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Curvature in Synthetic Differential Geometry of Groupoids

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Abstract. We study the fundamental properties of curvature in groupoids within the framework of synthetic differential geometry. As is usual in synthetic differential geometry, its combinatorial nature is emphasized. In particular, the classical Bianchi identity is deduced from a combinatorial version of it.

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1. Introduction

The notion of curvature, which is one of the most fundamental concepts in differential geometry, retrieves its combinatorial or geometric meaning in synthetic differential geometry. It was Kock [5] who studied it up to the second Bianchi identity synthetically for the first time. In particular, he has revealed the combinatorial nature of the second Bianchi identity by deducing it from an abstract one.

Kock [5] trotted out first neighborhood relations, which are indeed to be seen in formal manifolds, but which are no longer expected to be seen in microlinear spaces in general. Since we believe that microlinear spaces should play the same role in synthetic differential geometry as smooth manifolds have done in classical differential geometry, we have elevated his ideas to a microlinear context in [11].

Recently we got accustomed to groupoids, which encouraged us to attack the same problem once again. Within the framework of groupoids, we find it pleasant to think multiplicatively rather than additively (cf. Nishimura [14]), which helps grasp the nature of the second Bianchi identity firmly. Now we are to the point. What we have to do in order to illicit the classical second Bianchi identity from the combinatorial one is only to note some commutativity on the infinitesimal level, though groupoids are, by and large, highly noncommutative. Our present experience is merely an example of the familiar wisdom in mathematics that a good generalization reveals the nature.

2. Preliminaries

2.1. Synthetic differential geometry

Our standard reference on synthetic differential geometry is Chapters 1–5 of Lavendhomme [7]. We will work internally within a good topos, in which the intended set $\mathbb{R}$ of real numbers is endowed with a cornucopia of nilpotent infinitesimals pursuant to the general Kock-Lawvere axiom. To see how to build such a good topos, the reader is referred to Kock [2] or Moerdijk and Reyes [9]. Any space mentioned in this paper will be assumed to be microlinear. We denote by D the set $\{d \in \mathbb{R} \mid d^2 = 0\}$, as is usual in synthetic differential geometry.

Given a group G, we denote by $\mathcal{A}G$ the tangent space of G at its identity, i.e., the totality of mappings $t : D \to G$ such that t_0 is the identity of G. We will often write t_d rather than $t(d)$ for any $d \in D$. As we will see shortly, $\mathcal{A}G$ is more than an $\mathbb{R}$-module.

Proposition 1. *For any $t \in \mathcal{A}G$ and any $(d_1, d_2) \in D(2)$, we have*

$$t_{d_1+d_2} = t_{d_1} t_{d_2} = t_{d_2} t_{d_1}$$

so that t_{d_1} and t_{d_2} commute.

Proof. By the same token as in Proposition 3 of §3.2 of Lavendhomme [7].

As an easy corollary of this proposition, we can see that

$$t_{-d} = (t_d)^{-1}$$

since we have $(d, -d) \in D(2)$.

Proposition 2. *For any $t_1, t_2 \in \mathcal{A}G$, we have*

$$(t_1 + t_2)_d = (t_2)_d (t_1)_d = (t_1)_d (t_2)_d$$

for any $d \in D$, so that $(t_1)_d$ and $(t_2)_d$ commute.

Proof. By the same token as in Proposition 6 of §3.2 of Lavendhomme [7].

As an easy consequence of this proposition, we can see, by way of example, that $(t_1)_{d_1 d_2}$ and $(t_2)_{d_1 d_3}$ commute for any $d_1, d_2, d_3 \in D$, since we have

$$(t_1)_{d_1 d_2} (t_2)_{d_1 d_3} = (d_2 t_1)_{d_1} (d_3 t_2)_{d_1} = (d_3 t_2)_{d_1} (d_2 t_1)_{d_1} = (t_2)_{d_1 d_3} (t_1)_{d_1 d_2}.$$

Proposition 3. *For any $t_1, t_2 \in \mathcal{A}G$, there exists a unique $s \in \mathcal{A}G$ such that*

$$s_{d_1 d_2} = (t_2)_{-d_2}(t_1)_{-d_1}(t_2)_{d_2}(t_1)_{d_1}$$

for any $d_1, d_2 \in D$.

Proof. By the same token as in pp. 71–72 of Lavendhomme [7].

We will write $[t_1, t_2]$ for the above s.

Theorem 4. *The $\mathbb{R}$-module $\mathcal{A}G$ endowed with the above Lie bracket $[\cdot,\cdot]$ is a Lie algebra over $\mathbb{R}$.*

Proof. By the same token as in our previous paper [13].

Remark 5. The idea that group-theoretic commutators lead to Lie algebras has long been known in standard differential geometry, and the reader is referred to p. 57 of [16] for its first synthetic treatment. However we should stress that general Jacobi structures discovered by Nishimura [10] are more fundamental than Lie algebras in synthetic differential geometry. The latter can easily be derived from the former in case that groups are available, but the former can be available without the latter in sight, for which the reader is referred to Nishimura [15].

2.2. Groupoids

Groupoids are, roughly speaking, categories whose morphisms are always invertible. Our standard reference on groupoids is MacKenzie [8]. Given a groupoid G over a base M with its object inclusion map $\mathrm{id} : M \to G$ and its source and target projections $\alpha, \beta : G \to M$, we denote by $\mathfrak{B}(G)$ the totality of bisections of G, i.e., the totality of mappings $\sigma : M \to G$ such that $\alpha \circ \sigma$ is the identity mapping on M and $\beta \circ \sigma$ is a bijection of M onto M. It is well known that $\mathfrak{B}(G)$ is a group with respect to the operation $*$, where for any $\sigma, \rho \in \mathfrak{B}(G)$, $\sigma * \rho \in \mathfrak{B}(G)$ is defined to be

$$(\sigma * \rho)(x) = \sigma((\beta \circ \rho)(x))\rho(x)$$

for any $x \in M$. It can easily be shown that the space $\mathfrak{B}(G)$ is microlinear, provided that both M and G are microlinear, for which the reader is referred to Proposition 6 of Nishimura [13].

Given $x \in M$, we denote by $\mathcal{A}_x^n G$ the totality of mappings $\gamma : D^n \to G$ with $\gamma(0,\ldots,0) = \mathrm{id}_x$ and $(\alpha \circ \gamma)(d_1,\ldots,d_n) = x$ for any $(d_1,\ldots,d_n) \in D^n$. We denote by $\mathcal{A}^n G$ the set-theoretic union of $\mathcal{A}_x^n G$'s for all $x \in M$. In particular, we usually write $\mathcal{A}_x G$ and $\mathcal{A}G$ in place of $\mathcal{A}_x^1 G$ and $\mathcal{A}^1 G$ respectively. It is easy to see that $\mathcal{A}G$ is naturally a vector bundle over M. A morphism $\varphi : H \to G$ of groupoids over M naturally gives rise to a morphism $\varphi_* : \mathcal{A}H \to \mathcal{A}G$ of vector bundles over M. As in §3.2.1 of Lavendhomme [7], where three distinct but equivalent viewpoints of vector fields are presented, the totality $\Gamma(\mathcal{A}G)$ of sections of the vector bundle $\mathcal{A}G$ can canonically be identified with the totality of tangent vectors to $\mathfrak{B}(G)$ at id, for which the reader is referred to Nishimura [13]. We will enjoy this identification freely, and we dare to write $\Gamma(\mathcal{A}G)$ for the totality of tangent vectors to $\mathfrak{B}(G)$ at id. Given $X, Y \in \Gamma(\mathcal{A}G)$, we define a microsquare $Y * X$ to $\mathfrak{B}(G)$ at id to be

$$(Y * X)(d_1, d_2) = Y_{d_2} * X_{d_1}$$

for any $(d_1, d_2) \in D^2$.

Given $\gamma \in \mathcal{A}^{n+1}G$ and $e \in D$, we define $\gamma_e^i \in \mathcal{A}^n G$ $(1 \leq i \leq n+1)$ to be

$$\gamma_e^i(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, e, d_i, \ldots, d_n)\gamma(0, \ldots, 0, \underset{i}{e}, 0, \ldots, 0)^{-1}$$

for any $(d_1, \ldots, d_n) \in D^n$. For our later use in the last section of this paper, we introduce a variant of this notation. Given $\gamma \in \mathcal{A}^{n+2}G$ and $e_1, e_2 \in D$, we define $\gamma_{e_1,e_2}^{i,j} \in \mathcal{A}^n G$ $(1 \leq i < j \leq n+2)$ to be

$$\gamma_{e_1,e_2}^{i,j}(d_1, \ldots, d_n) =$$
$$\gamma(d_1, \ldots, d_{i-1}, e_1, d_i, \ldots, d_{j-2}, e_2, d_{j-1}, \ldots, d_n)\gamma(0, \ldots, 0, \underset{i}{e_1}, 0, \ldots, 0, \underset{j}{e_2}, 0, \ldots, 0)^{-1}$$

Given $\gamma \in \mathcal{A}^2 G$, we define $\tau_\gamma^1 \in \mathcal{A}^2 G$ to be

$$\tau_\gamma^1(d_1, d_2) = \gamma(d_1, 0)$$

for any $(d_1, d_2) \in D^2$. Similarly, given $\gamma \in \mathcal{A}^2 G$, we define $\tau_\gamma^2 \in \mathcal{A}^2 G$ to be

$$\tau_\gamma^2(d_1, d_2) = \gamma(0, d_2)$$

for any $(d_1, d_2) \in D^2$. Given $\gamma \in \mathcal{A}^2 G$, we define $\Sigma\gamma \in \mathcal{A}^2 G$ to be

$$(\Sigma\gamma)(d_1, d_2) = \gamma(d_2, d_1)$$

for any $(d_1, d_2) \in D^2$.

Any $\gamma \in \mathcal{A}^2 G$ can canonically be identified with the mapping $e \in D \mapsto \gamma_e^1 \in \mathcal{A}G$, so that we can identify $\mathcal{A}^2 G$ and $(\mathcal{A}G)^D$. As is expected, this identification enables us to define $\gamma_2 \underset{1}{-} \gamma_1 \in \mathcal{A}^2 G$ for $\gamma_1, \gamma_2 \in \mathcal{A}^2 G$, provided that $\gamma_1(0, \cdot) = \gamma_2(0, \cdot)$. Similarly, we can define $\gamma_2 \underset{2}{-} \gamma_1 \in \mathcal{A}^2 G$ for $\gamma_1, \gamma_2 \in \mathcal{A}^2 G$, provided that $\gamma_1(\cdot, 0) = \gamma_2(\cdot, 0)$. Given $\gamma_1, \gamma_2 \in \mathcal{A}^2 G$, their strong difference $\gamma_2 \dot{-} \gamma_1 \in \mathcal{A}G$ is defined, provided that $\gamma_1 \mid_{D(2)} = \gamma_2 \mid_{D(2)}$. Lavendhomme's [7] treatment of strong difference $\dot{-}$ in §3.4 carries over mutatis mutandis to our present context. We note in passing the following simple proposition on strong difference $\dot{-}$, which is not to be seen in our standard reference [7] on synthetic differential geometry.

Proposition 6. *For any $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{A}^2 G$ with $\gamma_1 \mid_{D(2)} = \gamma_2 \mid_{D(2)} = \gamma_3 \mid_{D(2)}$, we have*

$$(\gamma_2 \dot{-} \gamma_1) + (\gamma_3 \dot{-} \gamma_2) + (\gamma_1 \dot{-} \gamma_3) = 0.$$

2.3. Differential forms

Given a groupoid G and a vector bundle E over the same space M, the space $\mathbf{C}^n(G, E)$ of *differential n-forms with values in E* consists of all mappings ω from $\mathcal{A}^n G$ to E whose restriction to $\mathcal{A}^n_x G$ for each $x \in M$ takes values in E_x satisfying the following n-homogeneous and alternating properties:

1. We have
$$\omega(a \underset{i}{\cdot} \gamma) = a\omega(\gamma) \quad (1 \leq i \leq n)$$
for any $a \in \mathbb{R}$ and any $\gamma \in \mathcal{A}^n_x G$, where $a \underset{i}{\cdot} \gamma \in \mathcal{A}^n_x G$ is defined to be
$$(a \underset{i}{\cdot} \gamma)(d_1, \ldots, d_n) = \gamma(d_1, \ldots, d_{i-1}, ad_i, d_{i+1}, \ldots, d_n)$$
for any $(d_1, \ldots, d_n) \in D^n$.
2. We have
$$\omega(\gamma \circ D^\theta) = \mathrm{sign}(\theta)\omega(\gamma)$$
for any permutation θ of $\{1, \ldots, n\}$, where $D^\theta : D^n \to D^n$ permutes the n coordinates by θ.

3. Connections

Let $\pi : H \to G$ be a morphism of groupoids over M. Let L be the kernel of π with its canonical injection $\iota : L \to H$. It is clear that L is a group bundle over M. These entities shall be fixed throughout the rest of the paper. Thus we have an exact sequence of groupoids as follows:

$$0 \to L \xrightarrow{\iota} H \xrightarrow{\pi} G.$$

A *connection ∇ with respect to* π is a morphism $\nabla : \mathcal{A}G \to \mathcal{A}H$ of vector bundles over M such that the composition $\pi_* \circ \nabla$ is the identity mapping of $\mathcal{A}G$. A connection ∇ with respect to π shall be fixed throughout the rest of the paper. If G happens to be $M \times M$ (the pair groupoid of M) with π being the projection $h \in H \mapsto (\alpha(h), \beta(h)) \in M \times M$, our present notion of connection degenerates into the classical one of infinitesimal connection.

Given $\gamma \in \mathcal{A}^{n+1}G$, we define $\gamma_i \in \mathcal{A}G$ $(1 \leq i \leq n+1)$ to be

$$\gamma_i(d) = \gamma(0, \ldots, 0, \underset{i}{d}, 0, \ldots, 0)$$

for any $d \in D$. As in our previous paper [14], we have:

Theorem 7. *Given $\omega \in \mathbf{C}^n(G, \mathcal{A}L)$, there exists a unique $\mathbf{d}_\nabla \omega \in \mathbf{C}^{n+1}(G, \mathcal{A}L)$ such that*

$$\begin{aligned}&((\mathbf{d}_\nabla \omega)(\gamma))_{d_1 \cdots d_{n+1}} \\ &= \prod_{i=1}^{n+1} \{(\omega(\gamma_0^i))_{d_1 \cdots \widehat{d_i} \cdots d_{n+1}} ((\nabla \gamma_i)_{d_i})^{-1} (\omega(\gamma_{d_i}^i))_{-d_1 \cdots \widehat{d_i} \cdots d_{n+1}} (\nabla \gamma_i)_{d_i}\}^{(-1)^i}\end{aligned}$$

for any $\gamma \in \mathcal{A}^{n+1}G$ and any $(d_1, \ldots, d_{n+1}) \in D^{n+1}$.

Remark 8. The above formula, if it is rewritten additively, is essentially the standard familiar formula for coboundary of cubical cochains with values in a group bundle as follows:

$$
\begin{aligned}
&((\mathbf{d}_\nabla \omega)(\gamma))_{d_1 \cdots d_{n+1}} \\
&= \sum_{i=1}^{n+1} (-1)^i \{(\omega(\gamma_0^i))_{d_1 \cdots \widehat{d_i} \cdots d_{n+1}} + ((\nabla\gamma_i)_{d_i})^{-1} (\omega(\gamma_{d_i}^i))_{-d_1 \cdots \widehat{d_i} \cdots d_{n+1}} (\nabla\gamma_i)_{d_i}\}.
\end{aligned}
$$

We note that the $n+1$ main factors commute, and within each main factor the two subfactors commute. The former fact can be observed as in [14], and the latter fact can be observed by dint of Proposition 2.

4. A lift of the connection ∇ to microsquares

Let us define a mapping $\mathcal{A}^2 G \to \mathcal{A}^2 H$, which shall be denoted by the same symbol ∇ hopefully without any possible confusion, to be

$$\nabla\gamma(d_1, d_2) = (\nabla\gamma_{d_1}^1)_{d_2} (\nabla\gamma_0^2)_{d_1}$$

for any $\gamma \in \mathcal{A}^2 G$.

It is easy to see that

Proposition 9. *For any $\gamma \in \mathcal{A}^2 G$ and any $a \in \mathbb{R}$, we have*

$$
\begin{aligned}
\nabla(a \underset{1}{\cdot} \gamma) &= a \underset{1}{\cdot} \nabla\gamma \\
\nabla(a \underset{2}{\cdot} \gamma) &= a \underset{2}{\cdot} \nabla\gamma.
\end{aligned}
$$

Corollary 10. *For any $\gamma_1, \gamma_2 \in \mathcal{A}^2 G$, we have*

$$
\begin{aligned}
&\nabla(\gamma_2 \underset{1}{-} \gamma_1) = \nabla\gamma_2 \underset{1}{-} \nabla\gamma_1 \quad \textit{provided that} \quad \gamma_1(0, \cdot) = \gamma_2(0, \cdot); \\
&\nabla(\gamma_2 \underset{2}{-} \gamma_1) = \nabla\gamma_2 \underset{2}{-} \nabla\gamma_1 \quad \textit{provided that} \quad \gamma_1(\cdot, 0) = \gamma_2(\cdot, 0).
\end{aligned}
$$

Proof. This follows from the above proposition by Proposition 10 of §1.2 of Lavendhomme [7].

Proposition 11. *For any $t \in \mathcal{A}^1 G$, we define $\varepsilon_t \in \mathcal{A}^2 G$ to be*

$$\varepsilon_t(d_1, d_2) = t(d_1 d_2).$$

Then we have

$$(\nabla\varepsilon_t)(d_1, d_2) = (\nabla t)(d_1 d_2)$$

for any $d_1, d_2 \in D$.

Proof. It suffices to note that

$$(\nabla\varepsilon_t)(d_1, d_2) = (\nabla(d_1 t))(d_2) = (d_1 \nabla t)(d_2) = (\nabla t)(d_1 d_2).$$

Theorem 12. *For any* $\gamma_1, \gamma_2 \in \mathcal{A}^2 G$ *with* $\gamma_1 \mid_{D(2)} = \gamma_2 \mid_{D(2)}$, *we have*

$$\nabla(\gamma_2 \dot{-} \gamma_1) = \nabla\gamma_2 \dot{-} \nabla\gamma_1.$$

Proof. Let $d_1, d_2 \in D$. We have

$$(\nabla(\gamma_2 \dot{-} \gamma_1))(d_1 d_2) = (\nabla\varepsilon_{\gamma_2 \dot{-} \gamma_1})(d_1, d_2)$$

[by Proposition 11],

$$= (\nabla((\gamma_2 \underset{1}{-} \gamma_1) \underset{2}{-} \tau^2_{\gamma_1}))(d_1, d_2)$$

[by Proposition 7 of §3.4 of Lavendhomme [7]],

$$= ((\nabla\gamma_2 \underset{1}{-} \nabla\gamma_1) \underset{2}{-} \nabla\tau^2_{\gamma_1})(d_1, d_2)$$

[by Corollary 10],

$$= ((\nabla\gamma_2 \underset{1}{-} \nabla\gamma_1) \underset{2}{-} \tau^2_{\nabla\gamma_1})(d_1, d_2)$$
$$= \varepsilon_{\nabla\gamma_2 \dot{-} \nabla\gamma_1}(d_1, d_2)$$

[by Proposition 7 of §3.4 of Lavendhomme [7]],

$$= (\nabla\gamma_2 \dot{-} \nabla\gamma_1)(d_1 d_2)$$

[By Proposition 11].

Since $d_1, d_2 \in D$ were arbitrary, the desired conclusion follows at once.

5. The curvature form

Proposition 13. *For any* $\gamma \in \mathcal{A}^2 G$, *there exists a unique* $t \in \mathcal{A}^1 L$ *such that*

$$\iota(t_{d_1 d_2}) = ((\nabla\gamma^2_0)_{d_1})^{-1}((\nabla\gamma^1_{d_1})_{d_2})^{-1}(\nabla\gamma^2_{d_2})_{d_1}(\nabla\gamma^1_0)_{d_2}$$

for any $d_1, d_2 \in D$.

Proof. Let $\eta \in \mathcal{A}^2 H$ to be

$$\eta(d_1, d_2) = ((\nabla\gamma^2_0)_{d_1})^{-1}((\nabla\gamma^1_{d_1})_{d_2})^{-1}(\nabla\gamma^2_{d_2})_{d_1}(\nabla\gamma^1_0)_{d_2}$$

for any $d_1, d_2 \in D$. Then it is easy to see that

$$\eta(d, 0) = \eta(0, d) = \mathrm{id}_{\alpha(\eta(0,0))}.$$

Therefore there exists unique $t' \in \mathcal{A}^1 H$ such that

$$t'_{d_1 d_2} = \eta(d_1, d_2).$$

Furthermore we have

$$\begin{aligned}\pi(\eta(d_1,d_2)) &= \pi(((\nabla\gamma_0^2)_{d_1})^{-1})\pi(((\nabla\gamma_{d_1}^1)_{d_2})^{-1})\pi((\nabla\gamma_{d_2}^2)_{d_1})\pi((\nabla\gamma_0^1)_{d_2}) \\ &= ((\gamma_0^2)_{d_1})^{-1}((\gamma_{d_1}^1)_{d_2})^{-1}(\gamma_{d_2}^2)_{d_1}(\gamma_0^1)_{d_2} \\ &= \gamma(d_1,0)^{-1}(\gamma(d_1,d_2)\gamma(d_1,0)^{-1})^{-1}\gamma(d_1,d_2)\gamma(0,d_2)^{-1}\gamma(0,d_2) \\ &= \mathrm{id}_{\alpha(\eta(0,0))}.\end{aligned}$$

Therefore there exists a unique $t \in \mathcal{A}^1 L$ with $\iota(t) = t'$. This completes the proof.

We write $\Omega(\gamma)$ for the above t. Now we have

Proposition 14. *The mapping $\Omega : \mathcal{A}^2 G \to \mathcal{A}^1 L$ consists in $\mathbf{C}^2(G, \mathcal{A}L)$.*

Proof. We have to show that

$$\Omega(a \underset{1}{\cdot} \gamma) = a\Omega(\gamma) \tag{1}$$

$$\Omega(a \underset{2}{\cdot} \gamma) = a\Omega(\gamma) \tag{2}$$

$$\Omega(\Sigma\gamma) = -\Omega(\gamma) \tag{3}$$

for any $\gamma \in \mathcal{A}^2 G$ and any $a \in \mathbb{R}$. Now we deal with (1), leaving a similar treatment of (2) to the reader. Let $d_1, d_2 \in D$. We have

$$\begin{aligned}\iota(\Omega(a \underset{1}{\cdot} \gamma))_{d_1 d_2} &= ((\nabla(a \underset{1}{\cdot} \gamma)_0^2)_{d_1})^{-1}((\nabla(a \underset{1}{\cdot} \gamma)_{d_1}^1)_{d_2})^{-1}(\nabla(a \underset{1}{\cdot} \gamma)_{d_2}^2)_{d_1}(\nabla(a \underset{1}{\cdot} \gamma)_0^1)_{d_2} \\ &= ((\nabla\gamma_0^2)_{ad_1})^{-1}((\nabla\gamma_{ad_1}^1)_{d_2})^{-1}(\nabla\gamma_{d_2}^2)_{ad_1}(\nabla\gamma_0^1)_{d_2} \\ &= \iota(\Omega(\gamma))_{ad_1 d_2} \\ &= \iota(a\Omega(\gamma))_{d_1 d_2}.\end{aligned}$$

Now we deal with (3). We have

$$\begin{aligned}\iota(\Omega(\Sigma\gamma))\iota(\Omega(\gamma))_{d_1 d_2} &= \{((\nabla\gamma_0^1)_{d_2})^{-1}((\nabla\gamma_{d_2}^2)_{d_1})^{-1}(\nabla\gamma_{d_1}^1)_{d_2}(\nabla\gamma_0^2)_{d_1}\} \\ &\qquad \{((\nabla\gamma_0^2)_{d_1})^{-1}((\nabla\gamma_{d_1}^1)_{d_2})^{-1}(\nabla\gamma_{d_2}^2)_{d_1}(\nabla\gamma_0^1)_{d_2}\} \\ &= \mathrm{id}_{\alpha(\gamma(0,0))}.\end{aligned}$$

This completes the proof.

We call Ω the *curvature form* of ∇.

Proposition 15. *For any $\gamma \in \mathcal{A}^2 G$, we have*

$$\Omega(\gamma) = \Sigma\nabla\Sigma\gamma \stackrel{\cdot}{-} \nabla\gamma.$$

Proof. As in the proof of Proposition 8 of §3.4 of Lavendhomme [7], let us consider a function $\mathbf{l} : D^2 \vee D \to H$ given by

$$\mathbf{l}(d_1, d_2, e) = (\nabla\gamma^1_{d_1})_{d_2}(\nabla\gamma^2_0)_{d_1}\Omega(\gamma)_e$$

for any $(d_1, d_2, e) \in D^2 \vee D$. Then it is easy to see that $\mathbf{l}(d_1, d_2, 0) = (\nabla\gamma)(d_1, d_2)$ and $\mathbf{l}(d_1, d_2, d_1 d_2) = (\Sigma\nabla\Sigma\gamma)(d_1, d_2)$. Therefore we have

$$(\Sigma\nabla\Sigma\gamma \dot{-} \nabla\gamma)_e = \mathbf{l}(0, 0, e) = \Omega(\gamma)_e.$$

This completes the proof.

Now we deal with tensorial aspects of Ω. It is easy to see that

Proposition 16. *Let $X, Y \in \Gamma(\mathcal{A}G)$. Then we have*

$$\nabla(Y * X) = \nabla Y * \nabla X.$$

Now we have the following familiar form for Ω.

Theorem 17. *Let $X, Y \in \Gamma(\mathcal{A}G)$. Then we have*

$$\Omega(Y * X) = \nabla[X, Y] - [\nabla X, \nabla Y].$$

Proof. It suffices to note that

$$\Omega(Y * X) = \Sigma\nabla\Sigma(Y * X) \dot{-} \nabla(Y * X)$$

[by Proposition 15],

$$= \nabla\Sigma(Y * X) \dot{-} \Sigma\nabla(Y * X)$$

[by Proposition 6 of §3.4 of Lavendhomme [7]],

$$= \nabla(\Sigma(Y * X) \dot{-} X * Y) - (\Sigma\nabla(Y * X) \dot{-} \nabla(X * Y))$$

[by Proposition 6],

$$= \nabla(Y * X \dot{-} \Sigma(X * Y)) - (\nabla(Y * X) \dot{-} \Sigma\nabla(X * Y))$$

[by Proposition 6 of §3.4 of Lavendhomme [7]],

$$= \nabla(Y * X \dot{-} \Sigma(X * Y)) - (\nabla Y * \nabla X \dot{-} \Sigma(\nabla X * \nabla Y))$$

[by Proposition 16],

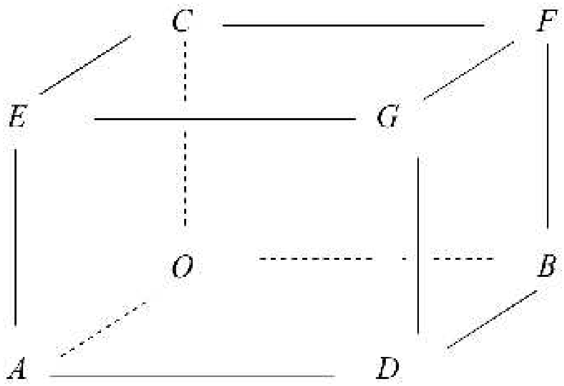
$$= \nabla[X, Y] - [\nabla X, \nabla Y]$$

[by Proposition 8 of §3.4 of Lavendhomme [7]].

6. The Bianchi identity

Let us begin with the following abstract Bianchi identity, which traces back to Kock [5], though our version is cubical, while Kock's one is simplicial. Our cubical Bianchi identity originated in [11].

Theorem 18. *Let the following figure be an arbitrary cube in a groupoid H.*



For each pair (X,Y) of adjacent vertices X,Y of the cube, $P_{YX}: X \to Y$ and $P_{XY}: Y \to X$ denote the mutually inverse morphisms of the edge. For any four vertices W,Z,Y,X of the cube rounding one of the six facial squares of the cube, R_{WZYX} denotes $P_{XW}P_{WZ}P_{ZY}P_{YX}$. Then we have

$$P_{OA}P_{AD}P_{DG}R_{DBFG}R_{FCEG}R_{EADG}P_{GD}P_{DA}P_{AO}R_{AECO}R_{CFBO}R_{BDAO} = \mathrm{id}_O. \quad (4)$$

Proof. Write over the desired identity exclusively in terms of P_{YX}'s, and write off all consective $P_{XY}P_{YX}$'s.

Notation 19. We will use the notation of the above theorem throughout the rest of this section.

Now we recall the Brown-Higgins cubical formula, for which the reader is referred to [1]. When we found out the formula (4) in [11] at the end of the previous century, we were not conscious of Brown and Higgins' work at all. It is the referee who has kindly turned our attention to their paper for comparison.

Theorem 20. *We have*

$$\begin{aligned}&(P_{OA}R_{DGEA}P_{AO})R_{AECO}(P_{OC}R_{EGFC}P_{CO})R_{CFBO}(P_{OB}R_{FGDB}P_{BO})R_{BDAO} \\ &= \mathrm{id}_O. \end{aligned} \quad (5)$$

Proof. Write over the desired identity exclusively in terms of P_{YX}'s, and write off all consecutive $P_{XY}P_{YX}$'s.

Remark 21. We compare the two combinatorial formulas established in the above two theorems. In (4) the three round tours beginning with G in conjugation together with the three round tours beginning with O appear with the first three and the last three grouped separately. In (5) the three round tours beginning with vertices adjacent to O but not encountering O in conjugation together with the three round tours beginning with O appear alternatingly. We are not sure whether (5) is derivable combinatorially from (4). Originally we based our proof of the second Bianchi identity on (4), but following the referee's suggestions, we give its proof based on (5) here, because it is shorter.

Now we would like to establish the second Bianchi identity in familiar form. To this end, we need two lemmas.

Lemma 22. *Let $x \in M$. If $s, t \in \mathcal{A}_x L$ are such that*

1. $s_d = f^{-1} s'_d f$ *($\forall d \in D$) for some $f : x \to y$ in H and some $s' \in \mathcal{A}_y L$, and*
2. $t_d = f^{-1} t'_d f$ *($\forall d \in D$) for some $f : x \to z$ in H and some $t' \in \mathcal{A}_z L$,*

then s_d and t_d commute for any $d \in D$.

Proof. This follows simply from Proposition 2.

We now express Theorem 7 in case of $n = 2$ geometrically.

Lemma 23. *Let $\gamma \in \mathcal{A}^3 G$. Let $d_1, d_2, d_3 \in D$. Using the cube in Theorem* 18, *we let the eight vertices O, A, B, C, D, E, F, G of the cube represent*

$$\beta(\gamma(0,0,0)), \beta(\gamma(d_1,0,0)), \beta(\gamma(0,d_2,0)), \beta(\gamma(0,0,d_3)),$$
$$\beta(\gamma(d_1,d_2,0)), \beta(\gamma(d_1,0,d_3)), \beta(\gamma(0,d_2,d_3)), \beta(\gamma(d_1,d_2,d_3))$$

in order, while we let the twelve edges of the cube represent

$$\begin{aligned}
&P_{AO} = (\nabla\gamma^{2,3}_{0,0})_{d_1}, P_{BO} = (\nabla\gamma^{1,3}_{0,0})_{d_2}, P_{CO} = (\nabla\gamma^{1,2}_{0,0})_{d_3}, P_{DA} = (\nabla\gamma^{1,3}_{d_1,0})_{d_2},\\
&P_{EA} = (\nabla\gamma^{1,2}_{d_1,0})_{d_3}, P_{DB} = (\nabla\gamma^{2,3}_{d_2,0})_{d_1}, P_{FB} = (\nabla\gamma^{1,2}_{0,d_2})_{d_3}, P_{EC} = (\nabla\gamma^{2,3}_{0,d_3})_{d_1},\\
&P_{FC} = (\nabla\gamma^{1,3}_{0,d_3})_{d_2}, P_{GD} = (\nabla\gamma^{1,2}_{d_1,d_2})_{d_3}, P_{GE} = (\nabla\gamma^{1,3}_{d_1,d_3})_{d_2}, P_{GF} = (\nabla\gamma^{2,3}_{d_2,d_3})_{d_1}.
\end{aligned} \tag{6}$$

Then we have

$$\begin{aligned}
&(\mathbf{d}_\nabla \Omega(\gamma))_{d_1 d_2 d_3}\\
&= (P_{OA} R_{DGEA} P_{AO}) R_{CFBO} (P_{OB} R_{FGDB} P_{BO}) R_{AECO} (P_{OC} R_{EGFC} P_{CO}) R_{BDAO}.
\end{aligned} \tag{7}$$

Remark 24. The reader should note that $(\nabla\gamma^{2,3}_{0,0})_{d_1}$ in (6) and $(\nabla\gamma_1)_{d_1}$ in Theorem 7 are the same, and so on.

Proof. It suffices to note the following:

$$R_{BDAO} = \Omega(\gamma^3_0)_{-d_1 d_2} \tag{8}$$

$$R_{CFBO} = \Omega(\gamma_0^1)_{-d_2 d_3} \tag{9}$$

$$R_{AECO} = \Omega(\gamma_0^2)_{d_1 d_3} \tag{10}$$

$$P_{OA}R_{DGEA}P_{AO} = ((\nabla\gamma_{0,0}^{2,3})_{d_1})^{-1}\Omega(\gamma_{d_1}^1)_{d_2 d_3}(\nabla\gamma_{0,0}^{2,3})_{d_1} \tag{11}$$

$$P_{OB}R_{FGDB}P_{BO} = ((\nabla\gamma_{0,0}^{1,3})_{d_2})^{-1}\Omega(\gamma_{d_2}^2)_{-d_1 d_3}(\nabla\gamma_{0,0}^{1,3})_{d_2} \tag{12}$$

$$P_{OC}R_{EGFC}P_{CO} = ((\nabla\gamma_{0,0}^{1,2})_{d_3})^{-1}\Omega(\gamma_{d_3}^3)_{d_1 d_2}(\nabla\gamma_{0,0}^{1,2})_{d_3}. \tag{13}$$

Now we are ready to establish the second Bianchi identity in familiar form.

Theorem 25. *We have*

$$\mathbf{d}_\nabla \Omega = 0.$$

Proof. We use the same notation as in Lemma 23. As you can see, the left-hand side of (5) and the right-hand side of (7) differ only in the order of their six factors (8)–(13). However we have

$$\begin{aligned}
&\mathrm{id}_O \\
&= (P_{OA}R_{DGEA}P_{AO})R_{AECO}(P_{OC}R_{EGFC}P_{CO})R_{CFBO}(P_{OB}R_{FGDB}P_{BO})R_{BDAO} \\
&\text{[by Theorem 20]}, \\
&= (P_{OA}R_{DGEA}P_{AO})R_{AECO}R_{CFBO}(P_{OC}R_{EGFC}P_{CO})(P_{OB}R_{FGDB}P_{BO})R_{BDAO} \\
&\text{[by Lemma 22]}, \\
&= (P_{OA}R_{DGEA}P_{AO})R_{CFBO}R_{AECO}(P_{OC}R_{EGFC}P_{CO})(P_{OB}R_{FGDB}P_{BO})R_{BDAO} \\
&\text{[by Proposition 2]}, \\
&= (P_{OA}R_{DGEA}P_{AO})R_{CFBO}(P_{OB}R_{FGDB}P_{BO})R_{AECO}(P_{OC}R_{EGFC}P_{CO})R_{BDAO} \\
&\text{[by Lemma 22]}.
\end{aligned}$$

This completes the proof.

Remark 26. In the course of the above proof we have realized that the six curvatures (8)–(13) commute by dint of Proposition 2 and Lemma 22.

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