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Projectivity and Flatness over the Endomorphism Ring of a Finitely Generated Comodule

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Abstract. Let k be a commutative ring, A a k-algebra, $\mathcal{C}$ an A-coring that is projective as a left A-module, ${}^*\mathcal{C}$ the dual ring of $\mathcal{C}$ and Λ a right $\mathcal{C}$-comodule that is finitely generated as a left ${}^*\mathcal{C}$-module. We give necessary and sufficient conditions for projectivity and flatness of a module over the endomorphism ring $\mathrm{End}^{\mathcal{C}}(\Lambda)$. If $\mathcal{C}$ contains a grouplike element, we can replace Λ with A.

1. Introduction

Throughout the paper, k is a commutative ring. Let A be a k-algebra and $\mathcal{C}$ an A-coring. The left A-linear maps $\mathcal{C} \to A$ have a ring structure. This allows one to define a dual ring ${}^*\mathcal{C}$. Let Λ be a right $\mathcal{C}$-comodule. Our starting point is the following: if $\mathcal{C}$ is finitely generated projective as a left A-module, then the category $\mathcal{M}^{\mathcal{C}}$ of right $\mathcal{C}$-comodules is isomorphic to ${}_{^*\mathcal{C}}\mathcal{M}$, the category of left ${}^*\mathcal{C}$-modules. Then [6] brings us necessary and sufficient conditions for an object of $\mathcal{M}^{\mathcal{C}}$ to be projective (resp. flat) as a module over the endomorphism ring $\mathrm{End}^{\mathcal{C}}(\Lambda)$ if Λ is finitely generated (resp. finitely presented) as a left ${}^*\mathcal{C}$-module. In this paper we will generalize these results to the category $\mathcal{M}^{\mathcal{C}}$, where $\mathcal{C}$ is projective as a left A-module (not necessarily finitely generated). If $\mathcal{C} = A$ then $\mathcal{M}^{\mathcal{C}}$ is the category of right A-modules, ${}^*\mathcal{C} = A$ and we recover the results from [6]. If $\mathcal{C}$ contains a grouplike element x, then A is a right $\mathcal{C}$-comodule that is cyclic as a left ${}^*\mathcal{C}$-module (see Lemma 2.5). So we can replace Λ with A in our main results; in this case, $\mathrm{End}^{\mathcal{C}}(\Lambda)$ is the subring of $(\mathcal{C}, x)$-coinvariants of A. Interesting examples of corings come from entwining structures. The language of corings allows us to generalize, unify and formulate more elegantly many results about generalized Hopf modules. This is due to the fact that Hopf modules and their generalizations (entwining modules, Doi-Hopf modules, relative Hopf modules, Yetter-Drinfeld modules, comodules over a coalgebra, etc. ...) are in fact comodules over a coring. Our techniques and methods are inspired from [5], [6] and [10].

2. Preliminary results

Let A be a k-algebra. An A-coring $\mathcal{C}$ is an (A, A)-bimodule together with two (A, A)-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that the usual coassociativity and counit properties hold. For more details on corings, we refer to [1], [2], [3], [4], [8] and [9]. Let $\mathcal{C}$ be an A-coring. We use the notation of Sweedler-Heyneman but we will omit the symbol $\sum$ and the parentheses on subscripts. A right $\mathcal{C}$-comodule is a right A-module M together with a right A-linear map $\rho_{M,\mathcal{C}} : M \to M \otimes_A \mathcal{C}; m \mapsto m_0 \otimes_A m_1$ such that

$$(id_M \otimes_A \epsilon_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = id_M, \quad \text{and} \quad (id_M \otimes_A \Delta_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = (\rho_{M,\mathcal{C}} \otimes_A id_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}}.$$

A morphism of right $\mathcal{C}$-comodules $f : M \to N$ is a right A-linear map such that

$$\rho_{N,\mathcal{C}} \circ f = (f \otimes_A id_M) \circ \rho_{M,\mathcal{C}}.$$

We denote the set of comodule morphisms between M and N by $\mathrm{Hom}^{\mathcal{C}}(M, N)$. Denote by $\mathcal{M}^{\mathcal{C}}$ the category formed by right $\mathcal{C}$-comodules and comodule morphisms and by $\mathcal{M}$ the category of k-modules. By [4, 18.3], the category $\mathcal{M}^{\mathcal{C}}$ has direct sums. We write ${}^*\mathcal{C} = {}_A\mathrm{Hom}({}_A\mathcal{C}, {}_AA)$, the left dual ring of $\mathcal{C}$. Then ${}^*\mathcal{C}$ is an associative ring with unit $\epsilon_{\mathcal{C}}$ (see [4, 17.8]); the multiplication is defined by

$$f\#g = f \circ (id_{\mathcal{C}} \otimes g) \circ \Delta_{\mathcal{C}} \quad \text{or equivalently} \quad f\#g(c) = f(c_1 g(c_2))$$

for all left A-linear maps f, g: $\mathcal{C} \to A$ and $c \in \mathcal{C}$; where $\Delta_{\mathcal{C}}(c) = c_1 \otimes_A c_2$. We will denote by ${}_{^*\mathcal{C}}\mathcal{M}$ the category of left ${}^*\mathcal{C}$-modules. Any right $\mathcal{C}$-comodule M is a left ${}^*\mathcal{C}$-module: the action is defined by $f.m = f(m_0)m_1$ (see [4, 19.1]). Assume that $\mathcal{C}$ is projective as a left A-module. By [4, 18.14], $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category and by [4, 19.3], it is a full subcategory of ${}_{^*\mathcal{C}}\mathcal{M}$; i.e.,

$$\mathrm{Hom}^{\mathcal{C}}(M, N) = {}_{^*\mathcal{C}}\mathrm{Hom}(M, N) \quad \text{for any} \quad M, N \in \mathcal{M}^{\mathcal{C}}.$$

As a consequence, an object of $\mathcal{M}^{\mathcal{C}}$ that is projective in ${}_{^*\mathcal{C}}\mathcal{M}$ is projective in $\mathcal{M}^{\mathcal{C}}$.

Given two right $\mathcal{C}$-comodules Λ and N, the k-module $\mathrm{Hom}^{\mathcal{C}}(\Lambda, N)$ is a module over the endomorphism ring $B = \mathrm{End}^{\mathcal{C}}(\Lambda)$ with the standard action

$$fb = f \circ b; \ \forall f \in \mathrm{Hom}^{\mathcal{C}}(\Lambda, N), b \in B.$$

This defines a functor $G = \mathrm{Hom}^{\mathcal{C}}(\Lambda, -) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_B$. The functor G has the left adjoint $F = - \otimes_B \Lambda : \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}}$, where for any $P \in \mathcal{M}_B$, $F(P) = P \otimes_B \Lambda$ is a right $\mathcal{C}$-comodule with the coaction $\rho_{P,\mathcal{C}} = id_P \otimes_B \rho_{\Lambda,\mathcal{C}}$; that is, there is a canonical isomorphism, for $N \in \mathcal{M}^{\mathcal{C}}$ and $P \in \mathcal{M}_B$

$$\mathrm{Hom}^{\mathcal{C}}(P \otimes_B \Lambda, N) \to \mathrm{Hom}_B(P, \mathrm{Hom}^{\mathcal{C}}(\Lambda, N)); \quad f \mapsto [p \mapsto f(p \otimes -)]$$

with inverse map $g \mapsto [p \otimes \lambda \mapsto g(p)(\lambda)]$.

The unit of the adjunction is given by

$$u_N : N \to \mathrm{Hom}^{\mathcal{C}}(\Lambda, N \otimes_B \Lambda), n \mapsto [\lambda \mapsto n \otimes \lambda]$$

for $N \in \mathcal{M}_B$ while the counit is the evaluation map

$$c_M : \mathrm{Hom}^{\mathcal{C}}(\Lambda, M) \otimes_B \Lambda \to M, f \otimes \lambda \mapsto f(\lambda)$$

for $M \in \mathcal{M}^{\mathcal{C}}$. The adjointness property means that we have

$$G(c_M) \circ u_{G(M)} = id_{G(M)}, \quad c_{F(N)} \circ F(u_N) = id_{F(N)};\ M \in \mathcal{M}^{\mathcal{C}}, N \in \mathcal{M}_B \quad (\star).$$

3. The main results

Let A be a k-algebra and $\mathcal{C}$ an A-coring. We keep the notations of the preceding sections. An object $\Lambda \in \mathcal{M}^{\mathcal{C}}$ is called semi-Σ-projective if the functor $\mathrm{Hom}^{\mathcal{C}}(\Lambda, -) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}$ sends an exact sequence of the form $\Lambda^{(J)} \to \Lambda^{(I)} \to N \to 0$ to an exact sequence (see [11]). Obviously, a projective object in $\mathcal{M}^{\mathcal{C}}$ is semi-Σ-projective.

Lemma 3.1. *Assume that $\mathcal{C}$ is projective as a left A-module. Let Λ be a right $\mathcal{C}$-comodule that is finitely generated as a left ${}^*\mathcal{C}$-module, and let $B = \mathrm{End}^{\mathcal{C}}(\Lambda)$. For every index set I,*

1. *the natural map $\kappa : B^{(I)} = \mathrm{Hom}^{\mathcal{C}}(\Lambda, \Lambda)^{(I)} \to \mathrm{Hom}^{\mathcal{C}}(\Lambda, \Lambda^{(I)})$ is an isomorphism;*
2. *$c_{\Lambda^{(I)}}$ is an isomorphism;*
3. *$u_{B^{(I)}}$ is an isomorphism;*
4. *if Λ is semi-Σ-projective in $\mathcal{M}^{\mathcal{C}}$, then u is a natural isomorphism; in other words, the induction functor $F = (-) \otimes_B \Lambda$ is fully faithful.*

Proof. (1) is easy.

(2) It is straightforward to check that the canonical isomorphism $B^{(I)} \otimes_B \Lambda \simeq \Lambda^{(I)}$ is nothing else than $c_{\Lambda^{(I)}} \circ (\kappa \otimes id_\Lambda)$. It follows from (1) that $\kappa \otimes id_\Lambda$ is an isomorphism. So $c_{\Lambda^{(I)}}$ is an isomorphism.

(3) Putting $M = \Lambda^{(I)}$ in $(\star)$ and using (1), we find

$$\mathrm{Hom}^{\mathcal{C}}(\Lambda, c_{\Lambda^{(I)}}) \circ u_{\mathrm{Hom}^{\mathcal{C}}(\Lambda,\Lambda^{(I)})} = id_{\mathrm{Hom}^{\mathcal{C}}(\Lambda,\Lambda^{(I)})};\ i.e.,$$

$$\mathrm{Hom}^{\mathcal{C}}(\Lambda, c_{\Lambda^{(I)}}) \circ u_{B^{(I)}} = id_{B^{(I)}}.$$

From (2), $\mathrm{Hom}^{\mathcal{C}}(\Lambda, c_{\Lambda^{(I)}})$ is an isomorphism, hence $u_{B^{(I)}}$ is an isomorphism.

(4) Take a free resolution $B^{(J)} \to B^{(I)} \to N \to 0$ of a right B-module N. Since u is natural, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{B^{(J)}} & \longrightarrow & \mathbf{B^{(I)}} & \longrightarrow & \mathbf{N} & \longrightarrow & 0 \\
\Big\downarrow u_{B^{(J)}} & & \Big\downarrow u_{B^{(I)}} & & \Big\downarrow u_N & & \\
\mathbf{GF}(\mathbf{B^{(J)}}) & \longrightarrow & \mathbf{GF}(\mathbf{B^{(I)}}) & \longrightarrow & \mathbf{GF}(\mathbf{N}) & \longrightarrow & 0
\end{array}$$

The top row is exact; the bottom row is exact, since

$$GF(B^{(I)}) = \mathrm{Hom}^{\mathcal{C}}(\Lambda, B^{(I)} \otimes_B \Lambda) = \mathrm{Hom}^{\mathcal{C}}(\Lambda, \Lambda^{(I)})$$

and Λ is semi-Σ-projective. By (3), $u_{B^{(I)}}$ and $u_{B^{(J)}}$ are isomorphisms; and it follows from the five lemma that u_N is an isomorphism. □

We can now give equivalent conditions for the projectivity and flatness of $P \in \mathcal{M}_B$.

Theorem 3.2. *Assume that $\mathcal{C}$ is projective as a left A-module. Let Λ be a right $\mathcal{C}$-comodule that is finitely generated as a left ${}^*\mathcal{C}$-module, and let $B = \mathrm{End}^{\mathcal{C}}(\Lambda)$. For $P \in \mathcal{M}_B$, we consider the following statements.*

(1) *$P \otimes_B \Lambda$ is projective in $\mathcal{M}^{\mathcal{C}}$ and u_P is injective;*
(2) *P is projective as a right B-module;*
(3) *$P \otimes_B \Lambda$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\Lambda^{(I)}$, and u_P is bijective;*
(4) *there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $\Lambda^{(I)}$, and $P \cong \mathrm{Hom}^{\mathcal{C}}(\Lambda, Q)$ in $\mathcal{M}_B$;*
(5) *$P \otimes_B \Lambda$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\Lambda^{(I)}$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If Λ is semi-Σ-projective in $\mathcal{M}^{\mathcal{C}}$, then (5) $\Rightarrow$ (3)*; if Λ is projective in $\mathcal{M}^{\mathcal{C}}$, then* (3) $\Rightarrow$ (1).

Proof. The proof uses Lemma 2.1 and is very similar to that of Theorem 2.2 in [6]. □

Over any ring A, a left module Λ is called finitely presented if there is an exact sequence $A^m \to A^n \to \Lambda \to 0$ for some natural integers m and n. Clearly a finitely presented module is finitely generated.

Theorem 3.3. *Assume that $\mathcal{C}$ is projective as a left A-module. Let Λ be a right $\mathcal{C}$-comodule that is finitely presented as a left ${}^*\mathcal{C}$-module, and let $B = \mathrm{End}^{\mathcal{C}}(\Lambda)$. For $P \in \mathcal{M}_B$, the following assertions are equivalent.*

(1) *P is flat as a right B-module;*

(2) $P \otimes_B \Lambda = \varinjlim Q_i$, *where* $Q_i \cong \Lambda^{n_i}$ *in* $\mathcal{M}^{\mathcal{C}}$ *for some positive integer* n_i, *and* u_P *is bijective;*

(3) $P \otimes_B \Lambda = \varinjlim Q_i$, *where* $Q_i \in \mathcal{M}^{\mathcal{C}}$ *is a direct summand of some* $\Lambda^{(I_i)}$ *in* $\mathcal{M}^{\mathcal{C}}$, *and* u_P *is bijective;*

(4) *there exists* $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$, *such that* $Q_i \cong \Lambda^{n_i}$ *for some positive integer* n_i *and* $\mathrm{Hom}^{\mathcal{C}}(\Lambda, Q) \cong P$ *in* $\mathcal{M}_B$*;*

(5) *there exists* $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$, *such that* Q_i *is a direct summand of some* $\Lambda^{(I_i)}$ *in* $\mathcal{M}^{\mathcal{C}}$, *and* $\mathrm{Hom}^{\mathcal{C}}(\Lambda, Q) \cong P$ *in* $\mathcal{M}_B$.

If Λ *is semi-*Σ*-projective in* $\mathcal{M}^{\mathcal{C}}$, *these conditions are also equivalent to conditions* (2) *and* (3), *without the assumption that* u_P *is bijective.*

Proof. The proof uses Lemma 2.1 and is very similar to that of Theorem 2.3 in [6]. $\square$

Remark 3.4. Assume that $\mathcal{C}$ is projective and finitely generated as a left A-module. By [4, 19.6], $M^{co\mathcal{C}} = {}_{^*\mathcal{C}}\mathcal{M}$. For every left ${}^*\mathcal{C}$-module Λ, we have $\Lambda^{co\mathcal{C}} = {}^{*\mathcal{C}}\Lambda$, the k-submodule of ${}^*\mathcal{C}$-invariants of Λ. So $\mathrm{End}^{\mathcal{C}}(\Lambda) = {}_{^*\mathcal{C}}\mathrm{End}(\Lambda)$, the endomorphism ring of the left ${}^*\mathcal{C}$-module Λ. Theorems 2.2 and 2.3 give necessary and sufficient conditions for projectivity and flatness of a module over ${}_{^*\mathcal{C}}\mathrm{End}(\Lambda)$. However these results are not new (see [6, Theorems 2.2 and 2.3]).

4. The coring contains a grouplike element

A grouplike element of $\mathcal{C}$ is an element $x \in \mathcal{C}$ such that $\Delta_{\mathcal{C}}(x) = x \otimes_A x$ and $\epsilon_{\mathcal{C}}(x) = 1_A$. Assume that $\mathcal{C}$ contains a grouplike element x. By [4, 28.2], A is an object of $\mathcal{M}^{\mathcal{C}}$: the $\mathcal{C}$-coaction is defined by $\rho_{A,\mathcal{C}}(a) = xa$; $\forall a \in A$.

For any right $\mathcal{C}$-comodule M, we call $M^{co\mathcal{C},x} = \{m \in M, \quad \rho_{M,\mathcal{C}}(m) = m \otimes_A x\}$ the k-submodule of $(\mathcal{C}, x)$-coinvariants of M. Clearly, $A^{co\mathcal{C},x} = \{a \in A, \quad ax = xa\}$ is a subring of A called the subring of $(\mathcal{C}, x)$-coinvariants.

We have $\mathrm{Hom}^{\mathcal{C}}(A, M) = M^{co\mathcal{C},x}$ and $\mathrm{End}^{\mathcal{C}}(A) = A^{co\mathcal{C},x}$. Set $B = A^{co\mathcal{C},x}$. Then replacing Λ with A, the functors F and G become

$$F = (-) \otimes_B A : \mathcal{M}_B \to \mathcal{M}^{\mathcal{C}} \quad \text{and} \quad G = (-)^{co\mathcal{C},x} : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_B.$$

The unit of the adjunction pair (F, G) is given by

$$u_N : N \to (N \otimes_B A)^{co\mathcal{C},x}; n \mapsto (n \otimes 1)$$

for every right B-module N while the counit is

$$c_M : M^{co\mathcal{C},x} \otimes_B A \to M; m \otimes a \mapsto ma$$

for every right $\mathcal{C}$-comodule M.

Lemma 4.1. *Assume that* $\mathcal{C}$ *contains a grouplike element* x. *Then* A *is a cyclic left* ${}^*\mathcal{C}$*-module under the action defined by* $f.a = f(xa)$ *for all* $f \in {}^*\mathcal{C}$ *and* $a \in A$.

Proof. Note that $\epsilon_{\mathcal{C}}.a = \epsilon_{\mathcal{C}}(xa) = \epsilon_{\mathcal{C}}(x)a = 1_A a = a$ and $\Delta_{\mathcal{C}}(xa) = x \otimes_A xa$. Now for $f, g \in {}^*\mathcal{C}$, we have $(f\#g).a = (f\#g)(xa) = f(xg(xa)) = f(x(g.a)) = f.(g.a)$. □

From Theorems 2.2 and 2.3 we easily obtain the following results.

Theorem 4.2. *Assume that $\mathcal{C}$ is projective as a left A-module and contains a grouplike element x. Set $B = A^{co\mathcal{C},x}$. For $P \in \mathcal{M}_B$, we consider the following statements.*

1. *$P \otimes_B A$ is projective in $\mathcal{M}^{\mathcal{C}}$ and u_P is injective;*
2. *P is projective as a right B-module;*
3. *$P \otimes_B \Lambda$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $A^{(I)}$, and u_P is bijective;*
4. *there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $A^{(I)}$, and $P \cong \mathrm{Hom}^{\mathcal{C}}(A, Q)$ in $\mathcal{M}_B$;*
5. *$P \otimes_B A$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $A^{(I)}$.*

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5)$.

If A is semi-Σ-projective in $\mathcal{M}^{\mathcal{C}}$, then $(5) \Rightarrow (3)$; if A is projective in $\mathcal{M}^{\mathcal{C}}$, then $(3) \Rightarrow (1)$.

Theorem 4.3. *Assume that $\mathcal{C}$ is projective as a left A-module and contains a grouplike element x, and that A is finitely presented as a left ${}^*\mathcal{C}$-module. Set $B = A^{co\mathcal{C},x}$. For $P \in \mathcal{M}_B$, the following assertions are equivalent.*

1. *P is flat as a right B-module;*
2. *$P \otimes_B A = \varinjlim Q_i$, where $Q_i \cong A^{n_i}$ in $\mathcal{M}^{\mathcal{C}}$ for some positive integer n_i, and u_P is bijective;*
3. *$P \otimes_B A = \varinjlim Q_i$, where $Q_i \in \mathcal{M}^{\mathcal{C}}$ is a direct summand of some $A^{(I_i)}$ in $\mathcal{M}^{\mathcal{C}}$, and u_P is bijective;*
4. *there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$, such that $Q_i \cong A^{n_i}$ for some positive integer n_i and $\mathrm{Hom}^{\mathcal{C}}(A, Q) \cong P$ in $\mathcal{M}_B$;*
5. *there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$, such that Q_i is a direct summand of some $A^{(I_i)}$ in $\mathcal{M}^{\mathcal{C}}$, and $\mathrm{Hom}^{\mathcal{C}}(A, Q) \cong P$ in $\mathcal{M}_B$.*

If A is semi-Σ-projective in $\mathcal{M}^{\mathcal{C}}$, these conditions are also equivalent to conditions (2) and (3), without the assumption that u_P is bijective.

The following lemma will enable us to improve Theorems 3.2 and 3.3 if P is a right A-module (for example, if P is a right $\mathcal{C}$-comodule).

Lemma 4.4. *Assume that $\mathcal{C}$ contains a grouplike element x. Let N be a right A-module. Then u_N is an injection.*

Proof. By [4, 18.9(1)], $\mathcal{C}$ is a right $\mathcal{C}$-comodule with the coaction $\Delta_{\mathcal{C}}$; and $N \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-comodule with the coaction $id_N \otimes \Delta_{\mathcal{C}}$. By [4, 18.10(1)], the k-linear map

$$\varphi : (N \otimes_A \mathcal{C})^{co\mathcal{C},x} \to N; n \otimes c \mapsto n\epsilon_{\mathcal{C}}(c)$$

is an isomorphism; i.e., $G(W) \cong N$, where $W = N \otimes_A \mathcal{C}$. Replace M with W in $(\star)$, then we get $G(c_W) \circ u_N = id_N$. So u_N is an injection. □

Remark 4.5. By Lemma 3.4, u_P is an injection for every $P \in \mathcal{M}_A$. So in the particular case where P is a right A-module, the injection assumption in Theorem 3.2 is superfluous. So we can replace "bijection" with "surjection" in Theorems 3.2 and 3.3. In other words, we get necessary and sufficient conditions for a right A-module P to be projective (resp. flat) as a right B-module.

5. The main example

Interesting examples of corings come from entwining structures. A right-right entwining structure over k is a triple (A, C, ψ) consisting of a k-algebra A, a coalgebra C over k (with comultiplication Δ_C and counit ϵ_C) and a k-module map $\psi : C \otimes A \to A \otimes C$ satisfying the following conditions (see [4, 32.1]):

$$\psi \circ (id_C \otimes \mu) = (\mu \otimes id_C) \circ (id_A \otimes \psi) \circ (\psi \otimes id_A)$$

$$(id_A \otimes \Delta_C) \circ \psi = (\psi \otimes id_C) \circ (id_C \otimes \psi) \circ (\Delta_C \otimes id_A)$$

$$(id_A \otimes \epsilon_C) \circ \psi = \epsilon_C \otimes id_A$$

$$\psi \circ (id_C \otimes \iota) = \iota \otimes id_C,$$

where μ is the multiplication of A and ι is its unit.

If we set $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$, these relations are respectively equivalent to

$$(aa')_\alpha \otimes c^\alpha = a_\alpha a'_\beta \otimes c^{\alpha\beta}$$

$$a_\alpha \otimes \Delta_C(c^\alpha) = a_{\alpha\beta} \otimes {c_1}^\beta \otimes {c_2}^\alpha$$

$$a_\alpha \epsilon_C(c^\alpha) = \epsilon_C(c)a$$

$$1_\alpha \otimes c^\alpha = 1 \otimes c.$$

The map ψ is called an entwining map, and A and C are said to be entwined by ψ. By [4, 32.1], $\mathcal{C} = A \otimes C$ is an A-coring with A-multiplications $a'(a \otimes c)a'' = a'a\psi(c \otimes a'')$, coproduct

$$\Delta_{\mathcal{C}} : A \otimes_k C \to A \otimes_k C \otimes_A A \otimes_k C \cong A \otimes_k C \otimes_k C; \quad a \otimes c \mapsto a \otimes \Delta_C(c)$$

and counit $\epsilon_{\mathcal{C}}(a \otimes c) = a\epsilon_C(c)$. The category $\mathcal{M}_A^C(\psi)$ of right-right (A, C, ψ)-entwined modules (see [4, 32.4] for the definition) is isomorphic to $\mathcal{M}^{\mathcal{C}}$. If k is a field, $\mathcal{C}$ is projective as a left A-module. An element $c \in C$ is called a grouplike element if $\Delta_C(c) = c \otimes c$ and $\epsilon_C(c) = 1$. If C contains a grouplike element c, then the coring $A \otimes_k C$ contains $x = 1_A \otimes c$ as a grouplike element. For more information about entwining structures we refer to [1], [2], [4] and [8].

Many entwining structures come from Doi-Hopf data.

Let k be a field, H a Hopf k-algebra with a bijective antipode S_H and C a coalgebra over k. We say that C is a right H-module coalgebra if C is a right H-module, and Δ_C and ϵ_C are right H-linear. A right H-comodule A that is also an algebra is called a right H-comodule algebra, if the unit and the multiplication are right H-colinear.

Definition 5.1. *Let A be a right H-comodule algebra and C a right H-module coalgebra. According to* [7], *we call the triple (H, A, C) a right-right Doi-Hopf datum.*

By [4, 33.4], any right-right Doi-Hopf datum (H, A, C) gives rise to an entwining structure (A, C, ψ): the map ψ is defined by $\psi(c \otimes a) = a_{[0]} \otimes (c \leftharpoondown a_{[1]})$, where $\rho_{A,H}(a) = a_{[0]} \otimes a_{[1]}$ and $\leftharpoondown$ denotes the H-action on C. By [4, 33.4], the category $\mathcal{M}^{\mathcal{C}}$ is isomorphic to $\mathcal{M}(H)^C_A$, the category of right-right (H, A, C)-Doi-Hopf modules.

Let us give some examples of Doi-Hopf data. In the examples, H is a Hopf algebra with a bijective antipode.

Examples 5.2. (i) Let I be a Hopf ideal of H. For any right H-comodule algebra A, the triple $(H, A, H/I)$, where the H-action on H/I is given by $(h' + I).h = (h'h) + I$ is a right-right Doi-Hopf datum. The coring $\mathcal{C} = A \otimes (H/I)$ contains $1_A \otimes (1_H + I)$ as a grouplike element. If $I = 0$, the category of right-right (H, A, H)-Doi-Hopf modules $\mathcal{M}(H)^H_A$ is nothing else than the category $\mathcal{M}^H_A$ of right-right relative (A, H)-Hopf modules. The coring $\mathcal{C} = A \otimes H$ contains $x = 1_A \otimes 1_H$ as a grouplike element. For every right-right relative (A, H)-Hopf module Λ, $\Lambda^{co\mathcal{C},x}$ is the vector subspace Λ^{coH} of H-coinvariants of Λ; and $A^{co\mathcal{C},x}$ is the subring A^{coH} of H-coinvariants of A. In the particular case $A = H$, we obtain the category of Hopf modules. Note that our results on projectivity are not new when $\Lambda = A$ and $I = 0$ (see [5, Theorem 2.1]).

(ii) Let A be an (H, H)-bicomodule algebra. We know from [8, page 182] that the triple $(H^{op} \otimes H, A, H)$ is a right-right Doi-Hopf datum. The coaction of $H^{op} \otimes H$ on A is given by

$$\rho_{H^{op} \otimes H, H}(a) = a_{[0]} \otimes S_H(a_{[-1]}) \otimes a_{[1]},$$

and the action of $H^{op} \otimes H$ on H is given by

$$l \leftharpoondown (h \otimes h') = hlh', \quad \text{for all} \quad h, h', l \in H.$$

The coring $\mathcal{C} = A \otimes H$ contains $x = 1_A \otimes 1_H$ as a grouplike element. The corresponding category of right-right $(H^{op} \otimes H, A, H)$-Doi-Hopf modules $\mathcal{M}(H^{op} \otimes H)^H_H$ is nothing else than the category $\mathcal{YD}(H, H)^H_A$ of right-right Yetter-Drinfeld modules. For every right-right Yetter-Drinfeld module Λ, $\Lambda^{co\mathcal{C},x}$ is the vector subspace Λ^{coH} of H-coinvariants of Λ; and $A^{co\mathcal{C},x}$ is the subring A^{coH} of H-coinvariants of A. In the particular case $A = H$, we obtain the category of classical Yetter-Drinfeld modules, also named crossed modules or quantum Yang-Baxter modules.

(iii) Let us consider the right-right Doi-Hopf datum (H, A, H), where the coaction of H on A is trivial, and the H-action on H is given by right multiplication. The corresponding category of right-right (H, A, H)-Doi-Hopf modules $\mathcal{M}(H)^H_A$ is nothing else than the category $\mathcal{L}^H_A$ of right-right generalized Long dimodules [8, page 193]. The coring $\mathcal{C} = A \otimes H$ contains $x = 1_A \otimes 1_H$ as a grouplike element, and for every Long dimodule Λ, $\Lambda^{co\mathcal{C},x}$ is the vector subspace Λ^{coH} of H-coinvariants of Λ; and $A^{co\mathcal{C},x}$ is the subring A^{coH} of H-coinvariants of A. In the particular case $A = H$, we obtain the category of classical Long dimodules.

(iv) Let $H = k$ be the trivial Hopf algebra. For any algebra A and any coalgebra C, the triple (k, A, C) is a right-right Doi-Hopf datum. If $A = k$ (see [8, page 49]), the category of right-right (k, k, C)-Doi-Hopf modules $\mathcal{M}(k)_k^C$ is nothing else than the category $\mathcal{M}^C$ of right C-comodules.

(v) For any right H-module coalgebra C, the triple (H, H, C) is a right-right Doi-Hopf datum.

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