Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 2, 599-605.

# Covering the Unit Cube by Equal Balls

**Antal Joós**

*College of Dunaújváros, 2400 Dunaújváros*
*Táncsics M. u. 1/a, Hungary*
*e-mail: ajoos@kac.poliod.hu*

**Abstract.** We give the minimal radius of 8 congruent balls, which cover the 4-dimensional unit cube.

MSC 2000: 52C17
Keywords: covering by equal balls, finite coverings

## 1. Introduction

We can read by Brass, Moser, Pach [3] about the problem of covering the $d$-dimensional unit cube by $n$ equal minimal balls. In [3], [1] one can find numerous results in the case $d = 2$. G. Kuperberg and W. Kuperberg [4] found the optimal solution in $d = 3, n = 2, 3, 4, 8$. In higher-dimensions G. Kuperberg and W. Kuperberg [4] found the case $d \geq 4, n = 4$.

One can read results about covering by $n$ equal balls a $d$-dimensional larger ball (Rogers [5], Verger-Gaugray [6]), and the $d$-dimensional crosspolytope (Böröczky, Jr., Fábián, Wintsche [2]).

## 2. Notations

Let $\mathbb{E}^d$ be the $d$-dimensional Euclidean space. Let $C^d := [0,1]^d$ be the $d$-dimensional unit cube. Let $B^d(a,r)$ be the $d$-dimensional ball with centre $a$ and radius $r$. $d(p,q)$ denotes the distance of the points $p, q$. Let $R_{a,b}$ be the ray with endpoint $a$ and containing $b$. $poq\angle$ denotes the convex angle determined by the three points $p, o, q$ in this order. Let $L_{a,b}$ be the straight line containing the different points $a, b$. $(E,F)\angle$ denotes the angle determined by the two rays $E, F$. Let $H(L,p)$ be the closed half plane bounded by the line $L$ and containing the point $p$.

Let

$$o_1\left(\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{6}\right),\ o_2\left(\tfrac{1}{6},\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6}\right),\ o_3\left(\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{6}\right),\ o_4\left(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{2}\right),\ o_5\left(\tfrac{5}{6},\tfrac{5}{6},\tfrac{5}{6},\tfrac{1}{2}\right),$$
$$o_6\left(\tfrac{5}{6},\tfrac{1}{2},\tfrac{1}{6},\tfrac{5}{6}\right),\ o_7\left(\tfrac{1}{2},\tfrac{1}{6},\tfrac{5}{6},\tfrac{5}{6}\right),\ o_8\left(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2},\tfrac{5}{6}\right),\ m\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right),\ o_1^3\left(\tfrac{5}{6},\tfrac{1}{6},\tfrac{1}{2},0\right),$$
$$o_2^3\left(\tfrac{1}{6},\tfrac{1}{2},\tfrac{5}{6},0\right),\ o_3^3\left(\tfrac{1}{2},\tfrac{5}{6},\tfrac{1}{6},0\right),\ o_4^3\left(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{6},0\right),\ o_5^3\left(\tfrac{5}{6},\tfrac{5}{6},\tfrac{5}{6},0\right).$$

**Theorem.** *The minimal radius of 8 congruent balls, which cover the 4-dimensional unit cube, is* $\sqrt{\frac{5}{12}}$.

## 3. Lemmas

**Lemma 1.** *The balls* $\mathbf{B}^4\left(o_i,\sqrt{\frac{5}{12}}\right)\subset\mathbb{E}^4$ $(i=1,2,\dots,8)$ *cover the cube* $C^4$.

*Proof.* Let $\mathbf{B}_i^4:=\mathbf{B}^4\left(o_i,\sqrt{\frac{5}{12}}\right)$ for $i=1,2,\dots,8$. Since $d(m,o_i)=\sqrt{\frac{1}{3}}<\sqrt{\frac{5}{12}}$ thus the center $m$ of $C^4$ lies in $\mathbf{B}_i^4$ for $i=1,2,\dots,8$.

We will show that the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$ cover every 3-dimensional face of the cube $C^4$. From this comes that $\mathbf{B}_i^4$ for $i=1,2,\dots,8$ cover $C^4$. (See Figure 1. The thick edges are the edges, which lie entirely in a ball $\mathbf{B}_i^4$.)

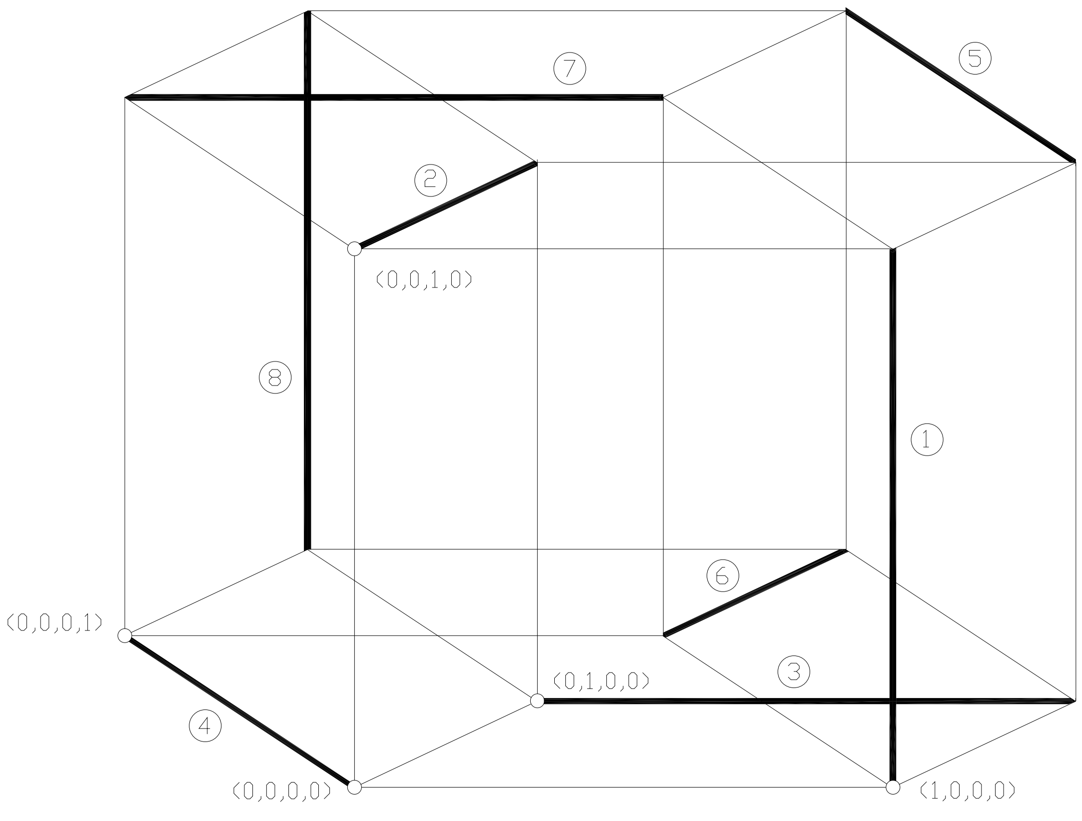


Figure 1. The four circle problem

We show that the balls $\mathbf{B}_i^4$ for $i=1,2,3,4,5$ cover the cube $[0,1]^3\times\{0\}$ (the cover of the other 3-dimensional faces of $C^4$ is similar).

The intersection of the 4-dimensional balls $\mathbf{B}_i^4$ for $i = 1,2,3,4,5$ and the hyperplane $x_4 = 0$ are the 3-dimensional balls $\mathbf{B}_i^3 := \mathbf{B}^3\left(o_i^3, \sqrt{\frac{7}{18}}\right)$ for $i = 1,2,3$ and $\mathbf{B}_i^3 := \mathbf{B}^3\left(o_i^3, \frac{1}{\sqrt{6}}\right)$ for $i = 4,5$, resp.

Firstly we show that the cube $C_0^3 := [0, \frac{1}{2}]^3 \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_i^3$ for $i = 1,2,3,4$ (see Figure 2).

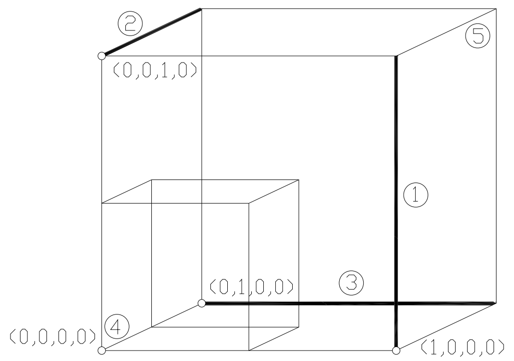


Figure 2.

Let $p(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Since $d(p, o_i^3) = \sqrt{\frac{11}{36}} < \sqrt{\frac{7}{18}}$ for $i = 1,2,3$ and $d(p, o_4^3) = \frac{1}{\sqrt{12}} < \frac{1}{\sqrt{6}}$ thus $p \in \mathbf{B}_i^3$ for $i = 1,2,3,4$. If we show that the 2-dimensional faces of the cube $C_0^3$ is covered by the balls $\mathbf{B}_i^3$ for $i = 1,2,3,4$ then we get that the cube $C_0^3$ is covered by the balls $\mathbf{B}_i^3$ for $i = 1,2,3,4$.

Let us see the 2-dimensional face with vertices $(0,0,0,0), (\frac{1}{2},0,0,0), (\frac{1}{2},\frac{1}{2},0,0), (0,\frac{1}{2},0,0)$. Since the region $\operatorname{conv}((0,0,0,0), (\frac{1}{2},0,0,0), (\frac{1}{2},\frac{1}{3},0,0), (\frac{1}{3},\frac{1}{2},0,0), (0,\frac{1}{2},0,0))$ is covered by $\mathbf{B}_4^3$, and the region $\operatorname{conv}\left((\frac{1}{2},\frac{1}{3},0,0), (\frac{1}{2},\frac{1}{2},0,0), (0,\frac{1}{2},0,0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above 2-dimensional face is covered by the balls $\mathbf{B}_3^3, \mathbf{B}_4^3$. Similarly the face with vertices $(0,0,0,0), (\frac{1}{2},0,0,0), (\frac{1}{2},0,\frac{1}{2},0), (0,0,\frac{1}{2},0)$ is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_4^3$, and the face with vertices $(0,0,0,0), (0,\frac{1}{2},0,0), (0,\frac{1}{2},\frac{1}{2},0), (0,0,\frac{1}{2},0)$ is covered by the balls $\mathbf{B}_2^3, \mathbf{B}_4^3$.

Let us consider the face with vertices $(\frac{1}{2},0,0,0), (\frac{1}{2},\frac{1}{2},0,0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0), (\frac{1}{2},0,\frac{1}{2},0)$. Since the region $\operatorname{conv}\left((\frac{1}{2},0,0,0), (\frac{1}{2},0,\frac{1}{2},0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0), (\frac{1}{2},\frac{1}{2},\frac{1}{3},0), (\frac{1}{2},\frac{1}{3},0,0)\right)$ is covered by $\mathbf{B}_1^3$, and the region $\operatorname{conv}\left((\frac{1}{2},\frac{1}{3},0,0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0), (\frac{1}{2},\frac{1}{2},0,0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above face is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_3^3$. Similarly the face with vertices $(0,0,\frac{1}{2},0), (\frac{1}{2},0,\frac{1}{2},0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0), (0,\frac{1}{2},\frac{1}{2},0)$ is covered by the balls $\mathbf{B}_1^3, \mathbf{B}_2^3$, and the face with vertices $(0,\frac{1}{2},0,0), (\frac{1}{2},\frac{1}{2},0,0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},0), (0,\frac{1}{2},\frac{1}{2},0)$ is covered by the balls $\mathbf{B}_2^3, \mathbf{B}_3^3$. Similarly the cube $[\frac{1}{2},1]^3 \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3, \mathbf{B}_2^3, \mathbf{B}_3^3, \mathbf{B}_5^3$.

Secondly we show that the cube $[\frac{1}{2},1] \times [0,\frac{1}{2}] \times [0,\frac{1}{2}] \times \{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3, \mathbf{B}_3^3$.

Since the region $\mathrm{conv}((1,0,0,0),(1,\frac{1}{2},0,0),(\frac{2}{3},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{3},0,0),(\frac{1}{2},0,0,0),(\frac{1}{2},\frac{1}{2},\frac{1}{3},0),(\frac{1}{2},\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},0,\frac{1}{2},0),(1,0,\frac{1}{2},0),(1,\frac{1}{2},\frac{1}{2},0))$ is covered by $\mathbf{B}_1^3$, and the region $\mathrm{conv}\left((\frac{1}{2},\frac{1}{3},0,0),(1,\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{2},0,0),(\frac{1}{2},\frac{1}{2},\frac{1}{2},0)\right)$ is covered by $\mathbf{B}_3^3$ thus the above cube is covered. Similarly the cube $[\frac{1}{2},1]^2\times[0,\frac{1}{2}]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]\times[\frac{1}{2},1]\times[0,\frac{1}{2}]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_2^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]\times[\frac{1}{2},1]^2\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_2^3,\mathbf{B}_3^3$, and the cube $[0,\frac{1}{2}]^2\times[\frac{1}{2},1]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_2^3$, and the cube $[\frac{1}{2},1]\times[0,\frac{1}{2}]\times[\frac{1}{2},1]\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_1^3,\mathbf{B}_2^3$. This implies that the cube $[0,1]^3\times\{0\}$ is covered by the 3-dimensional balls $\mathbf{B}_i^3$ for $i=1,2,3,4,5$, that is, $[0,1]^3\times\{0\}$ is covered by the 4-dimensional balls $\mathbf{B}_i^4$ for $i=1,2,3,4,5$.

Similarly the cube $[0,1]^3\times\{1\}$ is covered by the 4-dimensional balls $\mathbf{B}_4^4,\mathbf{B}_5^4,\mathbf{B}_6^4,\mathbf{B}_7^4,\mathbf{B}_8^4$, and the cube $\{0\}\times[0,1]^3$ is covered by the 4-dimensional balls $\mathbf{B}_2^4,\mathbf{B}_3^4,\mathbf{B}_4^4,\mathbf{B}_7^4,\mathbf{B}_8^4$, and the cube $\{1\}\times[0,1]^3$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_3^4,\mathbf{B}_5^4,\mathbf{B}_6^4,\mathbf{B}_7^4$, and the cube $[0,1]\times\{0\}\times[0,1]^2$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_2^4,\mathbf{B}_4^4,\mathbf{B}_6^4,\mathbf{B}_7^4$, and the cube $[0,1]\times\{1\}\times[0,1]^2$ is covered by the 4-dimensional balls $\mathbf{B}_2^4,\mathbf{B}_3^4,\mathbf{B}_5^4,\mathbf{B}_6^4,\mathbf{B}_8^4$, and the cube $[0,1]^2\times\{0\}\times[0,1]$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_3^4,\mathbf{B}_4^4,\mathbf{B}_6^4,\mathbf{B}_8^4$, and the cube $[0,1]^2\times\{1\}\times[0,1]$ is covered by the 4-dimensional balls $\mathbf{B}_1^4,\mathbf{B}_2^4,\mathbf{B}_5^4,\mathbf{B}_7^4,\mathbf{B}_8^4$. Then the 3-dimensional faces of the cube $C^4$ are covered by the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$, that is, the cube $C^4$ is covered by the balls $\mathbf{B}_i^4$ for $i=1,2,\dots,8$. □

**Lemma 2.** *Let $a_1,a_2\in\mathbf{B}^2(o,r)\subset\mathbb{E}^2$, $\left(\frac{1}{2}<r\right)$, $d(a_1,a_2)=1$. Let $R_{a_1,b_1},R_{a_2,b_2}$ be two rays perpendicular to $L_{a_1,a_2}$. If $d(o,L_{a_1,a_2}),r$ are fixed numbers then*

$$\mathrm{diam}\left(R_{a_1,b_1}\cap\mathbf{B}^2(o,r)\right)+\mathrm{diam}\left(R_{a_2,b_2}\cap\mathbf{B}^2(o,r)\right)$$

*is the greatest if $d(o,a_1)=d(o,b_1)$ and $R_{a_1,b_1},R_{a_2,b_2}$ lie in a closed half plane bounded by $L_{a_1,a_2}$ and containing $o$.*

*Proof.* Let $h_i$ be the point on the line $L_{a_i,b_i}$ and not contained $R_{a_i,b_i}$ for $i=1,2$. If $L_{a_1,a_2}$ does not contain $o$ and $R_{a_1,b_1}\not\subset H(L_{a_1,a_2},o)$ then $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))<\mathrm{diam}\,(R_{a_1,h_1}\cap\mathbf{B}^2(o,r))$. In this case we change $R_{a_1,b_1}$ for $R_{a_1,h_1}$ and we mark $R_{a_1,h_1}$ with $R_{a_1,b_1}$. Similarly if $L_{a_1,a_2}$ does not contain $o$ and $R_{a_2,b_2}\not\subset H(L_{a_1,a_2},o)$ then we change $R_{a_2,b_2}$ for $R_{a_2,h_2}$ and we mark $R_{a_2,h_2}$ with $R_{a_2,b_2}$. If $L_{a_1,a_2}$ contains $o$ and $H(L_{a_1,a_2},b_1)$ does not contain $b_2$ then we change $R_{a_2,b_2}$ for $R_{a_2,h_2}$ and we mark $R_{a_2,h_2}$ with $R_{a_2,b_2}$. With these changes $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))+\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))$ does not decrease.

Let $e$ be the straight line containing $o$ and parallel $L_{a_1,a_2}$ (Figure 3). Let $c_1,c_2$ be the intersection point of $e$ and $R_{a_1,b_1},R_{a_2,b_2}$, resp. If $o$ does not lie on the segment $c_1c_2$ and, say, $d(o,c_1)>d(o,c_2)$ then let $R_{a_1',b_1'}$ be the image of $R_{a_1,b_1}$ under the reflection with respect to the line $L_{a_2,b_2}$. In this case $\mathrm{diam}\,(R_{a_1,b_1}\cap\mathbf{B}^2(o,r))+\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))<\mathrm{diam}\left(R_{a_1',b_1'}\cap\mathbf{B}^2(o,r)\right)+\mathrm{diam}\,(R_{a_2,b_2}\cap\mathbf{B}^2(o,r))$. We use this method until $o$ lies between the images of the rays. Thus, we can assume that $o$ lies on the segment $c_1c_2$. Let $d_1,d_2$ be the intersection point of $\mathrm{bd}\,\mathbf{B}^2(o,r)$

and $R_{a_1,b_1}, R_{a_2,b_2}$, resp. Let $x := d(o, c_1)$ and

$$f(x) := \operatorname{diam}\left(R_{a_1,b_1} \cap \mathbf{B}^2(o,r)\right) + \operatorname{diam}\left(R_{a_2,b_2} \cap \mathbf{B}^2(o,r)\right) =$$
$$= d(a_1,d_1) + d(a_2,d_2) = 2d(o, L_{a_1,a_2}) + d(c_1,d_1) + d(c_2,d_2) =$$
$$= 2d(o, L_{a_1,a_2}) + \sqrt{r^2 - x^2} + \sqrt{r^2 - (1-x)^2}.$$

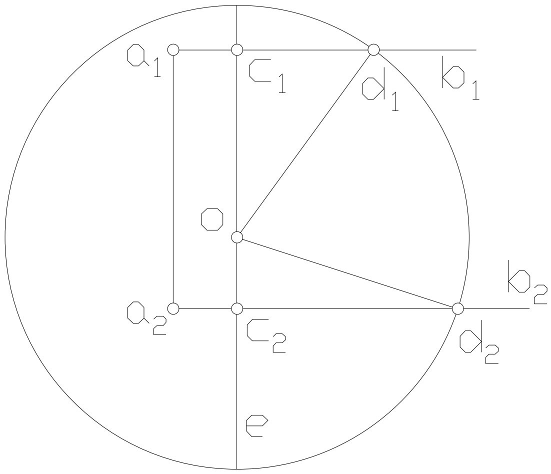


Figure 3.

By elementary calculus, the maximum value of $f(x)$ between 0 and 1 is achieved at $\frac{1}{2}$. This completes the proof of the lemma. □

**Lemma 3.** *Let* $R_{a,b_1}, R_{a,b_2}, R_{a,b_3} \subset \mathbb{E}^3$, $b_1ab_2\angle = \frac{\pi}{2}, b_1ab_3\angle = \frac{\pi}{2}, b_2ab_3\angle = \frac{\pi}{2}$ *and* $\mathbf{B}^3(o,r) \subset \mathbb{E}^3$. *Then*

$$\sum_{i=1,2,3} \operatorname{diam}\left(R_{a,b_i} \cap \mathbf{B}^3(o,r)\right) \le r\frac{3\sqrt{6}}{2}.$$

*Proof.* Let $c_i := R_{a,b_i} \cap \mathbf{B}^3(o,r)$ for $i = 1,2,3$. Of course,

$$d(a,c_1) + d(a,c_2) \le d(c_1,c_2)\sqrt{2},$$
$$d(a,c_2) + d(a,c_3) \le d(c_2,c_3)\sqrt{2},$$
$$d(a,c_1) + d(a,c_3) \le d(c_1,c_3)\sqrt{2}.$$

Thus

$$d(a,c_1) + d(a,c_2) + d(a,c_3) \le \frac{\sqrt{2}}{2}\left(d(c_1,c_2) + d(c_2,c_3) + d(c_1,c_3)\right).$$

Since

$$d(c_1,c_2) + d(c_2,c_3) + d(c_1,c_3) \le r3\sqrt{3}$$

thus

$$\sum_{i=1,2,3} \operatorname{diam}\left(R_{a,b_i} \cap \mathbf{B}^3(o,r)\right) = \sum_{i=1,2,3} d(a,c_i) \le r\frac{3\sqrt{6}}{2}.$$

This completes the proof of the lemma. □

Note that if $R_{a,b_1} \cap \mathbf{B}^3(o,r) + R_{a,b_2} \cap \mathbf{B}^3(o,r) + R_{a,b_3} \cap \mathbf{B}^3(o,r) = r\frac{3\sqrt{6}}{2}$ then $a \in \operatorname{int} \mathbf{B}^3(o,r)$.

**Lemma 4.** *Let $a_1, a_2 \in \mathbf{B}^4(o,r) \subset \mathbb{E}^4$, $\left(\frac{1}{2} < r < \sqrt{\frac{5}{12}}\right)$ and $d(a_1,a_2) = 1$. Let $R_{a_j,b_i^j}(i=1,2,3; j=1,2)$ such rays that $a_2a_1b_i^1\angle = \frac{\pi}{2}, b_i^j a_j b_k^j\angle = \frac{\pi}{2}, b_i^1 \parallel b_i^2$ and in the plane determined by the points $a_1, a_2, b_i^1$ the half plane $H(L_{a_1,a_2}, b_i^1)$ contains the point $b_i^2$ for any $i,k \in \{1,2,3\} (i \neq k)$, $j = 1,2$. Then*

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4(o,r)\right) < 4.$$

*Proof.* The value of

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4(o,r)\right)$$

is smaller than

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right).$$

Let $H$ be the hyper plane perpendicular to the segment $a_1a_2$ containing $o$. If the projection of the rays $R_{a_j,b_i^j}(i=1,2,3; j=1,2)$ onto the hyper plane $H$ is fixed then by Lemma 2

$$\sum_{i=1,2,3} \left(\operatorname{diam}\left(R_{a_1,b_i^1} \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right) + \operatorname{diam}\left(R_{a_2,b_i^2} \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right)\right)$$

is the greatest if $d(o,a_1) = d(o,a_2)$. Thus we can assume $d(o,a_1) = d(o,a_2)$.
Let $H_1, H_2$ be the hyper planes perpendicular to the segment $a_1a_2$ containing $a_1, a_2$, resp. In this case $\operatorname{diam}\left(H_1 \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right) = \operatorname{diam}\left(H_2 \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right) = \frac{2}{\sqrt{6}}$.
By Lemma 3

$$d(a_1,a_2) + \sum_{i=1,2,3;j=1,2} \operatorname{diam}\left(R_{a_j,b_i^j} \cap \mathbf{B}^4\left(o,\sqrt{\frac{5}{12}}\right)\right) \le 1 + 2\left(\frac{1}{\sqrt{6}}\frac{3\sqrt{6}}{2}\right) =$$

$$= 1 + 3 = 4.$$

This completes the proof of the lemma. □

## 4. Proof of the theorem

**Theorem.** *The minimal radius of* 8 *congruent balls, which cover the* 4*-dimensional unit cube, is* $\sqrt{\frac{5}{12}}$.

*Proof.* By Lemma 1 we have that 8 congruent balls with radius $\sqrt{\frac{5}{12}}$ can cover the cube $C^4$.

Let us assume that the 4-dimensional cube $C^4$ can cover 8 balls with radius $\frac{1}{2} < r < \sqrt{\frac{5}{12}}$ (of course, 8 balls with radius at most $\frac{1}{2}$ can not cover $C^4$).

Since in a ball with radius at most $\sqrt{\frac{5}{12}} < \frac{\sqrt{2}}{2}$ can not lie three vertices of $C^4$ thus in every ball lie exactly two vertices of $C^4$. By Lemma 4 the sum of the length of the edges of $C^4$ in a ball with radius $\frac{1}{2} < r < \sqrt{\frac{5}{12}}$ is smaller than 4, that is, 8 congruent balls with radius smaller than $\sqrt{\frac{5}{12}}$ can not cover the cube $C^4$ (the sum of the length of the edges of $C^4$ is 32); a contradiction. This completes the proof of the Theorem. □

## References


[1] Böröczky Jr., K.: *Finite packing and covering*. Cambridge Tracts in Mathematics **154**, Cambridge University Press, 2004. Zbl 1061.52011

[2] Böröczky Jr., K.; Fábián, I.; Wintsche, G.: *Covering the crosspolytope by equal balls*. Period. Math. Hung. **53** (2006) 103–113. Zbl 1127.52027

[3] Brass, P.; Moser, W.; Pach, J.: *Research problems in discrete geometry*. Springer Verlag, New York 2005. Zbl 1086.52001

[4] Kuperberg, G.; Kuperberg, W.: *Ball packings and coverings with respect to the unit cube*. (in preparation).

[5] Rogers, C. A.: *Covering a sphere with spheres*. Mathematika **10** (1963) 157–164. Zbl 0158.19603

[6] Verger-Gaugray, J.-L.: *Covering a ball with smaller equal balls in $R^n$*. Discrete Comput. Geom. **33** (2005), 143–155. Zbl 1066.52021


Received October 19, 2007