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Tame Homomorphisms of Polytopal Rings

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Abstract. The object of this paper is the tameness conjecture introduced in [3], which describes an arbitrary graded k-algebra homomorphism of polytopal rings. We will give further evidence of this conjecture by showing supporting results concerning joins, multiples and products of polytopes.

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1. Introduction

This paper studies the category of polytopal algebras over a field k, denoted $\mathrm{Pol}(k)$. We investigate the concept of *tameness*, as introduced in [3], where it is conjectured that every graded k-algebra homomorphism is tame. In short, such a homomorphism is called tame if it can be obtained by a composition of some standard homomorphisms, defined in [3].

In this work we will show the following: If graded homomorphisms between two polytopal rings are tame, then graded homomorphisms from rings obtained by taking multiples, joins, and products of the underlying polytopes are also tame. Thus, we extend the class of polytopal rings on which graded homomorphisms are tame, giving further evidence in support of the mentioned conjecture.

The objects of $\mathrm{Pol}(k)$, the *polytopal (monoid) rings*, are defined as follows. Let P be a convex lattice polytope in $\mathbb{R}^n$. Let $\mathrm{L}(P)$ denote the lattice points in

P, i.e. $\mathrm{L}(P) = P \cap \mathbb{Z}^n$, and let $\mathrm{S}(P)$ be the additive monoid of $\mathbb{Z}^{n+1}$ generated by $\{(x, 1) \mid x \in \mathrm{L}(P)\}$. A lattice point in $\mathrm{S}(P)$ can be represented as a monomial in $n+1$ variables by identifying its coordinate vector with the monomial's exponent vector. The degree of a monomial is in this case the last component of its exponent vector. The polytopal ring $k[P]$ is the monoid ring of $\mathrm{S}(P)$ with coefficients in k. $k[P]$ is a graded ring generated by its degree 1 monomials. These monomials correspond bijectively to the lattice points in P. The generators' relations are the binomial relations representing the affine dependencies in $\mathrm{L}(P)$.

A homomorphism in $\mathrm{Pol}(k)$ is a homomorphism of two polytopal rings as k-algebras preserving the grading. Such a homomorphism also has to preserve the binomial relationships among the generators. Therefore, $\mathrm{Hom}(k[P], k[Q])$, which is the set of all graded homomorphism between the two polytopal rings, is the zero set of a system of polynomials. Hence $\mathrm{Hom}(k[P], k[Q])$ gives rise to a Zariski closed set in the space of matrices $M_{mn}(k)$, where $m = \#L(Q)$ and $n = \#L(P)$, called the *Hom-variety*. We are aiming to describe the structure of the Hom-variety. The tameness conjecture is a geometric description of this set.

A tame homomorphism is a graded k-algebra homomorphism which can be obtained by a composition of four standard homomorphisms: polytope changes, homothetic blow-ups, Minkowski sums, and free extensions.

For the following definitions, assume $f : k[P] \to k[Q]$ is a graded homomorphism and $P, Q \subset \mathbb{R}^n$ are lattice polytopes.

First, suppose P' and Q' are lattice polytopes such that $P' \subset P$, $Q' \supset Q$, and $f(k[P']) \subset k[Q']$. This gives rise to a new homomorphism $f' : k[P'] \to k[Q']$ obtained from f in a natural way. Also, two polytopes $\tilde{P}$ and $\tilde{Q}$ that are isomorphic to P respectively Q as lattice polytopes, result in the homomorphism $\tilde{f} : k[\tilde{P}] \to k[\tilde{Q}]$, induced by f. The homomorphisms obtained these ways are called *polytope changes*.

Second, consider the normalization of $S(P)$, defined by

$$\overline{S(P)} = \{x \in \mathrm{gp}(S(P)) \mid x^m \in S(P) \text{ for some } m \in \mathbb{N}\}$$

and the normalization of $S(Q)$, defined similarly. The set $\mathrm{gp}(S(P))$ denotes the group of differences of $S(P)$. $S(P)$ is normal if and only if $S(P) = \overline{S(P)}$. It is well known that $k[\overline{P}] := k[\overline{S(P)}] = \overline{k[P]}$. If there are no monomials in the kernel of f, i.e. $\ker(f) \cap S(P) = \emptyset$, then f extends uniquely to the homomorphism $\overline{f} : k[\overline{P}] \to k[\overline{Q}]$ defined by

$$\overline{f}(x) = \frac{f(y)}{f(z)} \quad \text{where } x = \frac{y}{z},\ x \in \overline{S(P)}, \text{ and } y, z \in S(P).$$

In fact, $f(y)/f(z)$ belongs to $k[\overline{Q}]$ since $x \in \overline{S(P)}$ implies $x^c \in S(P)$, for some natural number c, which in turn implies $f(x^c) = (f(y)/f(z))^c \in k[Q]$. Let $k[\overline{P}]_c$ be the subring of $k[\overline{P}]$ generated by the homogeneous components of degree c in $\overline{S(P)}$ (and similarly for $k[\overline{Q}]_c$). Note that $k[\overline{P}]_c \simeq k[cP]$ and $k[\overline{Q}]_c \simeq k[cQ]$ in a natural way. Since f is graded, $\overline{f}$ restricts to the graded homomorphism $f^{(c)} : k[cP] \to k[cQ]$, which we call the *homothetic blow-up* of f.

Third, consider having two graded homomorphisms $f, g : k[P] \to k[Q]$. Let $\mathrm{N}(f(x))$ denote the Newton polytope of $f(x)$, i.e. the convex hull of the support monomials in $f(x)$ (and similarly for $\mathrm{N}(g(x))$). Assume f and g satisfy

$$\mathrm{N}(f(x)) + \mathrm{N}(g(x)) \subset Q \text{ for all } x \in L(P)$$

where $+$ denotes the Minkowski sum in $\mathbb{R}^n$. We have $z^{-1}f(x)g(x) \in k[Q]$ where $z = (0, 0, \ldots, 1)$, and thus

$$f \star g : k[P] \to k[Q] \text{ such that } f \star g(x) = z^{-1}f(x)g(x)$$

for all $x \in L(P)$ defines a new graded homomorphism, called the *Minkowski sum* of f and g.

Lastly, assume P is a pyramid with base P_0 and vertex v such that $L(P) = \{v\} \cup L(P_0)$ and that we are given a homomorphism $f_0 : k[P_0] \to k[Q]$. This means $k[P]$ is a polynomial extension of $k[P_0]$. Thus, f_0 extends to a homomorphism $f : k[P] \to k[Q]$ by letting $f(v) = q$ for any $q \in k[Q]$ and $f(x) = f_0(x)$ for all $x \in \mathrm{L}(P_0)$. The homomorphism f is called a *free extension* of f_0.

The tameness conjecture, as found in [3], states the following.

Conjecture 1. (W. Bruns, J. Gubeladze) *Every homomorphism in* $\mathrm{Pol}(k)$ *is obtained by a sequence of taking free extensions, Minkowski sums, homothetic blow-ups, polytope changes and compositions, starting from the identity mapping* $k \to k$*. Moreover, there are normal forms of such sequences.*

Certain subvarieties of the Hom-variety have already been described.

In [1] the subvariety corresponding to automorphisms has been described completely, showing that every automorphism is a composition of some basic automorphisms. This can be viewed as a polytopal generalization of the linear algebra fact that every invertible matrix can be written as the product of elementary, permutation, and diagonal matrices. The result is stronger than the notion of tameness, thus implying that automorphisms are tame.

In [2] the variety corresponding to codimension-1 retractions of polygons has been described. It is conjectured that the result generalizes to arbitrary dimensions of the polytope. This result is also stronger than tameness; hence, such retractions are tame.

In showing that these results imply tameness of the corresponding morphisms, some other classes of homomorphisms are shown to be tame in [3]. Among these classes, the following two will be used. First, homomorphisms respecting monomial structures are tame. Second, homomorphisms from $k[c\Delta_n]$ (where Δ_n is the n-simplex, c, $n \in \mathbb{N}$) are tame. We will also use the notion of face retractions, which are idempotent endomorphisms of $k[P]$ defined by

$$\pi_F(x) = \begin{cases} x, & \text{if } x \in F \\ 0, & \text{if } x \notin F \end{cases}$$

for all $x \in \mathrm{L}(P)$, where F is a face of P (note that face retractions are tame since they respect the monomial structures).

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2. Main results

The results of this work concern joins, multiples, and Segre products.

The join of two lattice polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ of dimension n respectively m is a subset of $\mathbb{R}^{n+m+1}$. Consider the embeddings ι_1 and ι_2 of $\mathbb{R}^n$ and $\mathbb{R}^m$ into $\mathbb{R}^{n+m+1}$ defined as

$$\iota_1 : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0) \qquad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n$$

$$\iota_2 : (y_1, \ldots, y_m) \mapsto (0, \ldots, 0, y_1, \ldots, y_m, 1) \quad \text{for all } (y_1, \ldots, y_m) \in \mathbb{R}^m .$$

The join of P and Q is defined as the convex hull of the image of P under ι_1 and the image of Q under ι_2, i.e. $\text{join}(P, Q) = \text{conv}(\iota_1(P), \iota_2(Q))$.

We denote the c^{th} multiple of P by cP, i.e. $cP = \{cx \mid x \in P\}$. Since the lattice points in $S(P)$ sitting on height c may correspond to a proper subset of the lattice points in cP, $k[cP]$ is in general an overring of the c^{th} Veronese subring of $k[P]$ (the ring generated by the homogeneous components of degree c in $S(P)$). The two rings coincide when P is normal.

The Segre product of $k[P]$ and $k[Q]$ is $k[P \times Q]$ where $P \times Q = \{(x, y) | \, x \in P, y \in Q\}$.

Theorem 1. *If every graded homomorphism from $k[P]$ respectively $k[Q]$ is tame, then*

(a) *every graded homomorphism from $k[\text{join}(P, Q)]$ is tame,*

(b) *every graded homomorphism from $k[cP]$, where $c \in \mathbb{N}$, is tame,*

(c) *every graded homomorphism from $k[P \times Q]$ is tame.*

To prove Theorem 1, the following lemma will be used. The lemma is stated and proved in the case of Segre products, but similar arguments hold true for joins and multiples.

Lemma 1. *Let f be a graded homomorphism from $k[P \times Q]$ and assume the hypothesis of Theorem 1. To show that f is tame we can without loss of generality assume that $\ker(f) \cap S(P \times Q) = \emptyset$.*

Proof. Let $f : k[P \times Q] \to k[R]$ be a graded homomorphism (where R is some lattice polytope). First we observe that the hypothesis of Theorem 1 descends to the faces of the polytopes. That is, if every graded homomorphism from $k[P]$ is tame and F is a face of P, then any graded homomorphism from $k[F]$ is tame. Note that such a homomorphism $f' : k[F] \to k[R]$ is the composition of the homomorphisms

$$k[F] \xrightarrow{x \mapsto x} k[P] \xrightarrow{x \mapsto \pi_F(x)} k[F] \xrightarrow{x \mapsto f'(x)} k[R]$$

The first map is tame since it maps monomials to monomials. The composition of the last two maps is tame since it is a homomorphism from $k[P]$, tame by assumption.

Now, assume there exist a monomial $m \in \ker(f)$. Since $\ker(f)$ is a prime ideal, the ideal $I = (\ker(f) \cap S(P \times Q)) \subset k[P \times Q]$ is a monomial prime ideal containing m. However, monomial prime ideals are exactly the kernels of face retractions [4]. Thus, f is a composition of a face retraction and a map $g : k[F'] \to k[R]$ where F' is a face of $P \times Q$. Face retractions are known to be tame. Hence, since a composition of tame homomorphisms is tame, tameness of g will imply tameness of f. Note that there are no monomials in the kernel of g. Also, a face of $P \times Q$ is of the form $P' \times Q'$ where P' is a face of P and Q' a face of Q [6].

Thus, Theorem 1(c) is proved by establishing tameness of every graded homomorphism g from $k[P' \times Q']$ such that $\ker(g) \cap S(P' \times Q') = \emptyset$, given that homomorphisms from $k[P']$ and $k[Q']$ are tame. □

From here on, it is assumed that there are no monomials in the kernel of the homomorphisms in question.

3. Translation into discrete objects

A graded homomorphism in $\mathrm{Hom}(k[P], k[Q])$ can be viewed as an affine map from $\mathrm{L}(P)$ to $\mathbb{Z}_+^l$, and vice versa. In this case, “affine” refers to a map admitting an affine extension to the corresponding affine hulls. By viewing a graded homomorphism as an affine map, polynomials in a polytopal ring translate into discrete objects, which will aid in proving Theorem 1.

Suppose P and Q are lattice polytopes such that $\mathrm{L}(P) \subset \mathbb{Z}_+^d$ and $\mathrm{L}(Q) \subset \mathbb{Z}_+^e$ (this can always be assumed by a polytope change). Let $f : k[P] \to k[Q]$ be a graded homomorphism. $\mathrm{L}(Q)$ is a subset of $\{X_1^{a_1} \cdots X_e^{a_e} Z | a_i \in \mathbb{Z}_+\}$ and thus the polynomials $\varphi_x = f(x)Z^{-1}$ belong to $k[X_1, \dots, X_e]$ for all $x \in L(P)$. Clearly, since f respects the binomial relations in $L(P)$, so does the map $x \mapsto \varphi_x$.

The polynomial ring $k[X_1, \dots, X_e]$ is a unique factorization domain. Therefore $(k[X_1, \dots, X_e] \setminus \{0\})/k^*$ is a free commutative monoid. Hence, there is a subset $\mathcal{P} \subset (k[X_1, \dots, X_e] \setminus \{0\})/k^*$ of irreducibles such that for each class $[\varphi_x] \in (k[X_1, \dots, X_e] \setminus \{0\})/k^*$ we have $[\varphi_x] = P_1^{a_1} \cdots P_l^{a_l}$ for some class $P_i \in \mathcal{P}$, $l \in \mathbb{N}$, and uniquely determined $a_i \geq 0$. Since only finitely many irreducibles are needed to represent any $[\varphi_x]$ there exists an $l \in \mathbb{N}$ such that $[\varphi_x] \in \{P_1^{a_1} \cdots P_l^{a_l} | P_i \in \mathcal{P}, a_i \in \mathbb{Z}_+\}$ for all $x \in \mathrm{L}(P)$. Note that $\{P_1^{a_1} \cdots P_l^{a_l} | P_i \in \mathcal{P}, a_i \in \mathbb{Z}_+\}$ is isomorphic to $\mathbb{Z}_+^l$. Each polynomial $f(x)$ with $x \in L(P)$ therefore gives rise to an integral vector by the correspondence

$$f(x) \mapsto [\varphi_x] = P_1^{a_1} \cdots P_l^{a_l} \mapsto (a_1, \dots, a_l).$$

Consequently, f gives rise to an affine map from $\mathrm{L}(P)$ to $\mathbb{Z}_+^l$ which respects the binomial relations in $\mathrm{L}(P)$.

Conversely, suppose α is an affine map $\mathrm{L}(P) \to \mathbb{Z}_+^l$. For each $x \in \mathrm{L}(P)$, $\alpha(x) \in \mathbb{Z}_+^l$ gives rise to a polynomial p_x in $k[X_1, \ldots, X_e]$ under the correspondence

$$\alpha(x) = (a_1, \ldots, a_l) \mapsto P_1^{a_1} \cdots P_l^{a_l} = p_x.$$

Therefore, for each $x \in L(P)$, $\varphi_x = t_x p_x$ for some nonzero scalar t_x. Note that the scalars t_x are clearly subject to the same binomial relations as the lattice points in P. Letting $p'_x = t_x p_x$ for all $x \in L(P)$, results in $f(x) = p'_x Z$, thus recovering the homomorphism f from α.

Also, α can be extended to an affine integral map $P \to \mathbb{R}_+^l$. If $\mathrm{L}(P) = \{x_0, \ldots, x_n\}$ then $P = \mathrm{conv}(x_0, \ldots, x_n)$ and any $x \in P$ can be represented as $x = c_0 x_0 + \cdots + c_n x_n$ for some nonnegative real numbers c_i satisfying $\sum_{i=0}^n c_i = 1$. Hence, if $f(x_i)$ corresponds to $a_i \in \mathbb{Z}_+^l$ for each $i \in \{0, 1, \ldots, n\}$, $f(x)$ corresponds to $c_0 a_0 + \cdots + c_n a_n \in \mathbb{R}_+^l$. The correspondence is well-defined since f respects the binomial relations in $\mathrm{L}(P)$.

4. Joins

The proof of Theorem 1(a) is mainly derived from well-known properties of joins. It is known that there are no relations between the lattice points coming from P and the lattice points coming from Q, in $\mathrm{join}(P, Q)$. Any two polytopal rings $k[P]$ and $k[Q]$ therefore satisfy $k[\mathrm{join}(P, Q)] \simeq k[P] \otimes k[Q]$. Also, by definition, there are no new lattice points in $\mathrm{join}(P, Q)$ since if $x \in \mathrm{L}(\mathrm{join}(P, Q))$ then $x \in \mathrm{L}(\iota_1(P)) \simeq \mathrm{L}(P)$ or $x \in \iota_2(\mathrm{L}(Q)) \simeq \mathrm{L}(Q)$. Hence, two graded homomorphisms $f : k[P] \to k[L]$ and $g : k[Q] \to k[L]$ define a new graded homomorphism $F : k[\mathrm{join}(P, Q)] \to k[L]$ by letting

$$F(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{L}(P) \\ g(x) & \text{if } x \in \mathrm{L}(Q) \end{cases} \quad \text{for all } x \in \mathrm{L}(\mathrm{join}(P, Q)).$$

Conversely every homomorphism $F : k[\mathrm{join}(P, Q)] \to k[L]$ is necessarily of this form.

Theorem 1(a) is proved once it is shown that this "pasting" of the two tame homomorphisms is also tame.

Proof of Theorem 1(a). Suppose P and Q are lattice polytopes.
Let $F : k[\mathrm{join}(P, Q)] \to k[R]$ be a graded homomorphism where R is a lattice polytope in $\mathbb{R}^d$. Assume $Z = (0, \ldots, 0, 1) \in \mathrm{L}(R)$, after a polytope change. Further assume $\mathrm{L}(P) = \{x_0, \ldots, x_n\}$ and $\mathrm{L}(Q) = \{y_0, \ldots, y_m\}$. There exist homomorphisms $f : k[P] \to k[R]$ and $g : k[Q] \to k[R]$ such that $F(x_i) = f(x_i)$ for $i \in \{0, 1, \ldots, n]$ and $F(y_j) = g(y_j)$ for $j \in \{0, 1, \ldots, m]$.

Consider the homomorphisms

$$a_P : k[\mathrm{join}(P, Q)] \to k[R] \text{ such that } a_P(x_i) = f(x_i),\ a_P(y_j) = Z$$

$$a_Q : k[\mathrm{join}(P, Q)] \to k[R] \text{ such that } a_Q(x_i) = Z,\ a_Q(y_j) = g(y_j)$$

for $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, m\}$. Note that a_P factors through the homomorphism $k[\mathrm{join}(P, Q)] \to k[\mathrm{join}(P, y_0)]$ such that $x_i \mapsto x_i$ and $y_j \mapsto y_0$ and the homomorphism $k[\mathrm{join}(P, y_0)] \to k[R]$ such that $x_i \mapsto f(x_i)$ and $y_0 \mapsto Z$. The first map is tame since it maps monomials to monomials and the second is tame since it is a free extension of f which is tame by assumption. Thus a_P is tame. Similarly a_Q factors through the homomorphisms $k[\mathrm{join}(P, Q)] \to k[\mathrm{join}(x_0, Q)]$ such that $y_j \mapsto y_j$ and $x_i \mapsto x_0$ and $k[\mathrm{join}(x_0, Q)] \to k[R]$ such that $y_j \mapsto f(y_j)$ and $x_0 \mapsto Z$. These maps are tame for the same reasons and hence a_Q is also a tame homomorphism.

F is obtained by the Minkowski sum of a_P and a_Q since for all $v \in \mathrm{L}(\mathrm{join}(P, Q))$

$$(a_P \star a_Q)(v) = a_P(v) a_Q(v) Z^{-1} = \begin{cases} f(v), & \text{if } v \in \mathrm{L}(P) \\ g(v), & \text{if } v \in \mathrm{L}(Q) \end{cases}$$

which is precisely F. F is tame as desired. □

5. Multiples

Graded homomorphisms from $k[c\Delta_n]$, where $c\Delta_n$ is the c^{th} multiple of the n-simplex, are tame, as shown in [3]. A similar argument is used to prove its generalization concerning the c^{th} multiple of a general lattice polytope, stated as Theorem 1(b). The following lemma is used.

Lemma 2. *Suppose P is a lattice polytope and $\alpha : \mathrm{L}(cP) \to \mathbb{Z}^l_+$ is an affine map. Then there exists a vector $v \in \mathbb{Z}^l_+$ and an affine map $\beta : \mathrm{L}(P) \to \mathbb{Z}^l_+$ such that $\alpha(cx) = v + c\beta(x)$ for all $x \in \mathrm{L}(P)$.*

Proof of Lemma 2. Assume $\mathrm{L}(P) = \{x_0, \dots, x_n\}$. Suppose $\alpha(cx_i) = (a_{i1}, \dots, a_{il})$ for $i \in \{0, 1, \dots, n\}$. Let $v = (\min\{a_{i1}\}_{i=0}^n, \dots, \min\{a_{il}\}_{i=0}^n) \in \mathbb{Z}^l_+$. We will show that $\alpha(cx_i) - v$ are c^{th} multiples of integral vectors for all i.

Let $i \in \{0, 1, \dots, n\}$. Note that for all $k \in \{1, 2, \dots, l\}$ the k^{th} component of either $\alpha(cx_i) - v$ or $\alpha(cx_j) - v$, for some $j \neq i$, is 0. If the k^{th} component of $\alpha(cx_i) - v$ is 0 we are done.

Assume the k^{th} component of $\alpha(cx_j) - v$ is 0 for some $j \neq i$. Since α is an affine map, $\alpha(cx_i) - \alpha(cx_j) = c(\alpha(x_i) - \alpha(x_j))$ is a c^{th} multiple of an integral vector. But $\alpha(cx_i) - \alpha(cx_j) = (\alpha(cx_i) - v) - (\alpha(cx_j) - v)$. Since the k^{th} component of $\alpha(cx_j) - v$ is 0, the k^{th} component of $\alpha(cx_i) - \alpha(cx_j)$ is the k^{th} component of $\alpha(cx_i) - v$. Consequently, the k^{th} component is a c^{th} multiple of an integer and it is clearly nonnegative by the choice of v.

Hence $\alpha(cx_i) - v$ are c^{th} multiples of integral vectors for all i. Denote $\frac{1}{c}(\alpha(cx_i) - v)$ by $\beta(x_i)$ for each i. This results in the affine map $\beta : \mathrm{L}(P) \to \mathbb{Z}^l_+$ such that $\alpha(cx) = v + c\beta(x)$ for all $x \in \mathrm{L}(P)$, as desired. □

Note that the positivity condition in Lemma 2 is needed so that the integral vectors can be identified with polynomials through the process explained in Section 3.

Proof of Theorem 1(b). Suppose $\mathrm{L}(P) = \{x_0, \dots, x_n\}$, $\mathrm{L}(cP) \subset \mathbb{Z}^d_+$ and $\mathrm{L}(Q) \subset \mathbb{Z}^m_+$, so that $\mathrm{L}(Q) \subset \{X_1^{a_1} \cdots X_m^{a_m} Z \mid a_i \geq 0\}$. After changing the lattice of reference $\mathrm{gp}(S(P)) = \mathbb{Z}^{d+1}$. Thus, for all $x \in L(cP)$, $x = a_0x_0 + \cdots + a_nx_n$ for some integers a_i such that $\sum_{i=0}^n a_i = c$. Let $f : k[cP] \to k[Q]$ be a graded homomorphism.

By Section 3, $\varphi_x = f(x)Z^{-1}$ gives rise to the integral affine map $\alpha : cP \to \mathbb{R}^l_+$ which respects the binomial relations in $\mathrm{L}(cP)$. By Lemma 2 there exists a vector $v \in \mathbb{Z}^l_+$ and an affine integral map $\beta : P \to \mathbb{R}^l_+$ such that $\alpha = v + c\beta$. Now, $\beta(x_i)$ gives rise to a polynomial $\theta_i \in k[X_1, \dots, X_m]$ for each $i \in \{0, 1, \dots n\}$. Similarly v gives rise to a polynomial $\psi \in k[X_1, \dots, X_m]$. Then for $x \in L(cP)$, $\varphi_x = \psi\theta_0^{a_0} \cdots \theta_n^{a_n}$. Consider the c^{th} homothetic blow-up of

$$\Theta : k[P] \to k[Q'] \text{ such that } \Theta(x_i) = \theta_i Z$$

for all $x_i \in L(P)$. f is obtained by a polytope change applied to $\Psi \star \Theta^c$ where

$$\Psi : k[cP] \to k[Q'] \text{ such that } \Psi(x) = \psi Z$$

for all $x \in L(cP)$ and where Q' is a large enough lattice polytope to contain all relevant polytopes.

Θ is tame by assumption since it is a homomorphism from $k[P]$. Ψ factors as

$$k[cP] \xrightarrow{x \mapsto t} k[t] \xrightarrow{t \mapsto \psi Z} k[Q']$$

where the first factor is tame since it maps monomials to monomials, and the second factor is tame because it is a free extension of the identity map $k \to k$. As a result, f is tame. $\square$

6. Segre products

The first step in proving Theorem 1(c) is Lemma 3. Note that $\mathrm{L}(P \times Q) = \mathrm{L}(P) \times L(Q)$. Suppose $\mathrm{L}(P) = \{x_0, \dots, x_n\}$ and $\mathrm{L}(Q) = \{y_0, \dots, y_m\}$. The copy of P in $P \times Q$ with the lattice points $\{(x_0, y_j), \dots, (x_n, y_j)\}$ is denoted by $P \times y_j$. Similarly $x_i \times Q$ denotes the copy of Q with lattice points $\{(x_i, y_0), \dots, (x_i, y_m)\}$.

Lemma 3. *Suppose P and Q are lattice polytopes and $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}^l_+$ is an affine map. Then $\alpha = a_P + a_Q$ for affine maps a_P, $a_Q : \mathrm{L}(P \times Q) \to \mathbb{Z}^l_+$ satisfying $a_P(P \times y_i) = a_P(P \times y_j)$ and $a_Q(x_i \times Q) = a_Q(x_j \times Q)$ for all i, j*

Proof of Lemma 3. Suppose P and Q are lattice polytopes and let $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}^l_+$ be an affine map. Since it is enough to prove the lemma for each component of the maps involved, there is no loss of generality to assume that $l = 1$. Suppose α takes its minimum value at the vertex $(x', y') \in \mathrm{L}(P \times Q)$. Let $(x, y) \in \mathrm{L}(P \times Q)$. Sine α is affine,

$$\alpha(x, y) + \alpha(x', y') = \alpha(x, y') + \alpha(x', y).$$

In other words,

$$\alpha(x, y) = \alpha(x, y') + \alpha(x', y) - \alpha(x', y').$$

Define $a_P(x, y) = a(x, y')$ and $a_Q(x, y) = a(x', y) - a(x', y')$. By the choice of (x', y'), the nonnegativity requirements are satisfied. $\square$

Now we can prove the desired result.

Proof of Theorem 1(c). Let $f : k[P \times Q] \to k[R]$ be a graded homomorphism, where R is some lattice polytope. Assume $\mathrm{L}(P \times Q) = \mathrm{L}(P) \times \mathrm{L}(Q) \subset \mathbb{R}_+^{d+e}$ such that $\mathrm{L}(P) \cap \mathrm{L}(Q) = \{0\}$, and $\mathrm{L}(R) \subset \mathbb{R}_+^c$ (by applying a polytope change if necessary). Suppose $\mathrm{L}(P) = \{x_0, \dots, x_n\}$ and $\mathrm{L}(Q) = \{y_0, \dots, y_m\}$.

Since $k[R] \subset \{X_1^{a_1} \cdots X_c^{a_c} Z \mid a_i \geq 0\}$, the polynomials $\varphi(x_i, y_j) = f(x_i, y_j)Z^{-1}$ belong to $k[X_1, \dots, X_c]$. Thus, by Section 3, φ gives rise to an affine map $\alpha : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ for some $l \in \mathbb{N}$. Hence, by Lemma 3, $\alpha = a_P + a_Q$ where $a_P, a_Q : \mathrm{L}(P \times Q) \to \mathbb{Z}_+^l$ are affine maps such that $a_P(x_i, y_j) = p_i$ and $a_Q(x_i, y_j) = q_j$ for some $p_i,\ q_j \in \mathbb{Z}_+^l$.

Now, each p_i and q_j gives rise to polynomials $\pi_i,\ \varrho_j \in k[X_1, \dots, X_c]$, respectively. Then $\varphi(x_i, y_j) = \pi_i \cdot \varrho_j$ for all $(x_i, y_j) \in L(P \times Q)$. This means $f = f_P \star f_Q$ where

$$f_P : k[P \times Q] \to k[R'] \text{ such that } f_P(x_i, y_j) = \pi_i Z$$

$$f_Q : k[P \times Q] \to k[R'] \text{ such that } f_Q(x_i, y_j) = \varrho_j Z$$

for all $(x_i, y_j) \in \mathrm{L}(P \times Q)$ and where R' is a large enough lattice polytope to contain all relevant polytopes. f_P is a tame homomorphism since it factors as

$$k[P \times Q] \xrightarrow{(x_i, y_j) \mapsto x_i} k[P] \xrightarrow{x_i \mapsto \pi_i Z} k[R']$$

where the first factor is tame since it maps monomials to monomials and the second is tame since it is a homomorphism from $k[P]$. Similarly f_Q is tame since it factors as

$$k[P \times Q] \xrightarrow{(x_i, y_j) \mapsto y_j} k[Q] \xrightarrow{y_j \mapsto \varrho_j Z} k[R'].$$

Therefore f is a tame homomorphism, as desired. □

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