Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 2, 325-333.

Timelike Quaternion Frame of a Non-lightlike Curve

M. Özdemir A. A. Ergin

Department of Mathematics, Akdeniz University
07058 Antalya, Turkey
mozdemir07@gmail.com aaergin@akdeniz.edu.tr

Abstract. A rotation matrix can be generated by a unit timelike quaternion in the three dimensional Minkowski space. In this paper, we correspond columns of this rotation matrix to Frenet vectors of a non-lightlike curve and we find a reformulation of Frenet's formulas of non-lightlike curves in terms of the timelike quaternions in the Minkowski 3-space. Moreover, we do same things for the parallel transport frame.

MSC 2000: 53B30, 53A04, 15A66
Keywords: split quaternion, Frenet formulas, non-lightlike curves

1. Introduction

Rotations in the Euclidean 3-space can be expressed via unit quaternions. This way is more useful and natural way compared to other methods [3]. An orthonormal matrix can easily be generated by a unit quaternion. Identifying columns of this orthonormal matrix with Frenet vectors, Frenet's formulas can be given in terms of a linear equation in the quaternion variable. It is known that the rotation matrix generated by a unit quaternion is

$$R = \begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & -2q_1q_4+2q_2q_3 & 2q_1q_3+2q_2q_4 \\ 2q_2q_3+2q_4q_1 & q_1^2-q_2^2+q_3^2-q_4^2 & 2q_3q_4-2q_2q_1 \\ 2q_2q_4-2q_3q_1 & 2q_2q_1+2q_3q_4 & q_1^2-q_2^2-q_3^2+q_4^2 \end{bmatrix}. \tag{1}$$

If the columns of this matrix are identified with Frenet vectors $\vec{T}(s), \vec{N}(s), \vec{B}(s)$ respectively, the quaternion Frenet formulas can be expressed as

$$\begin{bmatrix} q_1'(s) \\ q_2'(s) \\ q_3'(s) \\ q_4'(s) \end{bmatrix} = \begin{bmatrix} 0 & -\tau(s) & 0 & -\kappa(s) \\ \tau(s) & 0 & -\kappa(s) & 0 \\ 0 & -\kappa(s) & 0 & \tau(s) \\ \kappa(s) & 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \\ q_4(s) \end{bmatrix} \tag{2}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion functions of the curve. Similarly, if the columns of the matrix are identified with the parallel transport frame $\{\vec{N}_1(s), \vec{T}(s), \vec{N}_2(s)\}$, the parallel transport frame equations can be expressed as

$$\begin{bmatrix} q_1'(s) \\ q_2'(s) \\ q_3'(s) \\ q_4'(s) \end{bmatrix} = \begin{bmatrix} 0 & -k_2(s) & 0 & k_1(s) \\ k_2(s) & 0 & -k_1(s) & 0 \\ 0 & k_1(s) & 0 & k_2(s) \\ -k_1(s) & 0 & -k_2(s) & 0 \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \\ q_4(s) \end{bmatrix} \tag{3}$$

where k_1, k_2 are principal curvatures with respect to $\vec{N}_1$ and $\vec{N}_2$ respectively ([2], [3]). Parallel transport frame and the comparing parallel transport frame with Frenet frame in the Euclidean 3-space were given by Bishop [1] and Hanson [2] in detail. The quaternion derivative formulas (2) and (3) allow us to do the same things we did the Frenet formulas and the parallel transport frame equations.

Split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, i, j, k\}$ satisfying the equalities $i^2 = -1$, $j^2 = k^2 = ijk = 1$. Split quaternions were also known as coquaternions and were put forward by James Cockle in 1849. For the detailed information about split quaternions, we refer to the references [5], [4], [7] and [8]. The split quaternion product of two split quaternions $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined by

$$p * q = p_1 q_1 + < \vec{V}p, \vec{V}q >_L + p_1 \vec{V}q + q_1 \vec{V}p + \vec{V}p \wedge_L \vec{V}q$$

where $\langle , \rangle_L$ and $\wedge_L$ are Lorentzian inner product and vector product respectively. Scalar and vector parts of split quaternion q are denoted by $Sq = q_1$ and $\vec{V}q = q_2 i + q_3 j + q_4 k$ respectively. Let $q = (q_1, q_2, q_3, q_4) = Sq + \vec{V}q$ be a split quaternion. The conjugate of a split quaternion, denoted Kq, is defined as $Kq = Sq - \vec{V}q$. The conjugate of the sum of quaternions is the sum of their conjugates.

A split quaternion q is called spacelike, timelike or lightlike, if $I_q < 0$, $I_q > 0$ or $I_q = 0$ respectively where $I_q = q * Kq = Kq * q = q_1^2 + q_2^2 - q_3^2 - q_4^2$. The norm of the $q = (q_1, q_2, q_3, q_4)$ is defined as $Nq = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}$. If $Nq = 1$ then q is called unit split quaternion and $q_0 = q/Nq$ is a unit split quaternion for $Nq \neq 0$. The set of timelike quaternions denoted by $\mathbb{T}\widehat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_1, q_2, q_3, q_4 \in \mathbb{R}, I_q > 0\}$ forms a group under the split quaternion product. Also, the set of unit timelike quaternions is represented by $\mathbb{T}\widehat{\mathbb{H}}_1$.

A similar relation to the relationship between quaternions and rotations in Euclidean space exists between split quaternions and Minkowski 3-space. Every 3 dimensional rotation in the Minkowski 3-space can be expressed via split quaternions. In this paper, we reformulate Frenet formulas and parallel transport frame equations in terms of timelike quaternions. First, we give basic notions in the Minkowski 3-space and split quaternions.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean space $\mathbb{E}^3$ provided with the Lorentzian inner product $\langle \vec{u}, \vec{v} \rangle_L = -u_1v_1+u_2v_2+u_1v_2$ where $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$. We say that a vector $\vec{u}$ in $\mathbb{E}_1^3$ is spacelike, lightlike or timelike if $\langle \vec{u}, \vec{u} \rangle_L > 0$, $\langle \vec{u}, \vec{u} \rangle_L = 0$ or $\langle \vec{u}, \vec{u} \rangle_L < 0$ respectively. The norm of the vector $\vec{u} \in \mathbb{E}_1^3$ is defined by $\|\vec{u}\| = \sqrt{|\langle \vec{u}, \vec{u} \rangle_L|}$. Associated to that inner product, for any $\vec{u} = (u_1, u_1, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in \mathbb{E}_1^3$, the Lorentzian vector product $\vec{u} \wedge_L \vec{v}$ of $\vec{u}$ and $\vec{v}$ is defined as follows:

$$\vec{u} \wedge_L \vec{v} = (-u_2v_3 + u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) .$$

An arbitrary curve $\alpha = \alpha(s) : I \to E_1^3$ is spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset R$. Let $\alpha(s)$ be a unit non-lightlike curve and $\vec{T}(s), \vec{N}(s), \vec{B}(s)$ are Frenet vectors, then Frenet formulas are as follows

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}'(s) \\ \vec{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ (\epsilon_{\vec{B}})\kappa(s) & 0 & \tau(s) \\ 0 & (\epsilon_{\vec{T}})\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{bmatrix} \tag{4}$$

where $\epsilon_{\vec{X}} =< \vec{X}, \vec{X} >_L$ and $\kappa(s)$, $\tau(s)$ are curvature and torsion functions of the curve respectively [10].

Parallel transport frame can be expressed as the following: for a unit non-lightlike curve $\alpha(s)$, we take tangent vector of $\alpha(s)$ and choose $\vec{N}_1(s)$ and $\vec{N}_2(s)$ unit vectors in the plane which is perpendicular to $\vec{T}$ such that derivatives of $\vec{N}_1(s)$ and $\vec{N}_2(s)$ depend only $\vec{T}(s)$ not each other. This frame is independent from the curvature of the curve. Using, $\vec{N}_1' = a\vec{T}$, $\vec{N}_2' = b\vec{T}$ and $\vec{T}' = c\vec{N}_1 + d\vec{N}_2$, we can obtain the parallel transport frame equations.

Let $\left\{\vec{T}(s), \vec{N}_1(s), \vec{N}_2(s)\right\}$ be parallel transport frame, then the evolution equations of this frame for a timelike curve are as follows;

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}_1'(s) \\ \vec{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & -k_2(s) \\ k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}_1(s) \\ \vec{N}_2(s) \end{bmatrix} \tag{5}$$

such that $\kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}$, $\theta(s) = \arctan \frac{k_2(s)}{k_1(s)}$ and $\tau(s) = \theta'(s)$. Also, the parallel transport frame equations for a spacelike curve are as follows

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{N}_1'(s) \\ \vec{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -(\epsilon_{\vec{N}_1})k_1(s) & 0 & 0 \\ -(\epsilon_{\vec{N}_2})k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}_1(s) \\ \vec{N}_2(s) \end{bmatrix} \tag{6}$$

such that $\epsilon_{\vec{X}} =< \vec{X}, \vec{X} >_L$, $\kappa(s) = \sqrt{(\epsilon_{\vec{N}_1})k_1^2 + (\epsilon_{\vec{N}_1})k_2^2}$, $\theta(s) = \arctan h \frac{k_2(s)}{k_1(s)}$ and $\tau(s) = \epsilon_{\vec{N}_1}\theta'(s)$ [9]. Note that we can use $\vec{T}' = k_1\vec{N}_1 + \epsilon_{\vec{T}}k_2\vec{N}_2$ for any non-lightlike curve and the causal characters of $\vec{N}_1(s)$ and $\vec{N}(s)$ are same as those of $\vec{N}_2(s)$ and $\vec{B}(s)$.

3. Timelike quaternion frame for a non-lightlike curve

In this section, we find a reformulation of the Frenet formulas and the parallel transport frame equations of a non-lightlike curve in terms of a linear equation in the quaternion variables. We do not deal with lightlike curves and the non-lightlike curves whose normal or binormal is null, since each column vector of the matrix (7) is spacelike or timelike.

Every rotation in the Minkowski 3-space can be expressed via unit timelike quaternions. If $q = (q_1, q_2, q_3, q_4)$ is a timelike quaternion, using the transformation law

$$(q * \vec{V}r * q^{-1})_i = \sum_{j=1}^{3} R_{ij}(\vec{V}r)_j$$

the corresponding rotation matrix can be found as

$$R_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix} \tag{7}$$

where $r = (Sr, \vec{V}r)$. All rows of this matrix are orthogonal in the Lorentzian mean. In addition, if we take a unit timelike quaternion $q \in \mathbb{T}\widehat{\mathbb{H}}_1$, we obtain an orthogonal rotation matrix in the Minkowski 3-space. Each rotation is represented by a rotation matrix with respect to the standard basis in $\mathbb{E}_1^3$. These matrices form the 3-dimensional special orthogonal group $SO(1,2)$. Moreover, the function $\varphi : S_2^3 \simeq \mathbb{T}\widehat{\mathbb{H}}_1 \to SO(1,2)$ which maps $q = (q_1, q_2, q_3, q_4)$ to the matrix R given in (7) is an homomorphism of group. The kernel of φ is $\{\pm 1\}$ so that rotation matrix corresponds to pair $\pm q$ of unit quaternion. In particular, $SO(1,2)$ is isomorphic to the quotient group $\mathbb{T}\widehat{\mathbb{H}}_1 / \{\pm 1\}$ from the first isomorphism theorem. In other words, for every rotation in the Minkowski 3-space $\mathbb{E}_1^3$, there are two unit timelike quaternions that determine this rotation. These timelike quaternions are q and $-q$ [8].

The quadratic form (7) for a general orthonormal $SO(1,2)$ frame suggests that the Frenet and parallel transport frame and their Frenet formulas might be expressible directly in terms of a linear equation in the quaternion variables. To generate a unique moving frame with its space curve for nonvanishing curvature, we can integrate the Frenet formulas. Similarly, we may integrate the much simpler quaternion equations.

The matrix (7) expresses all 3 dimensional rotations in the Minkowski 3-space. Since this matrix is an orthonormal matrix in the Minkowski 3-space, we can use it in identifying the orthonormal frame of a non-lightlike curve in the Minkowski 3-space. Therefore, we identify the columns of (7) with the elements of orthonormal frame. But, we must consider the case that the first column of (7) is timelike. Because of that, while we identify first column with $\vec{T}(s)$ for a timelike curve $\alpha(s)$, we should identify first column with $\vec{N}(s)$ for a spacelike curve with timelike normal. While we identify the columns with elements of the Frenet frame, order of $\vec{T}(s), \vec{N}(s)$ and $\vec{B}(s)$ is important. That is, for a spacelike curve with timelike binormal, we identify the columns in the order $\vec{N}(s), \vec{B}(s), \vec{T}(s)$ (not $\vec{N}(s), \vec{T}(s), \vec{B}(s)$). Similarly, we consider the same things for the parallel transport frame $\{\vec{T}(s), \vec{N}_1(s), \vec{N}_2(s)\}$.

3.1. Timelike quaternion frame for a timelike curve

Theorem 1. *Let $q = (q_1, q_2, q_3, q_4)$ be a unit timelike quaternion and let* (7) *be the corresponding rotation matrix. Then, the Frenet formulas* (4) *and the parallel transport frame equations* (5) *of a unit timelike curve can be expressed in terms of the q with*

$$\begin{bmatrix} q_1'(s) \\ q_2'(s) \\ q_3'(s) \\ q_4'(s) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\tau}{2}(s) & 0 & \frac{\kappa}{2}(s) \\ \frac{\tau}{2}(s) & 0 & -\frac{\kappa}{2}(s) & 0 \\ 0 & -\frac{\kappa}{2}(s) & 0 & \frac{\tau}{2}(s) \\ \frac{\kappa}{2}(s) & 0 & -\frac{\tau}{2}(s) & 0 \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \\ q_4(s) \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} q_1'(s) \\ q_2'(s) \\ q_3'(s) \\ q_4'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{k_2}{2}(s) & \frac{k_1}{2}(s) \\ 0 & 0 & -\frac{k_1}{2}(s) & \frac{k_2}{2}(s) \\ \frac{k_2}{2}(s) & -\frac{k_1}{2}(s) & 0 & 0 \\ \frac{k_1}{2}(s) & \frac{k_2}{2}(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1(s) \\ q_2(s) \\ q_3(s) \\ q_4(s) \end{bmatrix} \tag{9}$$

respectively.

Proof. Let us identify the columns of (7) with $\vec{T}(s), \vec{N}(s), \vec{B}(s)$ respectively.

$$\vec{T} = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 \\ 2q_2q_3 + 2q_4q_1 \\ 2q_2q_4 - 2q_3q_1 \end{bmatrix}, \; \vec{N} = \begin{bmatrix} 2q_1q_4 - 2q_2q_3 \\ q_1^2 - q_2^2 - q_3^2 + q_4^2 \\ 2q_2q_1 - 2q_3q_4 \end{bmatrix}, \; \vec{B} = \begin{bmatrix} -2q_1q_3 - 2q_2q_4 \\ -2q_3q_4 - 2q_2q_1 \\ q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix}.$$

It is easy to see that differentiation yields

$$\begin{aligned} \vec{T}'(s) &= 2[A][q'(s)], \\ \vec{N}'(s) &= 2[B][q'(s)], \\ \vec{B}'(s) &= 2[C][q'(s)], \end{aligned}$$

where $A = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ q_4 & q_3 & q_2 & q_1 \\ -q_3 & q_4 & -q_1 & q_2 \end{bmatrix}$, $B = \begin{bmatrix} q_4 & -q_3 & -q_2 & q_1 \\ q_1 & -q_2 & -q_3 & q_4 \\ q_2 & q_1 & -q_4 & -q_3 \end{bmatrix}$ and

$$C = \begin{bmatrix} -q_3 & -q_4 & -q_1 & -q_2 \\ -q_2 & -q_1 & -q_4 & -q_3 \\ q_1 & -q_2 & q_3 & -q_4 \end{bmatrix}.$$

Now, using the fact that Frenet's formulas (4) for a timelike curve $\alpha(s)$ are

$$\vec{T}' = \kappa\vec{N}, \quad \vec{N}' = \kappa\vec{T} + \tau\vec{B} \text{ and } \vec{B}' = -\tau\vec{N},$$

we find (8) by direct (but tedious) computations.

Similarly, identifying the parallel transport frame vectors $\vec{T}(s)$, $\vec{N}_1(s)$, $\vec{N}_2(s)$ (in that order) with the columns of (7) and using the parallel transport frame equations of a timelike curve, we get (9). □

Remark 2. In the equations (8) and (9), the matrices in the right side are semi antisymmetric for the semi-Euclidean space $\mathbb{E}^4_2$. Recall that a matrix A is semi antisymmetric in the semi-Euclidean space $\mathbb{E}^4_2$, provided that $I^*AI^* = -A^T$ where A^T is the transpose of the A and $I^* = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

3.2. Timelike quaternion frame for a spacelike curve

In this case, we examine the timelike quaternion frame of a spacelike curve according to the normal of the curve is timelike or spacelike.

i) If the normal $\vec{N}(s)$ of the spacelike curve is timelike, identifying the columns of (7) with $\vec{N}(s), \vec{B}(s), \vec{T}(s)$ (in that order) respectively, we find

$$\begin{aligned} \vec{N}'(s) &= 2[A][q'(s)], \\ \vec{B}'(s) &= 2[B][q'(s)], \\ \vec{T}'(s) &= 2[C][q'(s)]. \end{aligned}$$

So, using the fact that the Frenet formulas (4) for a unit spacelike curve with timelike normal are

$$\vec{T}' = \kappa\vec{N}, \; \vec{N}' = \kappa\vec{T} + \tau\vec{B} \text{ and } \vec{B}' = \tau\vec{N},$$

we obtain

$$q'(s) = \mathbb{B}_1(\kappa(s), \tau(s))\, q(s) \tag{10}$$

where

$$\mathbb{B}_1(\kappa(s), \tau(s)) = \begin{bmatrix} 0 & 0 & -\frac{\kappa}{2}(s) & \frac{\tau}{2}(s) \\ 0 & 0 & -\frac{\tau}{2}(s) & -\frac{\kappa}{2}(s) \\ -\frac{\kappa}{2}(s) & -\frac{\tau}{2}(s) & 0 & 0 \\ \frac{\tau}{2}(s) & -\frac{\kappa}{2}(s) & 0 & 0 \end{bmatrix}.$$

In the parallel transport frame with timelike $\vec{N}_1$, we correspond the columns of the matrix (7) to $\vec{N}_1(s), \vec{N}_2(s)$ and $\vec{T}(s)$ respectively. Thus, using the equations $\vec{N}_1'(s) = 2[A][q'(s)]$, $\vec{N}_2'(s) = 2[B][q'(s)]$, $\vec{T}'(s) = 2[C][q'(s)]$ and the parallel transport frame equations $\vec{T}' = k_1\vec{N}_1 + k_2\vec{N}_2$, $\vec{N}_1' = k_1\vec{T}$ and $\vec{N}_2' = -k_2\vec{T}$, we obtain

$$q'(s) = \mathbb{B}_2(k_1(s), k_2(s))\, q(s) \tag{11}$$

where

$$\mathbb{B}_2(k_1(s), k_2(s)) = \begin{bmatrix} 0 & \frac{k_2}{2}(s) & -\frac{k_1}{2}(s) & 0 \\ -\frac{k_2}{2}(s) & 0 & 0 & -\frac{k_1}{2}(s) \\ -\frac{k_1}{2}(s) & 0 & 0 & -\frac{k_2}{2}(s) \\ 0 & -\frac{k_1}{2}(s) & \frac{k_2}{2}(s) & 0 \end{bmatrix}.$$

ii) If the normal $\vec{N}(s)$ of the spacelike curve is spacelike, we identify the columns of (7) with $\vec{B}(s), \vec{T}(s), \vec{N}(s)$ (in that order) respectively and we find

$$\vec{B}'(s) = 2[A][q'(s)],\ \vec{T}'(s) = 2[B][q'(s)],\ \ \vec{N}'(s) = 2[C][q'(s)].$$

So, since the Frenet formulas (4) for a spacelike curve with timelike binormal are

$$\vec{T}' = \kappa\vec{N},\ \vec{N}' = -\kappa\vec{T} + \tau\vec{B} \text{ and } \vec{B}' = \tau\vec{N},$$

we find

$$q'(s) = \mathbb{B}_3(\kappa(s), \tau(s))\, q(s) \tag{12}$$

where

$$\mathbb{B}_3(\kappa(s), \tau(s)) = \begin{bmatrix} 0 & -\frac{\kappa}{2}(s) & -\frac{\tau}{2}(s) & 0 \\ \frac{\kappa}{2}(s) & 0 & 0 & -\frac{\tau}{2}(s) \\ -\frac{\tau}{2}(s) & 0 & 0 & \frac{\kappa}{2}(s) \\ 0 & -\frac{\tau}{2}(s) & -\frac{\kappa}{2}(s) & 0 \end{bmatrix}.$$

At last, in the parallel transport frame with timelike $\vec{N}_2$, we correspond the columns of the matrix (7) to $\vec{N}_1(s), \vec{T}(s)$ and $\vec{N}_2(s)$ respectively. Thus, using the equations $\vec{N}_2'(s) = 2[A][q'(s)]$, $\vec{T}'(s) = 2[B][q'(s)]$, $\vec{N}_1'(s) = 2[C][q'(s)]$ and the parallel transport frame equations $\vec{T}' = k_1\vec{N}_1 + k_2\vec{N}_2$, $\vec{N}_1' = -k_1\vec{T}$ and $\vec{N}_2' = k_2\vec{T}$, we obtain

$$q'(s) = \mathbb{B}_4(k_1(s), k_2(s))\, q(s) \tag{13}$$

where

$$\mathbb{B}_4(k_1(s), k_2(s)) = \begin{bmatrix} 0 & -\frac{k_1}{2}(s) & 0 & \frac{k_2}{2}(s) \\ \frac{k_1}{2}(s) & 0 & -\frac{k_2}{2}(s) & 0 \\ 0 & -\frac{k_2}{2}(s) & 0 & \frac{k_1}{2}(s) \\ \frac{k_2}{2}(s) & 0 & -\frac{k_1}{2}(s) & 0 \end{bmatrix}.$$

Remark 3. The matrices $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$ and $\mathbb{B}_4$ are semi antisymmetric for the semi-Euclidean space $\mathbb{E}_2^4$.

Now, we can express following theorems for unit spacelike curves in the 3-dimensional Minkowski space.

Theorem 4. *Let $q = (q_1, q_2, q_3, q_4)$ be a unit timelike quaternion and let (7) be corresponding rotation matrix. Then, the Frenet formulas (4) of a unit spacelike curve can be expressed in terms of the q as follows:*

i) $q'(s) = \mathbb{B}_1 q(s)$, *if normal of the curve is timelike,*

ii) $q'(s) = \mathbb{B}_3 q(s)$, *if normal of the curve is spacelike.*

Also, the parallel transport frame equations (6) of a unit spacelike curve can be expressed in terms of the q as follows:

i) $q'(s) = \mathbb{B}_2 q(s)$, *if N_1 is timelike,*

ii) $q'(s) = \mathbb{B}_4 q(s)$, *if N_1 is spacelike*

where $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$ and $\mathbb{B}_4$ are as above.

Thus, the Frenet formulas and the parallel transport frame equations are reformulated in terms of the timelike quaternion $q = (q_1, q_2, q_3, q_4)$. The matrices $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ and $\mathbb{B}_4$ are semi antisymmetric matrices in the semi-Euclidean space $\mathbb{E}_2^4$ and the equality $< q(s), q'(s) >_L = 0$ holds according to the construction. These properties guarantee that the unit timelike quaternions remain constrained to the unit semi-Euclidean sphere S_2^3.

For the given continuous functions $\kappa(s)$ and $\tau(s)$, it is known that there exists a parametric curve such that s is its arclength parameter and the given functions $\kappa(s)$ and $\tau(s)$ represent the curvature and the torsion of the curve respectively. So, we can give the following corollary.

Corollary 5. *If $\kappa(s)$ and $\tau(s)$ are given, the equations (8), (9), (10), (11), (12) and (13) can be integrated directly to find $q(s)$. So, the Frenet frame $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ or the parallel transport frame $\{\vec{T}(s), \vec{N}_1(s), \vec{N}_2(s)\}$ can be generated by using the matrix (7).*

References

[1] Bishop, R. L.: *There is more than one way to frame a curve.* Am. Math. Mon. **82**(3) (1975), 246–251. Zbl 0298.53001

[2] Hanson, A. J.; Ma, H.: *Quaternion frame approach to streamline visualization.* IEEE Trans. Visual. Comput. Graphics **1**(2) (1995), 164–174.

[3] Hanson, A. J.: *Visualizing Quaternions.* Morgan-Kaufmann/Elsevier, London 2006.

[4] Inoguchi, J.: *Timelike surfaces of constant mean curvature in Minkowski 3-space.* Tokyo J. Math. **21**(1) (1998), 141–152. Zbl 0930.53042

[5] Inoguchi, J.; Toda, M.: *Timelike minimal surfaces via loop groups.* Acta Appl. Math. **83**(3) (2004), 313–355. Zbl pre02128493

[6] Kantor, I. L.; Solodovnikov, A. S.: *Hypercomplex Numbers. An Elementary Introduction to Algebras.* Springer-Verlag, New York 1989. Zbl 0669.17001

[7] Kula, L.: *Split quaternions and geometrical applications*. Ankara University, Ph.D. Thesis, 2003.

[8] Özdemir, M.; Ergin, A. A.: *Rotations with unit timelike quaternions in Minkowski 3-space*. J. Geom. Phys., **56** (2006), 322–336. Zbl 1088.53010

[9] Özdemir, M.; Ergin, A. A.: *Parallel transport frame of non-lightlike curves*. Missouri J. Math. Sci., Preprint.

[10] Pei, D.; Sano, T.: *The focal developable and the binormal indicatrix of non-lightlike curve in Minkowski 3-space*. Tokyo J. Math. **23** (2000), 211–225. Zbl 0978.53004

Received December 12, 2006