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# Elementary Versions of the Sylvester-Gallai Theorem

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**Abstract.** A Sylvester-Gallai (SG) configuration is a set $S$ of $n$ points such that the line through any two points of $S$ contains a third point in $S$. L. M. Kelly (1986) positively settled an open question of Serre (1966) asking whether an SG configuration in a complex projective space must be planar. N. Elkies, L. M. Pretorius, and K. J. Swanepoel (2006) have recently reproved this result using elementary means, and have proved that SG configurations in a quaternionic projective space must be contained in a three-dimensional flat. We point out that these results hold in a setting that is much more general than $\mathbb{C}$ or $\mathbb{H}$, and that, for each individual value of $n$, there must be truly elementary proofs of these results. Kelly's result must hold in projective spaces over arbitrary fields of characteristic 0 and the new result of Elkies, Pretorius and Swanepoel must hold in all quaternionic skew-fields over a formally real center.

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In [4] the authors prove, by ingenious means that are significantly more elementary than those used in the original proof in [7], that Sylvester-Gallai (SG) configurations in projective spaces over $\mathbb{C}$ must be planar. They also prove a new result, namely that SG configurations in $\mathbb{H}$ must be contained in a three-dimensional flat.

The purpose of this note is to point out that these results (as well as the original Sylvester-Gallai theorem over $\mathbb{R}$) can be generalized to a purely algebraic setting, and thus be made genuinely elementary, in the strictly logical sense of being results regarding the validity of a first-order sentence in a first-order theory. However, in the original SG-theorem and in Kelly's version, this does not represent anything new, for, by a standard technique belonging to the algebraic geometry folklore, results over $\mathbb{R}$ and $\mathbb{C}$ can be transferred to real closed and to algebraically closed fields (the transfer principles are sometimes referred to as the Seidenberg-Tarski and the Lefschetz principle, having been first rigorously established by Tarski). In the case of the SG version over $\mathbb{H}$, we employ the same strategy with help from the less well-known designants of Heyting.

To show that the theories inside which the SG-theorems will hold are indeed elementary by being first-order theories, we shall first present the axiom system of Lenz [8] for dimension-free (unspecified dimension $\geq 3$) projective spaces.

Lenz's axiom system is expressed in the two-sorted language $\mathsf{L}_{\in}$, with variables for points and lines to be denoted by upper- and lowercase letters, and a binary relation $\in$ between points and lines, with $A \in l$ to be read as '$A$ is incident with $l$'. We shall use the following convenient abbreviations: $(A_1, \ldots, A_n \in l)$ for $A_1 \in l \wedge \cdots \wedge A_n \in l$, $A \in l_1, \ldots, l_n$ for $A \in l_1 \wedge \cdots \wedge A \in l_n$ , and $\neq (A_1 \ldots A_n)$ for $\bigwedge_{i\neq j} A_i \neq A_j$. Its axioms are:

L1 $(\forall AB)(\exists l)(\forall l')\, A \neq B \rightarrow (A, B \in l) \wedge [(A, B \in l' \rightarrow l' = l)]$

L2 $(\forall ABCDElmnp)(\exists P) \neq (ABCD) \wedge (A, B, E \in l) \wedge (C, D, E \in m)$
$\wedge (A, C \in n) \wedge (B, D \in p) \rightarrow (P \in n, p)$

L3 $(\forall l)(\exists ABC) \neq (ABC) \wedge (A, B, C \in l)$

L4 $(\exists lm)(\forall P)\, \neg(P \in l, m)$

Here L1 states that there is a unique line incident with two distinct points, L2 is Veblen's axiom, L3 states that there are three points on every lines, and L4 that there are two skew lines. We shall refer to the theory axiomatized by this axiom system as $\mathcal{L}$.

Given that all models of $\mathcal{L}$ are at least 3-dimensional, the theorem of Desargues holds in $\mathcal{L}$. Let $\mathbf{P}$ denote the axiom of Pappus (see e.g. [10]). It was shown in [8] that models of $\mathcal{L}$ can be coordinatized by means of skew fields, which are commutative precisely when $\mathbf{P}$ holds. All algebraic statements can thus be translated in the language of $\mathcal{L}$. Let $p$ be a prime and $\varphi_p$ be the $\mathsf{L}_{\in}$-statement corresponding to the algebraic statement that $p \neq 0$. Let $\pi_k(A_1, \ldots, A_n)$ stand for the $\mathsf{L}_{\in}$-statement that $A_1, \ldots, A_n$ lie in a $k$-dimensional flat. Let $\tau$ denote the $\mathsf{L}_{\in}$-statement corresponding to the algebraic statement that, for all $x, y, z$, $x(yz - zy)^2 = (yz - zy)^2 x$ (this statement plays an important role in characterizing the quaternions), and let $\psi_{m,n_1,\ldots,n_m}$ and $\varrho_m$ stand for the $\mathsf{L}_{\in}$-statements corresponding to

$$(\forall a_{1,1} \ldots a_{1,n_1} \ldots a_{m,1} \ldots a_{m,n_m}) \quad \sum_{i=1}^{m} \prod_{j=1}^{n_i} a_{i,j}^2 + 1 \neq 0$$

and

$$(\forall x_1 \ldots x_m)(\exists y) \quad (\bigvee_{i=1}^{m} x_i y \neq y x_i) \vee \sum_{i=1}^{m} x_i^2 + 1 \neq 0.$$

Let $SG_n^k$ stand for the statement

$$(\forall A_1 \dots A_n)\{\bigwedge_{1\le i<j\le n}[(\exists l_{ij})\bigvee_{h\notin\{i,j\}}(A_i, A_j, A_h \in l_{ij})]\} \to \pi_k(A_1, \dots, A_n).$$

The statement $\psi_{m,n_1,\dots n_m}$ states that the sum of products of squares is never $-1$, which, according to [11], is equivalent to the orderability of the skew field, $\varrho_m$ states that the centre of the skew field is formally real, and $SG_n^k$ that any set $\mathcal{S} = \{A_1, \dots A_n\}$ of $n$ points, with the property that for any two of them, say $A_i$ and $A_j$, there is a third point, $A_h$, in $\mathcal{S}$, which is collinear with $A_i$ and $A_j$, must lie in a $k$-dimensional flat.

With the above notations we have the following

**Theorem.**

(i) *For every positive integer $n$, there are positive integers $m(n)$ and $k(1, n), \dots, k(m(n), n)$, such that $SG_n^1$ holds in $\mathcal{L} \cup \{\psi_{m(n),k(1,n),\dots,k(m(n),n)}\}$.*

(ii) *For every positive integer $n$, there is a prime number $p(n)$ such that $SG_n^2$ holds in $\mathcal{L} \cup \{\mathbf{P}, \bigwedge_{p\le p(n),p \text{ prime}} \varphi_p\}$.*

(iii) *For every positive integer $n$, there is a positive integer $m(n)$ such that $SG_n^3$ holds in $\mathcal{L} \cup \{\tau, \varrho_{m(n)}\}$.*

*Proof.* (i) Notice that, according to [2], [3], SG holds for all ordered geometries, thus, in particular, it must hold for projective spaces over orderable skew-fields. The theorem now follows from the compactness theorem for first-order logic: Since $SG_n^1$ is true in projective spaces over skew fields satisfying all the $\psi_{m,n_1,\dots,n_m}$, for all choices of the numbers $m$, $n_1, \dots, n_m$ (as those skew fields are orderable by [11]), there must be a finite subset of those $\psi$'s from which it follows as well.

(ii) Let $A_1, \dots, A_n$ be $n$ points in a projective space over a commutative field $K$ of characteristic 0. Since they span a subspace of dimension at most $n-1$, we may assume that they lie in $\mathbb{P}^{n-1}(K)$. Suppose they span a subspace of dimension $d > 2$. This means that there is a set $S$ of $d+1$ among them that are projectively independent, which amounts to saying that a certain $(d+1)\times(d+1)$ determinant is $\neq 0$. Let $\overline{K}$ be the algebraic closure of $K$. The set $S$ remains projectively independent in $\mathbb{P}^{n-1}(\overline{K})$ as well, as the relevant determinant stays the same (this can also be seen by applying Hilbert's Nullstellensatz). Thus $SG_n^2$ is false for this particular choice of $A_1, \dots, A_n$ in $\mathbb{P}^{n-1}(\overline{K})$, and thus, by the "Lefschetz principle" it must be false in $\mathbb{P}^{n-1}(\mathbb{C})$ as well, contradicting Kelly's theorem. The theorem now follows by compactness.

(iii) Projective spaces of dimension $\geq 3$ which satisfy $\tau$ must have either the quaternionic skew-field (over an arbitrary field as center) or a field as their coordinate skew-field $F$ (see [10, **14**, p. 175]). If it satisfies the statements $\varrho_m$ for all positive integers $m$ as well, then $F$ must have a formally real center, i.e. $F$ must be the quaternion algebra $Q(K)$ over a formally real field $K$ or $F$ must be a formally real field. If $F$ is a formally real field, by (i), there is nothing left to prove. Suppose $F$ is the quaternion algebra $Q(K)$ over a formally real field $K$, and let $A_1, \ldots, A_n$ be $n \geq 5$ points in $\mathbb{P}^{n-1}(F)$. Suppose there is a set $S$ of 5 points among them which span a four-dimensional flat, i.e. which are projectively independent. We may think of them as lying in $\mathbb{P}^4(F)$. We want to translate this projective independence into an algebraic language. The linear algebra corresponding to skew fields has been first successfully dealt with by A. Heyting [6] (for alternative definitions of determinants for the skew-field case see [5]). There he introduced 'designants', a generalization of determinants to the skew field case, which coincide with determinants if the field multiplication is commutative. For our purposes, all we need to know about them is that they are algebraic functions of their entries, and that the necessary and sufficient condition for the projective independence of $n$ points $A_1, \ldots, A_n$, where $A_i$ has coordinates $(x_1^i, x_2^i, \ldots, x_n^i)$, is for one of the designants — obtained by varying the order in which $j$ goes through the numbers $1, 2, \ldots, n$ — whose rows consist of the entries $x_j^1, x_j^2, \ldots, x_j^n$, to be $\neq 0$. Thus, in our case, this amounts to saying that a certain $5\times 5$ designant is $\neq 0$. Let $K^c$ denote the real closure of $K$. The set $S$ remains projectively independent in $\mathbb{P}^{n-1}(Q(K^c))$ as well, as the relevant designant remains unchanged, given that the coordinates of the points in $S$ are the same. Thus $SG_n^3$ is false for this particular choice of $A_1, \ldots, A_n$ in $\mathbb{P}^{n-1}(Q(K^c))$. By the Tarski-Seidenberg transfer principle it must be false in $\mathbb{P}^{n-1}(Q(\mathbb{R}))$ as well, contradicting the theorem of Elkies, Pretorius and Swanepoel. Compactness now provides the desired version of our theorem. $\square$

There is no reason to believe that a common proof schema exists for all values of $n$, and that a proof of these theorems can be actually written down in this elementary setting. It may be that the proofs for each individual value of $n$ are different and not special cases of one 'idea'.

Other elementary forms of the original variant have been provided in [1] (see also [9]).

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