X| = 4$ and $\mathfrak{H} = \widetilde{\mathbf{G}_2(X)}$ or $\mathfrak{H} = \mathbf{G}_2^*(X)$, i.e. (cf. 4.4 and 4.7) the Fano 3-space and the structure $\mathbb{B}(4)$, have an interesting geometrical characterization, which is in fact a converse of the analysis 4.1.

Theorem 4.8. *Let* $\mathfrak{B}$ *be a* $(15_7\, 35_3)$*-configuration such that for some point* p *of* $\mathfrak{B}$ *any two lines through* p *yield a Fano subplane of* $\mathfrak{B}$*. Then either* $\mathfrak{B}$ *is a Fano 3-space* $PG(3,2)$ *or* $\mathfrak{B} \cong \widetilde{\mathbb{B}(4)}$.

Proof. From assumptions, the lines and planes through p yield a $(7_3\, 7_3)$-configuration i.e. a Fano plane. Set $X = \{1,2,3,4\}$.

Let L_1, L_2, L_3 be three noncoplanar lines through p; for every $1 \le i < j \le 3$ we have a Fano plane $\Pi_{i,j}$ spanned by L_i, L_j, and in every of these planes we have the third line $L_{i,j}$ through p. Let L_4 be the last, seventh line through p.

Let a_i, b_i be points of $\mathfrak{B}$ such that $L_i = \{p, a_i, b_i\}$ for $i \in X$. Let $1 \le i < j \le 3$; without loss of generality we can label the points on $L_{i,j}$ by the elements of $\wp_2(X)$ in such a way that $\{a_i, a_j, c_{\{i,j\}}\}$, $\{b_i, b_j, c_{\{i,j\}}\}$, $\{a_i, b_j, c_u\}$, and $\{a_j, b_i, c_u\}$ ($u = X \setminus \{i,j\}$) are lines of $\Pi_{i,j}$.

Next, for k such that $\{i,j,k\} = \{1,2,3\}$ we have the plane which contains the lines $L_4, L_k, L_{i,j}$ and thus either

$$\overline{a_4, a_k},\ \overline{b_4, b_k} \text{ pass through } c_{\{4,k\}} \text{ and } \overline{a_4, b_k},\ \overline{b_4, a_k} \text{ pass through } c_{\{i,j\}}$$

or

$\overline{a_4, a_k}$, $\overline{b_4, b_k}$ pass through $c_{\{i,j\}}$ and $\overline{a_4, b_k}$, $\overline{b_4, a_k}$ pass through $c_{\{4,k\}}$.

We define the graph $\mathcal{P}$ on X putting: $\{i,j\} \in \mathcal{P}$ for all $1 \le i < j \le 3$ and for $k \in \{1,2,3\}$ we have $\{k,4\} \in \mathcal{P}$ iff $\overline{a_4, a_k}$ passes through $c_{\{4,k\}}$.

The last plane through p contains the lines $L_{1,2}$, $L_{1,3}$, and $L_{2,3}$. Consider the line $\overline{c_{\{1,2\}}, c_{\{1,3\}}}$; its third point is $c_{\{2,3\}}$ or $c_{\{1,4\}}$. Direct verification shows that in the first case the points c_x yield the structure $\mathbf{G}_2(X)$, and in the second case they yield $\mathbf{G}_2^*(X)$. In any case we see that $\mathfrak{B} \cong \mathbb{M}^X \widetilde{\triangleright_{\mathcal{P}} \mathbf{G}_2}(X)$ or $\mathfrak{B} \cong \mathbb{M}^X \widetilde{\triangleright_{\mathcal{P}} \mathbf{G}_2^*}(X)$, which closes the proof. $\square$

Next, we pass to the completions of some less regular and "less projective" multiveblen configurations.

Proposition 4.9. *The completions of two configurations* $\mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(2)$ *and* $\mathbb{M}^4 \triangleright_{\{\{3,4\}\}} \mathbb{B}(2)$ *are isomorphic.*

Proof. Let $X = \{1,2,3,4\}$. It suffices to observe that the map

$$\begin{array}{cccccccccc} p & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & a_4 & b_4 & c_x,\ x \in \wp_2(X) \\ p & a_3 & b_3 & b_4 & a_4 & a_1 & b_1 & b_2 & a_2 & c_x \end{array}$$

embeds $\mathbb{M}^X \triangleright_{\{\{1,2\}\}} \mathbb{B}(\{1,2\})$ into $\mathbb{M}^X \widetilde{\triangleright_{\{\{3,4\}\}}} \mathbb{B}(\{1,2\})$. $\square$

Let us analyze in some details the geometry of the completions of the structures in the family $\mathbb{M}^4 \triangleright_{\mathcal{P}} \mathbb{B}(2)$.

Analysis 4.10. Let $X = \{1,2,3,4\}$, $r = \{1,2\}$, and $\mathcal{P}$ be a graph defined on X of the form (1). Write $\mathfrak{A} = \mathbb{M}^X \triangleright_{\mathcal{P}} \mathbb{B}(r)$ and let $\mathfrak{B}$ be the completion of $\mathfrak{A}$.

(I) Let us number the new lines through p as follows: $L_7 = \overline{c_{\{1,2\}}, c_{\{3,4\}}}$, $L_8 = \overline{c_{\{1,3\}}, c_{\{2,3\}}}$, $L_9 = \overline{c_{\{1,4\}}, c_{\{2,4\}}}$. Then the three triples (L_7, L_8, L_9), (L_7, L_1, L_2), and (L_7, L_3, L_4) span in $\mathfrak{B}$ three Fano planes, which have the line L_7 in common. The corresponding Fano planes will be denoted by $\Pi_{\mathcal{C}}$, $\Pi_{1,2}$, and $\Pi_{3,4}$ respectively. The union of the above three planes is the point set of $\mathfrak{B}$.

(II) No two other lines through p span in $\mathfrak{B}$ a Fano plane, and (besides the above) only pairs (L_i, L_j) with $i, j \le 4$ yield in $\mathfrak{B}$ Veblen figures (each pair yields two Veblen figures).

(III) Every line through a point a_i crosses the plane $\Pi_{\mathcal{C}}$. Let M_1, M_2 be two lines through a point a_i (or through a point b_i) and let Γ be the Fano plane which contains a_i and L_7 ($\Gamma = \Pi_{i,i'}$, where $\{i,i'\} = \{1,2\}$ or $\{i,i'\} = \{3,4\}$). Assume that $\mathcal{P} = \emptyset$. Then

- either $M_1, M_2 \subset \Gamma$ and then M_1, M_2 span in $\mathfrak{B}$ the Fano plane Γ, or
- $M_j \cap \Gamma = \{a_i\}$ for $j = 1,2$ and then there is no Veblen figure inscribed into M_1, M_2, or
- $M_{j_1} \subset \Gamma$, $M_{j_2} \cap \Gamma = \{a_i\}$, where $\{j_1, j_2\} = \{1,2\}$. Then there is exactly one Veblen figure inscribed into the M_j.

Analogous analysis repeated for the cases $\mathcal{P} = \{1,2\}$ and $\mathcal{P} = \{1,3\}$ proves that in every of the above cases there is no Fano figure spanned by a pair of lines through a_i such that at least one of these lines is not contained in Γ. We write shortly $\Pi_1 = \Pi_{1,2}$ and $\Pi_2 = \Pi_{3,4}$.

(IV) Let q be a point of $\mathfrak{B}$ not on L_7. Then $\mathfrak{B}$ contains exactly one Fano subplane through q. Indeed, for $q = a_i$ the claim follows from (III). If $q = c_u$ and Π is a Fano plane through q distinct from $\Pi_{\mathcal{C}}$, then there is in Π a point of the form a_i (or b_i resp.), so Π would be a Fano plane through a_i distinct from the three listed in (I), which contradicts (III).

(V) Clearly, in view of 3.1 every $f \in \mathrm{Aut}(\mathfrak{A})$ has the unique extension $\widetilde{f}$ to an automorphism of $\mathfrak{B}$. From (IV) we infer immediately that an arbitrary automorphism f of $\mathfrak{B}$ leaves the line L_7 invariant. Consequently, f permutes the planes in $\{\Pi_1, \Pi_2, \Pi_{\mathcal{C}}\}$ and it permutes the points on L_7. Moreover, if $f(p) = q$ then there are two planes Γ_1, Γ_2 in $\{\Pi_1, \Pi_2, \Pi_{\mathcal{C}}\}$ with the following property (cf. (I), (II)):

Let $M_i, M_{3+i} \subset \Gamma_i$ be two lines through q distinct from L_7 for $i = 1, 2$. For every $1 \le j_1 < j_2 \le 4$ there are in $\mathfrak{B}$ two Veblen figures inscribed into M_{j_1}, M_{j_2}. $(*)$

(i) Let $\mathcal{P} = \emptyset$. Direct verification shows that the following permutation is an automorphism of $\mathfrak{B}$:

$$\begin{matrix} p & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & a_4 & b_4 & c_{\{1,2\}} & c_{\{1,3\}} & c_{\{1,4\}} & c_{\{2,3\}} & c_{\{2,4\}} & c_{\{3,4\}} \\ c_{\{3,4\}} & a_3 & b_4 & a_4 & b_3 & b_1 & b_2 & a_1 & a_2 & p & c_{\{2,3\}} & c_{\{1,4\}} & c_{\{2,4\}} & c_{\{1,3\}} & c_{\{1,2\}} \end{matrix}.$$

Consequently, no point on L_7 is distinguished in terms of the geometry of $\mathfrak{B}$.

(ii) Let $\mathcal{P} = \{\{1,2\}\}$. From 2.3(iii) and 4.7 it follows that $\mathfrak{B}$ has an automorphism f which interchanges p and $c_{\{3,4\}}$ and thus $(*)$ holds for $q = c_{\{3,4\}}$. But $(*)$ does not hold for $q = c_{\{1,2\}}$. Indeed, $\overline{a_3, a_4} \subset \Pi_2$, $\overline{b_1, b_2} \subset \Pi_1$ both pass through q and do not yield a Veblen figure, $\overline{a_1, a_2} \subset \Pi_1$, $\overline{c_{\{1,3\}}, c_{\{2,4\}}} \subset \Pi_{\mathcal{C}}$ do not yield a Veblen figure, and $\overline{b_3, b_4} \subset \Pi_2$, $\overline{c_{\{1,3\}}, c_{\{1,3\}}} \subset \Pi_{\mathcal{C}}$ do not yield a Veblen figure. Thus every $f \in \mathrm{Aut}(\mathfrak{B})$ fixes $c_{\{1,2\}}$.

(iii) Let $\mathcal{P} = \{\{1,3\}\}$. Then $(*)$ does not hold for $q = c_{\{1,2\}}$ and $q = c_{\{3,4\}}$. Namely, the following pairs of lines through $c_{\{1,2\}}$ falsify $(*)$: $\big(\overline{b_3, b_4} \subset \Pi_2,\ \overline{a_2, b_1} \subset \Pi_1\big)$, $\big(\overline{b_3, b_4} \subset \Pi_2,\ \overline{c_{\{1,3\}}, c_{\{2,4\}}} \subset \Pi_{\mathcal{C}}\big)$, and $\big(\overline{a_2, b_1} \subset \Pi_1,\ \overline{c_{\{1,3\}}, c_{\{2,4\}}} \subset \Pi_{\mathcal{C}}\big)$. And the pairs $\big(\overline{a_1, a_2} \subset \Pi_1, \overline{a_4, b_3} \subset \Pi_2\big)$, $\big(\overline{a_4, b_3} \subset \Pi_2, \overline{c_{\{1,3\}}, c_{\{1,4\}}} \subset \Pi_{\mathcal{C}}\big)$, and $\big(\overline{b_1, b_2} \subset \Pi_1, \overline{c_{\{1,3\}}, c_{\{1,4\}}} \subset \Pi_{\mathcal{C}}\big)$ of lines through $c_{\{3,4\}}$ falsify $(*)$. Thus every $f \in \mathrm{Aut}(\mathfrak{B})$ fixes the point p.

From (i)–(iii) we learn that the three Steiner triple systems – completions of multiveblen configurations – $\mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(\{1,2\})$, $\mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(\{1,2\})$, and $\mathbb{M}^4 \triangleright_{\{\{1,3\}\}} \mathbb{B}(\{1,2\})$ are pairwise nonisomorphic.

It is also worth to mention another general construction, which yields, in some particular cases, representations of our linear completions.

Construction 4.11. Let $\mathfrak{M} = \langle S, \mathcal{L}\rangle$ be a Steiner triple system, and let M be its subspace. We write $D := S \setminus M$, $S' := D \times \{0,1,2\}$, $S_i := M \cup D \times \{i\}$ for $i = 0,1,2$, and $S^* := M \cup S'$.

Finally, let $\circledcirc$ be a binary operation defined on D with left and right subtraction (the equations $a \circledcirc x = b$ and $x \circledcirc a = b$ are uniquely solvable for any given $a, b \in D$). We define the family $\mathcal{L}^*$ consisting of the following 3-subsets of S^*.

- A line $L \in \mathcal{L}$ contained in M belongs to $\mathcal{L}^*$.
- If $L \in \mathcal{L}$ and $L \cap M = \emptyset$ then $L \times \{i\} \in \mathcal{L}^*$ for every $i = 0,1,2$.
- If $L \in \mathcal{L}$ and $L \cap M = \{p\}$ then $\{p\} \cup (L \cap D) \times \{i\} \in \mathcal{L}^*$ for $i = 0,1,2$.
- If $a, b \in D$ then the set $\{(a,1),(b,2),(a \circledcirc b, 0)\}$ belongs to $\mathcal{L}^*$.

The obtained structure

$$\Xi(\mathfrak{M}, M, \circledcirc) := \langle S^*, \mathcal{L}^*\rangle$$

is a Steiner triple system. The above construction is motivated by the following observations. The three subsets S_i of S^* yield three subspaces of $\Xi(\mathfrak{M}, M, \circledcirc)$ which intersect in the subspace M; for fixed $i, j \in C_3$ the operation $(x,i) \mapsto (x,j)$ with $x \in D$ is a (restriction of a) collineation of S_i onto S_j which fixes M; for every $x \in D$ and $\{i,j,k\} = \{0,1,2\}$, the lines through (x,i) yield a "perspective" from S_j onto S_k.

More precisely, we can compute the following

Proposition 4.12. *Let a Steiner triple system $\mathfrak{M}$ be a $\big(\boldsymbol{\nu}_{\boldsymbol{r}}\, \boldsymbol{b}_3\big)$-configuration. Let its subspace M yield a $\big((\boldsymbol{\nu}_0)_{\boldsymbol{r}_0}\, (\boldsymbol{b}_0)_3\big)$-configuration, and let $\circledcirc$ be a binary operation as required in* 4.11. *Then the structure $\Xi(\mathfrak{M}, M, \circledcirc)$ is a $\big((\boldsymbol{\nu}_0 + 3(\boldsymbol{\nu} - \boldsymbol{\nu}_0))_{(\boldsymbol{r}_0 + 3(\boldsymbol{r} - \boldsymbol{r}_0))}\, (\boldsymbol{b}_*)_3\big)$-configuration, where $\boldsymbol{b}_* = 9\boldsymbol{b} + 4\boldsymbol{b}_0 - 2(\boldsymbol{r}_0\boldsymbol{\nu} + \boldsymbol{r}\boldsymbol{\nu}_0) = \boldsymbol{b}_0 + 3(\boldsymbol{b} - \boldsymbol{b}_0) + (\boldsymbol{\nu} - \boldsymbol{\nu}_0)^2$.*

Proof. Adopt the notation of 4.11. Let $a \in M$; every line through a not contained in M is triplicated and thus the degree of a in $\Xi(\mathfrak{M}, M, \circledcirc)$ is $\boldsymbol{r}_0 + 3(\boldsymbol{r} - \boldsymbol{r}_0)$. Let $a \in D$ and $q = (a,i)$ ($i = 1$ or $i = 2$). The lines of $\Xi(\mathfrak{M}, M, \circledcirc)$ through q fall into the two classes: those which are determined by the lines through a and those which join q with the points of the form $(b, 3 - i)$, $b \in D$. There are $\boldsymbol{r}$ lines in the first class and $\boldsymbol{\nu} - \boldsymbol{\nu}_0$ lines in the second class and thus the degree of q is $\boldsymbol{r} + (\boldsymbol{\nu} - \boldsymbol{\nu}_0)$. Analogously we compute the degree of $(a,0)$. From the assumptions we have $2\boldsymbol{r}_0 = \boldsymbol{\nu}_0 - 1$ and $2\boldsymbol{r} = \boldsymbol{\nu} - 1$, which gives $\boldsymbol{r}_0 + 3(\boldsymbol{r} - \boldsymbol{r}_0) = \boldsymbol{r} + (\boldsymbol{\nu} - \boldsymbol{\nu}_0)$. □

A structure of the form $\Xi(\mathfrak{F}, L, \circledcirc)$, where $\mathfrak{F}$ is the Fano plane, L is a line of $\mathfrak{F}$, and $\circledcirc$ is a suitable binary operation will be called *a twisted Fano space.*

The construction of $\Xi(\mathfrak{M}, M, \circledcirc)$ will be applied to represent the linear completion $\mathfrak{B} = \widetilde{\mathfrak{A}}$ of a structure of the form $\mathfrak{A} = \mathbb{M}^4 \triangleright_{\mathcal{P}} \mathfrak{H}$ as a twisted Fano space. Let us consider, first, three Fano planes Π_C, Π_A, and Π_B of $\mathfrak{B}$ with a common line L, which is always possible in view of 4.1(IV) and 4.10(I) (the symbols A, B, C play the role of the elements of $\{0, 1, 2\}$, $A \sim 1$, $B \sim 2$, and $C \sim 0$). Say $L = L_7 = L_{1,2}$ and let the points on L be

$$p_1 = p, \quad p_2 = c_{\{1,2\}}, \quad \text{and} \quad p_3 = c_{\{3,4\}}.$$

We choose a point $X_0 \in \Pi_X \setminus L$ for every $X \in \{A, B, C\}$ such that A_0, B_0, C_0 are collinear; without loss of generality we can take

$$A_0 = a_1, \quad B_0 = b_3.$$

After that we label the remaining points in $\Pi_X \setminus L$ by the symbols X_i $(i = 1, 2, 3)$ in such a way that the maps $A_i \mapsto B_i$ and $A_i \mapsto C_i$ are collineations constant on L (thus the elements of Z_4 can be identified with the elements of $\Pi_A \setminus L$). Without loss of generality we can assume that the following triples are collinear

$$(X_0, X_2, p_1), (X_0, X_1, p_3), (X_0, X_3, p_2), (X_2, X_1, p_2), (X_2, X_3, p_3), (X_1, X_3, p_1).$$

For every $i, j \leq 3$ we have the unique $k \leq 3$ such that $C_k \in \overline{A_i, B_j}$ so we have the binary operation $\circledcirc$ defined on Z_4 by the condition $C_{i \circledcirc j} \in \overline{A_i, B_j}$. Finally, we see that

$$\mathfrak{B} \quad \cong \quad \Xi(\Pi_A, L, \circledcirc). \tag{4}$$

Let us analyze some particular cases. In accordance with the above rules we relabel the points on the three Fano planes of $\mathfrak{B} = \widetilde{\mathfrak{A}}$ spanned by $L_1 \cup L_2$, $L_3 \cup L_4$, and the set $\mathcal{C}$, respectively, as follows:

(i) $\mathfrak{A} = \mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)$: $C_0 = c_{\{1,3\}}$, $A_1 = a_2$, $A_2 = b_1$, $A_3 = b_2$, $B_1 = a_4$, $B_2 = a_3$, $B_3 = b_4$, $C_1 = c_{\{1,4\}}$, $C_2 = c_{\{2,3\}}$, and $C_3 = c_{\{2,4\}}$.

(ii) $\mathfrak{A} = \mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2(4)$: $C_0 = c_{\{1,3\}}$, the A_i and B_i are as in (i), $C_1 = c_{\{1,4\}}$, $C_2 = c_{\{2,4\}}$, $C_3 = c_{\{2,3\}}$.

(iii) $\mathfrak{A} = \mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)$: $C_0 = c_{\{2,4\}}$, $A_1 = b_2$, $A_2 = b_1$, $A_3 = a_2$, $B_1 = b_4$, $B_2 = a_3$, $B_3 = a_4$, $C_1 = c_{\{2,3\}}$, $C_2 = c_{\{1,3\}}$, $C_3 = c_{\{1,4\}}$.

(iv) $\mathfrak{A} = \mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(2)$: $C_0 = c_{\{1,3\}}$, $A_1 = a_2$, $A_2 = b_1$, $A_3 = b_2$, $B_1 = b_4$, $B_2 = a_3$, $B_3 = a_4$, $C_1 = c_{\{2,4\}}$, $C_2 = c_{\{2,3\}}$, $C_3 = c_{\{1,4\}}$ (here the labeling of the points $c_{\{1,2\}}, c_{\{3,4\}}$ by the symbols p_2, p_3 was interchanged).

(v) $\mathfrak{A} = \mathbb{M}^4 \triangleright_{\{\{1,3\}\}} \mathbb{B}(2)$: $C_0 = c_{\{2,4\}}$, $A_1 = a_2$, $A_2 = b_1$, $A_3 = b_2$, $B_1 = a_4$, $B_2 = a_3$, $B_3 = b_4$, $C_1 = c_{\{2,3\}}$, $C_2 = c_{\{1,4\}}$, $C_3 = c_{\{1,3\}}$.

The corresponding operation $\circledcirc$ such that $\widetilde{\mathfrak{A}} = \Xi(\Pi_A, L_7, \circledcirc)$ is defined by one of the following tables:

$\circledcirc$	0	1	2	3
0	0	2	3	1
1	2	0	1	3
2	3	1	0	2
3	1	3	2	0

Table 2.
$\circledcirc$ in $\mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)$

$\circledcirc$	0	1	2	3
0	0	3	2	1
1	3	0	1	2
2	2	1	0	3
3	1	2	3	0

Table 3.
$\circledcirc$ in $\mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2(4)$

$\circledcirc$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 4.
$\circledcirc$ in $\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)$

$\circledcirc$	0	1	2	3
0	0	2	1	3
1	3	1	2	0
2	1	3	0	2
3	2	0	3	1

Table 5.
$\circledcirc$ in $\mathbb{M}^4 \triangleright_{\{\{1,2\}\}} \mathbb{B}(2)$

$\circledcirc$	0	1	2	3
0	0	1	3	2
1	1	3	2	0
2	3	2	0	1
3	2	0	1	3

Table 6.
$\circledcirc$ in $\mathbb{M}^4 \triangleright_{\{\{1,3\}\}} \mathbb{B}(2)$

The structure $\mathfrak{B}$ does not determine uniquely a multiplication $\circledcirc$ such that $\mathfrak{B} \cong \Xi(\Pi_A, L, \circledcirc)$. Firstly, the labeling of the planes through L by the symbols A, B, C is arbitrary. Secondly, the numbering of the points A_i in $\Pi_A \setminus L$ is arbitrary. Thirdly, the choice of the point $B_0 \in \Pi_B \setminus L$ is arbitrary. Once B_0 was chosen, the numbering of the other points in $\Pi_B \setminus L$ and of the points in $\Pi_C \setminus L$ is determined. Every such numbering defines other multiplication table. The obtained operations are not necessarily isomorphic, and every one of these operations defines the same structure $\mathfrak{M}$.

It is worth to note that the considered multiplication tables are latin squares of the size 4×4. Consequently, our procedure can be performed with an arbitrary latin square defining $\circledcirc$, possibly generalizing our investigations, and our configurations can be studied in this language as well.

The multiplication table (Table 4) which defines the structure $PG(3,2) = \widetilde{\mathbf{G}_2(6)}$ is isomorphic to the addition table of the group $C_2 \oplus C_2$. It is an expected general result:

Proposition 4.13. *Let V be a k-dimensional vector space over the field $GF(2)$ with the zero vector $\mathbf{0}$, and Y be a $(k-1)$-dimensional subspace of V. Recall that points of the projective space $PG(k-1,2)$ are simply nonzero vectors of V and then $Y \setminus \{\mathbf{0}\}$ is a hyperplane of this space. The additive group of Y is isomorphic to C_2^{k-1}. Let $\boldsymbol{e}$ be a fixed vector in $V \setminus Y$. Every vector of $V \setminus Y$ can be written in the form $\boldsymbol{e} + y$ with $y \in Y$ so we can define on $V \setminus Y$ the operation $\circledcirc$: $(\boldsymbol{e} + y_1) \circledcirc (\boldsymbol{e} + y_2) = (\boldsymbol{e} + (y_1 + y_2))$. Clearly, $\langle V \setminus Y, \circledcirc \rangle \cong C_2^{k-1}$. Then $\Xi(PG(k-1,2), Y \setminus \{\mathbf{0}\}, \circledcirc) \cong PG(k,2)$.*

Proof. It suffices to note that the points of $PG(k,2)$ are nonzero vectors of a $(k+1)$-dimensional vector space W over $GF(2)$; without loss of generality we can assume that $W = Y \oplus \langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle$ for some vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$. Every vector $u \in W \setminus Y$ can be written in the form $u = y + \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2$ with $y \in Y$ and $\alpha_1, \alpha_2 \in \{0,1\}$. Let us put for $a = y + \boldsymbol{e} \in V \setminus Y$

$$(a,1) := y + \boldsymbol{e}_1,\ (a,2) := y + \boldsymbol{e}_2, \text{ and } (a,0) := y + \boldsymbol{e}_1 + \boldsymbol{e}_2.$$

Direct verification shows that the above identification establishes an isomorphism of $\Xi(PG(k-1,2), Y \setminus \{\mathbf{0}\}, \circledcirc)$ on $PG(k,2)$. $\square$

The trick used above enables us to introduce into our completions some analytical methods. The most "similar to the projective" structure obtained as the linear completion of $\mathbb{B}(4)$ has an interesting analytical characterization.

Representation 4.14. Let us consider the 4-dimensional vector space Z_2^4 and let Y be its subspace spanned by the vectors $[1,0,0,0]$ and $[0,0,0,1]$. Let us take the points $A_0 = [0,1,0,0]$, $B_0 = [0,0,1,0]$, $C_0 = [0,1,1,0]$ and the corresponding projective planes spanned by Y and the above points. The remaining projective points that are not on Y are then labeled as follows:

$$\begin{array}{llll} \text{on } \Pi_A: & A_1 = [1,1,0,1] & A_2 = [1,1,0,0] & A_3 = [0,1,0,1] \\ \text{on } \Pi_B: & B_1 = [1,0,1,1] & B_2 = [1,0,1,0] & B_3 = [0,0,1,1] \\ \text{on } \Pi_C: & C_1 = [1,1,1,1] & C_2 = [1,1,1,0] & C_3 = [0,1,1,1]. \end{array}$$

Clearly, the operation $\circledcirc$ defined by Table 4 characteristic for $\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)$ yields the original projective structure in Z_2^4.

Let $\circledcirc$ be defined by Table 3; then $\Xi(\Pi_A, Y, \circledcirc) = \widetilde{\mathbb{B}(4)} =: \mathfrak{B}$. Evidently, the projective lines over Z_2^4 contained in Π_X with $X \in \{A,B,C\}$ remain lines of $\mathfrak{B}$. In particular, since every projective line which crosses Y is contained in one of these planes, every such a line is a line of $\mathfrak{B}$. One can directly verify that projective lines through C_0 and through C_2 remain lines of $\mathfrak{B}$ as well. The only distinction concerns the lines through C_1 and C_3: if L is a projective line through C_i not contained in Π_C ($i \in \{1,3\}$) then $\mathfrak{B}$ contains as a line the set $(L \setminus \{C_i\}) \cup \{C_{4-i}\}$.

Finally, let us apply to the above points the multiplication defined in Table 2; then $\mathfrak{B} := \Xi(\Pi, Y, \circledcirc) = \widetilde{\mathbb{M}^4 \triangleright_{N_4} \mathbb{B}(2)}$. As above, the projective lines over Z_2^4 contained in Π_X with $X \in \{A,B,C\}$ remain lines of $\mathfrak{B}$. The projective lines through C_0 also remain lines of $\mathfrak{B}$. If L is a projective line through C_i not contained in Π_C ($i \neq 0$) then the set $(L \setminus \{C_i\}) \cup \{C_{\alpha(i)}\}$ is a line of $\mathfrak{B}$, where α is the cycle $(1,2,3)$.

In any case, the structure $\mathfrak{B}$ results from the Fano space $PG(3,2)$ by replacing original lines missing a fixed line L by some new family of 3-subsets of the point set of $PG(3,2)$. This justifies the term "twisted" that is used for structures of this form.

The obtained information concerning structures discussed in the paper, their automorphisms, and completions are summarized in Table 7.

5. Remarks on closing configurations

Let $\mathfrak{A}$ be a $\big(15_4\, 20_3\big)$-multiveblen configuration and $\widetilde{\mathfrak{A}}$ be its linear completion. The $\big(15_3\, 15_3\big)$-configuration consisting of the points of $\mathfrak{A}$ and of the lines of $\widetilde{\mathfrak{A}}$ not in $\mathfrak{A}$ will be called the *closing configuration* of $\mathfrak{A}$ and will be denoted by $\mathfrak{A}^\infty$. Without going into details concerning the geometry of closing configurations of the multiveblen configurations considered in the paper we note only the following remarks

(i) $\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)^\infty \cong \mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2^*(4)^\infty$, but $\widetilde{\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2(4)} \not\cong \widetilde{\mathbb{M}^4 \triangleright_{K_4} \mathbf{G}_2^*(4)}$;

(ii) $\mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2(4)^\infty \cong \mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2^*(4)^\infty$, but $\widetilde{\mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2(4)} \not\cong \widetilde{\mathbb{M}^4 \triangleright_{N_4} \mathbf{G}_2^*(4)}$;

structure	Aut	completion	other representations
$\mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2(4)$	S_6	$PG(3,2)$ $\Lambda(PG(2,2),0)$	$\mathbf{G}_2(6)$
$\mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2(4)$	$C_2\oplus S_4$	$\widetilde{\mathbb{B}(4)}$	$\mathbb{B}(4)$, $\mathbf{V}_3(4)^\circ$
$\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2(4)$	C_2^3	$\widetilde{\mathbb{B}(4)}$	$\mathbb{M}^4\triangleright_{K_4}\mathbb{B}(2)$
$\mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2^*(4)$	$C_2\oplus S_4$	$\widetilde{\mathbb{B}(4)}$	
$\mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2^*(4)$	$C_2\oplus S_4$	$PG(3,2)$ $\Lambda(PG(2,2),1)$	
$\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2^*(4)$	$C_2\oplus C_2^2$	$\widetilde{\mathbb{B}(4)}$	
$\mathbb{M}^4\triangleright_{N_4}\mathbb{B}(2)$	$C_2\oplus C_2^2$	$\mathfrak{B}_1$	
$\mathbb{M}^4\triangleright_{\{\{1,2\}\}}\mathbb{B}(2)$	C_2^3	$\mathfrak{B}_2$	
$\mathbb{M}^4\triangleright_{\{\{3,4\}\}}\mathbb{B}(2)$	$C_2\oplus D_4$	$\mathfrak{B}_2$	
$\mathbb{M}^4\triangleright_{\{\{1,3\}\}}\mathbb{B}(2)$	C_2^2	$\mathfrak{B}_3$	

Table 7. A review of the properties of the considered structures (the symbols $\mathfrak{B}_i$ merely indicate the isomorphism type).

(iii) $\widetilde{\mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2}(4) \cong \widetilde{\mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2^*}(4)$, but $\mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2(4)^\infty \ncong \mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2^*(4)^\infty$;

(iv) $\widetilde{\mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2}(4) \cong \widetilde{\mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2^*}(4)$, but $\mathbb{M}^4\triangleright_{N_4}\mathbf{G}_2(4)^\infty \ncong \mathbb{M}^4\triangleright_{K_4}\mathbf{G}_2^*(4)^\infty$;

(v) $\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2(4)^\infty \cong \mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2^*(4)^\infty$ and $\widetilde{\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2}(4) \cong \widetilde{\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2^*}(4)$, but $\mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2(4) \ncong \mathbb{M}^4\triangleright_{L_4}\mathbf{G}_2^*(4)$.

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