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Vertices of Simple Modules for the Symmetric Groups in Blocks of Small Weights

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Abstract. With one exception, we determine the vertices of simple modules for symmetric groups in characteristic 2 and 3 which are contained in blocks of weight at most 4 and at most 3, respectively. Moreover, we provide some information on the sources of the modules in question.

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1. Introduction

Consider an algebraically closed field F of prime characteristic p, and denote the symmetric group of degree $n \in \mathbb{N}$ by $\mathfrak{S}_n$. It is well known that the isomorphism classes of simple $F\mathfrak{S}_n$-modules are labelled by the p-regular partitions of n. Moreover, each block B of $F\mathfrak{S}_n$ is associated with a certain nonnegative integer w, the *p-weight* of B, and a certain p-regular partition κ of $n-pw$, the *p-core* of B. This characterization of $\mathfrak{S}_n$-blocks is the content of Nakayama's conjecture, proved by R. Brauer and G. de B. Robinson in [3]. In [22] J. Scopes proves the following:

Theorem 1.1. *Up to Morita equivalence, there are only finitely many blocks of symmetric groups over F of given weight $w \geq 0$.*

The Morita equivalence used by Scopes has an explicit combinatorial description which we will explain in Section 4. As usual, we will refer to this Morita equivalence as the *Scopes equivalence*. Scopes also shows that this equivalence induces a bijection between the isomorphism classes of simple modules belonging to the respective blocks which preserves the lexicographic ordering on the corresponding p-regular partitions. In fact, even more is true. Namely, that bijection preserves vertices and sources of the simple modules as well. Thus, in order to determine the vertices of all simple modules of symmetric groups over F belonging to blocks of a given weight w, it suffices to consider simple $F\mathfrak{S}_n$-modules in blocks of weight w where $n \leq N$, for some $N \in \mathbb{N}$ depending on w.

While the simple $F\mathfrak{S}_n$-modules as well as the blocks of $F\mathfrak{S}_n$ possess explicit combinatorial descriptions, nothing similar is known for the vertices of simple $F\mathfrak{S}_n$-modules, in general. For that reason, in this note we investigate simple $F\mathfrak{S}_n$-modules belonging to blocks of "small" weights. More precisely, with one exception, we determine the vertices of simple modules of symmetric groups in characteristic 2 and 3 which are contained in blocks of weight at most 4 and at most 3, respectively. We obtain our results as follows: firstly, we determine the vertices of simple $F\mathfrak{S}_n$-modules, for $n \leq 14$ in case $p = 2$, and $n \leq 15$ in case $p = 3$ belonging to blocks of the desired weights. For this, several computations have been carried out using the computer algebra system MAGMA [2]. Afterwards, in order to settle the general case, we then apply the Scopes equivalence and Kleshchev's modular branching rules to our results for the small symmetric groups. Details on the algorithms actually used to perform our computations are given in [7], and can also be found in [5] and [23].

A comment on the state of the art in characteristic $p > 3$ is in order. Our methods for the actual vertex computation do work for arbitrary characteristics, even for arbitrary finite groups. Nevertheless, the simple modules for the symmetric groups have always been constructed as composition factors of appropriate Specht modules. The internal MAGMA function `SymmetricRepresentation` facilitates the construction of Specht modules. Thus we heavily build on the decomposition numbers for $\mathfrak{S}_n$ which, for $n \geq 18$, are currently unknown. However, the computation of vertices of simple $F\mathfrak{S}_n$-modules is of interest only for $n \geq p^2$. Otherwise, the vertices are known to be the defect groups of the respective blocks, by Knörr's theorem [17]. For these reasons, we cannot provide any computational data for $p \geq 5$ so far.

This article is structured as follows. We begin by recalling some fundamental facts about vertices of indecomposable modules over group algebras, about the Scopes equivalence and Kleshchev's modular branching rules. We also give a short overview of our computational methods. In Section 6 we present our results on vertices of simple $F\mathfrak{S}_n$-modules in blocks of small weights, for the primes 2 and 3. Throughout this paper, F is an algebraically closed field of characteristic $p > 0$, a group G is always supposed to be finite, and any FG-module is a left FG-module of finite F-dimension. We assume the reader to be familiar with the representation theory of the symmetric groups. For an extensive introduction to this subject, we refer to [14] and [15].

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2. Vertices and sources

Vertices and sources of indecomposable modules over group algebras had been introduced by J. A. Green in [13]. We briefly collect the facts which will be necessary throughout this paper. For further details on this subject, and proofs of the results stated here, we refer to [13].

Let G be a group. An FG-module M is said to be *relatively H-projective*, for some subgroup H of G, if $M | \operatorname{Ind}_H^G(\operatorname{Res}_H^G(M))$. In case M is indecomposable, a subgroup P of G which is minimal subject to the condition that M is relatively P-projective is called a *vertex* of M. The vertices of an indecomposable FG-module form a G-conjugacy class of p-subgroups of G, and, given a vertex P of an indecomposable FG-module M, there are a Sylow p-subgroup S of G and a defect group Δ of the block containing M such that $P \leq \Delta \leq S$ and $|S : P| \,\big|\, \dim(M)$. Moreover, there exists an indecomposable FP-module L with vertex P such that $L | \operatorname{Res}_P^G(M)$ and $M | \operatorname{Ind}_P^G(L)$. Then L is called a *source* of M, and L is determined up to isomorphism and $N_G(P)$-conjugacy. Furthermore, we have the following well known result, a proof of which can for example be found in [21], Lemma 4.3.4, Theorem 4.3.6.

Lemma 2.1. *Let M be an indecomposable FG-module with vertex P, and let N be an indecomposable FH-module with vertex Q, where H is a subgroup of G. Then:*

(i) *If $M | \operatorname{Ind}_H^G(N)$ then $P \leq_G Q$.*
(ii) *If $N | \operatorname{Res}_H^G(M)$ then $Q \leq_G P$.*
(iii) *If $M | \operatorname{Ind}_H^G(N)$ and $N | \operatorname{Res}_H^G(M)$ then M and N have both a vertex and a source in common.*

In the course of Section 6 we will also be faced with outer tensor products of indecomposable modules and their vertices. For this reason, we mention the following lemma, a proof of which is given by Külshammer in [18].

Lemma 2.2. *Let G_1 and G_2 be groups. Furthermore, let M_1 be an indecomposable FG_1-module, and M_2 an indecomposable FG_2-module with vertex P_1 and P_2, respectively. Then the indecomposable $F[G_1 \times G_2]$-module $M_1 \boxtimes M_2 := M_1 \otimes_F M_2$ has vertex $P_1 \times P_2$. If L_1 is a source of M_1, and L_2 is a source of M_2, then $L_1 \boxtimes L_2$ is a source of $M_1 \boxtimes M_2$.*

The following results are very powerful tools for dealing with simple FG-modules.

Theorem 2.3. (R. Knörr [17]) *Let M be a simple FG-module belonging to a block B of FG. Moreover, let P be a vertex of M. Then there is a block b of $F[PC_G(P)]$ with defect group P such that $b^G = B$. Thus there is a defect group Δ of B such that $C_\Delta(P) \leq P \leq \Delta$.*

Theorem 2.4. (K. Erdmann [8]) *Let M be a simple FG-module belonging to a block B of FG with cyclic vertex P. Then P is a defect group of B.*

Hence the above theorems particularly imply that simple FG-modules belonging to blocks with abelian defect groups have those defect groups as vertices, and simple modules belonging to blocks with noncyclic defect groups cannot have cyclic vertices. As far as simple modules for the symmetric groups are concerned, we will also need the results below, due to J. Grabmeier and D. J. Benson, respectively. For a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of n, we will write S^λ to denote the corresponding *$F\mathfrak{S}_n$-Specht module.* Furthermore, one defines the *Young subgroup* $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ as

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_s},$$

with different direct factors acting on disjoint subsets of $\{1, \ldots, n\}$. Given a fixed decomposition of the $F\mathfrak{S}_n$-permutation module $M^\lambda := \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(F)$ into indecomposable direct summands, there is exactly one among these indecomposable summands which contains S^λ as a submodule. This module is up to isomorphism uniquely determined, and denoted by Y^λ (cf. [19], Section 4.6). Thus, in particular, this *Young module* Y^λ is a trivial source module. Provided that λ is a p-regular partition of n, we denote the corresponding simple $F\mathfrak{S}_n$-module by D^λ. For convenience, we may now allow to add zero parts to any partition λ of $n \in \mathbb{N}$.

Theorem 2.5. ([J. Grabmeier [12], Section 7.8) *Let $n \in \mathbb{N}$, and let $\lambda = (\lambda_1, \ldots, \lambda_s, 0)$ be a partition of n such that $\lambda_s > 0$. Then the Sylow p-subgroups of the Young subgroup*

$$\prod_{i=1}^{s} (\mathfrak{S}_{\lambda_i - \lambda_{i+1}})^i$$

of $\mathfrak{S}_n$ are vertices of the $F\mathfrak{S}_n$-Young module Y^λ.

Theorem 2.6. (D. J. Benson [1], Theorem 1.1) *Let $p = 2$, $n \in \mathbb{N}$, and let $\lambda = (\lambda_1, \ldots, \lambda_s, 0)$ be a 2-regular partition of n such that $\lambda_s > 0$. Then $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^\lambda) = E_+^\lambda \oplus E_-^\lambda$ with non-isomorphic simple $F\mathfrak{A}_n$-modules E_+^λ and E_-^λ if and only if the following conditions are satisfied:*

(1) *$\lambda_{2j-1} - \lambda_{2j} \leq 2$, for all $j > 0$.*

(2) *$\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod 4$, for all $j > 0$.*

Otherwise, $\mathrm{Res}_{\mathfrak{A}_n}^{\mathfrak{S}_n}(D^\lambda)$ is simple.

Provided $p = 2$, a 2-regular partition λ of n satisfying conditions (1) and (2) above, is also called an *S-partition.* Note that the simple $F\mathfrak{A}_n$-modules E_+^λ and

E^λ_- are $\mathfrak{S}_n$-conjugate to each other. In particular, $\dim(E^\lambda_+) = \dim(E^\lambda_-)$, and the vertices of E^λ_+ are in $\mathfrak{S}_n$ conjugate to those of E^λ_-. As a direct consequence of Benson's result and Green's Indecomposability Theorem (cf. [13], Theorem 8) we have:

Corollary 2.7. *Let $p = 2$ and $n \in \mathbb{N}$. Suppose further that λ is a 2-regular partition of n. Then the simple $F\mathfrak{S}_n$-module D^λ is relatively $\mathfrak{A}_n$-projective if and only if λ is an S-partition.*

3. The computational methods

We now briefly mention the basic ideas behind our computational methods. For this, consider a group G, a finite field $\mathbb{F}_q$ of characteristic p, and an indecomposable $\mathbb{F}_qG$-module M. In order to determine the vertices of M, the idea is to restrict M to some subgroup H of G to which M is relatively projective. One then computes a direct sum decomposition $\mathrm{Res}^G_H(M) = a_1N_1 \oplus \cdots \oplus a_mN_m$ with $a_1, \ldots, a_m \in \mathbb{N}$ and pairwise non-isomorphic indecomposable $\mathbb{F}_qH$-modules $N_1, \ldots, N_m$, and tests whether one of the summands $N_1, \ldots, N_m$ is relatively projective to some maximal subgroup of H. Of course it suffices to consider representatives for the H-conjugacy classes of maximal subgroups of H. If H is a p-group, and if N_i is not relatively projective to any maximal subgroup of H, for some $i \in \{1, \ldots, m\}$, then both N_i and M have vertex H. Otherwise, for each $j \in \{1, \ldots, m\}$, there is some maximal subgroup H_j of H such that N_j is relatively H_j-projective. In this case one tests whether N_j is also relatively projective to some maximal subgroup of H_j, for $j \in \{1, \ldots, m\}$. Iterating this, one deduces the vertices of $N_1, \ldots, N_m$. The maximal among these are also vertices of M.

In practice, H is mostly a Sylow p-subgroup of G or a defect group of the block containing M. Nevertheless, we allow H to be arbitrary. One reason for this is that the endomorphism algebra $\mathrm{End}_{FH}(\mathrm{Res}^G_H(M))$ becomes larger, and hence the computation of a direct sum decomposition of $\mathrm{Res}^G_H(M)$ requires more time and memory, the closer H gets to p-groups. Indecomposable direct summands can be computed using the MAGMA function `IndecomposableSummands`; our test for relative projectivity builds on Higman's Criterion which involves the relative trace map (cf. [21], Theorem 4.2.2.).

In [23] the second author developed a series of MAGMA functions which facilitate the computation of vertices of indecomposable modules over group algebras. For modules of "small" dimensions, less that 1000 say, the vertex computation could be carried out automatically via these programs. Both the computation of relative traces and the determination of direct sum decompositions of modules requires the computation of endomorphism algebras. However, the dimensions of the modules under consideration grow fast, and the relevant subgroups of G are p-groups which leads to high dimensional endomorphism algebras, and thus to an enormous memory request, as already mentioned above. In these cases, the computations had to be carried out rather interactively. In [7] we have presented an algorithm for stripping indecomposable direct summands with cyclic vertices and, in particular, indecomposable projective direct summands off an $\mathbb{F}_q H$-module when H is a p-group. That algorithm actually does without computing any direct sum decompositions. In this way we can considerably shrink the dimensions of the modules whose vertices have to be computed explicitly. Moreover, in order to reduce the number of subgroups of G which may possibly occur as vertices of a given indecomposable module, we applied a result which builds on the theory of rank varieties of elementary abelian p-groups. For details we refer to [7], Theorem 3.7.

We should emphasize that all actual computations have clearly been carried out over finite fields. Keeping the notation of the first paragraph, let K be any extension field of $\mathbb{F}_q$. Then the KG-module $K \otimes_{\mathbb{F}_q} M$ may split into direct summands. However, each of its indecomposable direct summands has the same vertices as the $\mathbb{F}_q G$-module M. A proof of this result can, for instance, be found in [10], L. III.4.14. Moreover, at some stages of our computations we had to decide whether an indecomposable $\mathbb{F}_q G$-module M is also absolutely indecomposable. To overcome this problem, we made use of the following result which is an immediate consequence of Wedderburn's theorem.

Proposition 3.1. *Let K be any field of prime characteristic p, and let M be a KG-module. Furthermore, set $E := \mathrm{End}_{KG}(M)$, and suppose that all composition factors of the E-module M are isomorphic to a one-dimensional E-module D. Then the KG-module M is absolutely indecomposable.*

Endomorphism rings have been determined via the MAGMA function `EndomorphismAlgebra`. When it comes to the symmetric groups, the situation is rather comfortable, since every field is a splitting field of $\mathfrak{S}_n$, by [15], Theorem 2.1.12. Thus it is safe to construct simple $\mathbb{F}_q\mathfrak{S}_n$-modules and determine their vertices. When extending the field, these modules remain simple, and also their vertices do not change. Nevertheless, the dimensions of the sources might change when extending the coefficient field. All subsequent assertions on sources are therefore formulated over algebraically closed fields. As mentioned in the introduction, the simple $\mathbb{F}_q\mathfrak{S}_n$-modules have been constructed as composition factors of appropriate $\mathbb{F}_q\mathfrak{S}_n$-Specht modules.

4. The Scopes equivalence

In this section we summarize the basic facts about the Scopes equivalence (cf. [22]). For the following, we fix $n \in \mathbb{N}$. Moreover, let B be a block of $F\mathfrak{S}_n$ of p-weight $w > 0$ and p-core $\kappa = (\kappa_1, \ldots, \kappa_r)$ where $\kappa_r > 0$. We then set $t := pw + r$. Following [15], pp. 77–79, this yields a so called sequence of *β-numbers* $\beta_t(\kappa) = (\beta_1, \ldots, \beta_t)$ for κ which is defined as follows:

$$\beta_i := \begin{cases} \kappa_i - i + t, & \text{if } i \leq r \\ -i + t, & \text{if } i > r, \end{cases}$$

for $i = 1, \ldots, t$. This sequence of β-numbers is then displayed on an abacus with p runners having a bead at each of the positions $\beta_1, \ldots, \beta_t$.

Suppose further, that there exist some $k \geq w$ and $i \in \{1, \dots, p-1\}$ such that there are exactly k more beads on runner i than on runner $i-1$ of the abacus. Moving all these k beads from runner i to runner $i-1$ yields an abacus display of another sequence of β-numbers $\beta_t(\bar{\kappa})$ where the partition $\bar{\kappa}$ is again a p-core. Moreover, there is a block $\overline{B}$ of $F\mathfrak{S}_{n-k}$ with p-weight w and p-core $\bar{\kappa}$.

Example 4.1. Consider the case $p = 3$, $n = 10$ and the block B of $F\mathfrak{S}_{10}$ of 3-weight 2, labelled by the 3-core $\kappa = (3, 1)$. Then, in the notation above, we have $t = 8$ and $\beta_8(\kappa) = (10, 7, 5, 4, 3, 2, 1, 0)$. The corresponding abacus has the following form:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & \bullet & \bullet \\ | & | & | \\ - & \bullet & - \\ | & | & | \\ - & \bullet & - \end{array} \qquad \longmapsto \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & - & - \\ | & | & | \\ \bullet & - & - \end{array}$$

Moving the two lowest beads on runner 1 to runner 0, yields the abacus display of $\beta_8(\bar{\kappa})$ where $\bar{\kappa} = (2)$ labels the principal block $\overline{B}$ of $F\mathfrak{S}_8$.

With the above notation one has the following:

Theorem 4.2. (J. Scopes [22]) *The blocks B and $\overline{B}$ are Morita equivalent. Furthermore, there is a bijection Φ between the isomorphism classes of simple modules in B and the isomorphism classes of simple modules in $\overline{B}$ which preserves the lexicographic ordering of the corresponding partitions.*

In the situation of the previous theorem, we say that the blocks B and $\overline{B}$ form a $[w : k]$-*pair*. In particular, the 3-blocks of $\mathfrak{S}_{10}$ and $\mathfrak{S}_8$ mentioned in Example 4.1 form a $[2 : 3]$-pair. The bijection Φ can be described combinatorially as follows: given a simple $F\mathfrak{S}_n$-module D^λ belonging to B, we display the sequence of β-numbers $\beta_t(\lambda)$ with $t = pw + r$ on an abacus with p runners. Then the partition $\bar{\lambda}$ of $n - k$ with $\Phi(D^\lambda) \cong D^{\bar{\lambda}}$ is obtained by interchanging the ith and $(i-1)$st runner of the abacus. In [22], Section 4, Scopes obtains the Morita equivalence between the blocks B and $\overline{B}$, via exact F-linear functors

$$\Psi : B\text{−}\mathbf{mod} \longrightarrow \overline{B}\text{−}\mathbf{mod} \quad \text{and} \quad \Psi' : \overline{B}\text{−}\mathbf{mod} \longrightarrow B\text{−}\mathbf{mod},$$

between the category of finitely generated B-modules and the category of finitely generated $\overline{B}$-modules, which are both left and right adjoint to each other. In fact, Ψ assigns to a simple module D^λ belonging to B the simple module $\Phi(D^\lambda)$, and Ψ' assigns to a simple module D^μ belonging to $\overline{B}$ the simple module $\Phi^{-1}(D^\mu)$. From now on, we will refer to this specific Morita equivalence as the Scopes equivalence.

If, for some $m \leq n$, we are given any Scopes equivalent blocks B and $\overline{B}$ of $F\mathfrak{S}_n$ and $F\mathfrak{S}_m$, respectively, we denote the bijection between the isomorphism classes of simple modules in B and those in $\overline{B}$ by Φ as well.

In fact, the bijection Φ mentioned in the theorem is vertex and source preserving in the following sense: given a simple $F\mathfrak{S}_n$-module D^λ belonging to B, we again denote the corresponding simple $F\mathfrak{S}_{n-k}$-module $\Phi(D^\lambda)$ belonging to $\overline{B}$ by $D^{\bar{\lambda}}$. Then

$$\overline{B} \cdot \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}}(D^\lambda) \cong \bigoplus_{k!} D^{\bar{\lambda}} \qquad \text{and} \qquad B \cdot \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-k}}(D^{\bar{\lambda}}) \cong \bigoplus_{k!} D^\lambda,$$

by [22], proof of Corollary 3.3. Consequently, D^λ and $D^{\bar{\lambda}}$ have common vertices and common sources. Iterating this argument, we get:

Theorem 4.3. *Let B and $\overline{B}$ be Scopes equivalent blocks of p-weight $w > 0$. Moreover, let D^λ be a simple module in B. Then D^λ and $\Phi(D^\lambda)$ have common vertices and common sources.*

5. Modular branching rules

Beside the Scopes equivalence, Kleshchev's modular branching rules will be the most important tool for our vertex computations. We therefore recall the basic facts which we will need in Section 6, and refer to [16] for further details on this subject.

Remark 5.1. In the following, we set $I := \{0, \dots, p-1\}$ and identify I with the field $\mathbb{F}_p$. Moreover, for $z \in \mathbb{Z}$, we denote its residue class modulo p by $\bar{z}$. For any partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of n with $\lambda_s > 0$, we consider the corresponding Young diagram

$$[\lambda] = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | 1 \le x \le s,\ 1 \le y \le \lambda_x\}.$$

Furthermore, let $i \in I$.

(a) A node $(x, y) \in [\lambda]$ is called an *i-removable node* if $\overline{y - x} = i$ and $[\lambda] \setminus \{(x, y)\}$ is the Young diagram of a partition of $n - 1$. Analogously, $(u, v) \notin [\lambda]$ is called an *i-addable node* of $[\lambda]$ if $\overline{v - u} = i$ and $[\lambda] \cup \{(u, v)\}$ is the Young diagram of a partition of $n + 1$.

(b) We now label the i-removable nodes of $[\lambda]$ by a "$-$" and the i-addable nodes by a "$+$". The sequence of "$+$" and "$-$" one gets when going along the rim of $[\lambda]$, from bottom left to top right, is called the *i-signature* of λ. Repeatedly cancelling all terms of the form $-+$ then yields the *reduced i-signature* of λ which is thus always a sequence of the form $+ \cdots + - \cdots -$.

(c) A removable node of $[\lambda]$ corresponding to a "$-$" in the reduced i-signature is called *i-normal*, and an addable node of $[\lambda]$ corresponding to a "$+$" in the reduced i-signature is called *i-conormal*. The total number of i-normal nodes of $[\lambda]$ will be denoted by $\varepsilon_i(\lambda)$, and the total number of i-conormal nodes of $[\lambda]$ will be denoted by $\varphi_i(\lambda)$.

In the course of the next section, we will make extensive use of the following observation:

Proposition 5.2. *Let λ be a p-regular partition of n, and let $i \in I$. Furthermore, let μ be the p-regular partition of $n - \varepsilon_i(\lambda)$ which is obtained by removing all i-normal nodes from $[\lambda]$, and let ν be the p-regular partition of $n + \varphi_i(\lambda)$ which is obtained by adding all i-conormal nodes to $[\lambda]$.*

(i) *Provided $\varepsilon_i(\lambda) = \varphi_i(\mu)$, the modules D^λ and D^μ have both a vertex and a source in common.*

(ii) *Provided $\varepsilon_i(\nu) = \varphi_i(\lambda)$, the modules D^λ and D^ν have both a vertex and a source in common.*

Proof. By [16], Theorems 11.2.10 and 11.2.11, we obtain that $D^\mu | \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-\varepsilon_i(\lambda)}}(D^\lambda)$ as well as $D^\nu | \operatorname{Ind}^{\mathfrak{S}_{n+\varphi_i(\lambda)}}_{\mathfrak{S}_n}(D^\lambda)$. Assuming $\varphi_i(\mu) = \varepsilon_i(\lambda)$, application of Kleshchev's induction and restriction functors as introduced in [16], 11.2, and the Frobenius Reciprocity Theorem immediately yields that $[\lambda]$ is obtained by adding all i-conormal nodes to $[\mu]$. Hence also $D^\lambda | \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-\varepsilon_i(\lambda)}}(D^\mu)$, by [16], Theorem 11.2.11. Assertion (i) now follows from Lemma 2.1.

Similarly, if $\varphi_i(\lambda) = \varepsilon_i(\nu)$ then $[\lambda]$ is obtained by removing all i-normal nodes from $[\nu]$, and thus $D^\lambda | \operatorname{Res}^{\mathfrak{S}_{n+\varphi_i(\lambda)}}_{\mathfrak{S}_n}(D^\nu)$, by [16], Theorem 11.2.10. Consequently, we then deduce (ii) from Lemma 2.1 again. □

Remark 5.3. In the situation of Proposition 5.2 (i) and (ii), respectively, we will from now on use the notation $\lambda \longleftrightarrow_i \mu$ and $\nu \longleftrightarrow_i \lambda$, respectively.

6. Blocks of small weights

In this section we determine the vertices of simple $F\mathfrak{S}_n$-modules belonging to p-blocks of "small" p-weights w. More precisely, we settle the cases $p = 2,\ w \leq 4$ and $p = 3,\ w \leq 3$ with one exception. From now on, given $n \in \mathbb{N}$, by $\mathfrak{P}_n$ and $\mathfrak{Q}_n$, respectively, we denote a Sylow p-subgroup of $\mathfrak{S}_n$ and $\mathfrak{A}_n$, respectively.

The case $p = 2$

First of all we consider the case where $p = 2$. The partitions which are 2-cores are precisely the *triangular partitions*, i.e. partitions of the form $\kappa_x := (x, x-1, x-2, \ldots, 1)$, for $x \geq 0$. Using the abacus notation as explained in Section 4, one obtains the following:

Lemma 6.1. *Let B_x be a 2-block of $F\mathfrak{S}_n$ with 2-weight $w > 0$ and 2-core κ_x where $x \geq w - 1$. Then B_x is Scopes equivalent to the 2-block $B_{\kappa_{w-1}}$ with 2-core κ_{w-1} and 2-weight w.*

Proof. We prove the assertion by induction on x. In case $x = w - 1$ it clearly holds, and we may thus assume $x \geq w$. Consider the sequence of β-numbers $\beta_{x+2w}(\kappa_x) = (2w + 2x - 1, 2w + 2x - 3, \ldots, 2w + 1, 2w - 1, 2w - 2, \ldots, 0)$. Then the corresponding abacus has exactly x more beads on runner 1 than on runner 0.

Moving these x beads to runner 0 yields the abacus configuration of $\beta_{x+2w}(\kappa_{x-1})$. Hence B_x is Scopes equivalent to the block of $F\mathfrak{S}_{n-x}$ with 2-core κ_{x-1}, and the assertion follows by induction. $\square$

Thus, due to Theorem 4.3, in order to determine the vertices of all simple $F\mathfrak{S}_n$-modules belonging to 2-blocks of 2-weight $w > 0$, it is sufficient to consider 2-blocks of 2-weight w which are labelled by 2-cores κ_x where $x \leq w-1$. In the following, this will be done, for $w \leq 4$. Note that, in our subsequent theorems, both the numbers of simple modules as well as their labels are of course known, for all blocks under consideration.

Theorem 6.2. *Let $p = 2$, $n \in \mathbb{N}$, and let B be a 2-block of $F\mathfrak{S}_n$ of 2-weight $w \leq 1$. Then the unique simple module in B has vertex $\mathfrak{P}_{pw}$ and trivial source.*

Proof. By [13], Theorem 12, the defect groups of a block are precisely the maximal vertices of all simple modules belonging to the block. Since B contains exactly one simple module D^λ, its vertices have to be the defect groups of B. Moreover, $D^\lambda \cong S^\lambda \cong Y^\lambda$, and D^λ has thus trivial source. $\square$

Theorem 6.3. *Let $p = 2$, $n \in \mathbb{N}$, and let B be a 2-block of $F\mathfrak{S}_n$ of 2-weight $w = 2$, and with 2-core κ_x. For the simple modules belonging to B we then obtain the following:*

(i) *in case $n = 4$*
- $D^{(4)}$ *has vertex* $\mathfrak{P}_4$,
- $D^{(3,1)}$ *has vertex* $\mathfrak{Q}_4$.

Both modules have trivial sources.

(ii) *in case $n \geq 5$*
- $D^{(x+4,x-1,x-2,\ldots,0)}$ *has vertex* $\mathfrak{P}_4$ *and trivial source,*
- $D^{(x+2,x+1,x-2,x-3,\ldots,0)}$ *has vertex* $\mathfrak{Q}_4$ *and source* $\mathrm{Res}^{\mathfrak{A}_5}_{\mathfrak{Q}_4}(E_+^{(3,2)})$.

Proof. Clearly, $D^{(4)}$ has vertex $\mathfrak{P}_4$ and trivial source, and $D^{(3,1)}$ is relatively $\mathfrak{Q}_4$-projective, by Corollary 2.7. Since any proper subgroup of $\mathfrak{Q}_4$ is cyclic, $D^{(3,1)}$ has vertex $\mathfrak{Q}_4$, by Theorem 2.4. Moreover, $\dim(E_+^{(3,1)}) = \dim(E_-^{(3,1)}) = 1$ so that $D^{(3,1)}$ has trivial source. For $x = 1$, we have $D^{(x+4,x-1)} = D^{(5)}$ with vertex $\mathfrak{P}_4$ and $D^{(x+2,x+1)} = D^{(3,2)}$. Since $(3,2)$ is an S-partition, $D^{(3,2)}$ is relatively $\mathfrak{Q}_4$-projective, by Corollary 2.7. By Theorem 2.4, we then obtain that $D^{(3,2)}$ has vertex $\mathfrak{Q}_4$, since all proper subgroups of $\mathfrak{Q}_4$ are cyclic. Furthermore, the simple direct summands $E_-^{(3,2)}$ and $E_+^{(3,2)}$ of $\mathrm{Res}^{\mathfrak{S}_5}_{\mathfrak{A}_5}(D^{(3,2)})$ remain indecomposable when restricted to $\mathfrak{Q}_4$, and these restrictions are therefore sources of $D^{(3,2)}$. The assertion for $x \geq 2$ then follows from Lemma 6.1 and Theorem 4.3. $\square$

Theorem 6.4. *Let $p = 2$, $n \in \mathbb{N}$, and let B be a 2-block of $F\mathfrak{S}_n$ of 2-weight $w = 3$, and with 2-core κ_x. For the simple modules in B we then obtain the following:*

(i) *in case $n = 6$*

- $D^{(6)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2 = \mathfrak{P}_6$,
- $D^{(5,1)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2$,
- $D^{(4,2)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2$.

Furthermore, the restrictions of these modules to $\mathfrak{P}_4 \times \mathfrak{P}_2$ *are also sources.*

(ii) *in case* $n = 7$

- $D^{(7)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2$ *and trivial source,*
- $D^{(5,2)}$ *has vertex* $(\mathfrak{P}_2)^3$ *and trivial source,*
- $D^{(4,2,1)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2$ *and source* $\mathrm{Res}^{\mathfrak{S}_6}_{\mathfrak{P}_4 \times \mathfrak{P}_2}(D^{(4,2)})$.

(iii) *in case* $n \geq 9$

- $D^{(x+6,x-1,x-2,\ldots,0)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{P}_2$ *and trivial source,*
- $D^{(x+4,x+1,x-2,x-3,x-4,\ldots,0)}$ *has vertex* $(\mathfrak{P}_2)^3$ *and trivial source,*
- $D^{(x+2,x+1,x,x-3,x-4,\ldots,0)}$ *has vertex* $\mathfrak{Q}_4 \times \mathfrak{P}_2$ *and source* $\mathrm{Res}^{\mathfrak{A}_5 \times \mathfrak{S}_2}_{\mathfrak{Q}_4 \times \mathfrak{P}_2}(E_+^{(3,2)} \boxtimes D^{(2)})$.

Proof. Consider the cases $n = 6, 7$ first. The assertion trivially holds for $D^{(6)}$ and $D^{(7)}$, and $D^{(5,1)}$ has vertex $\mathfrak{P}_4 \times \mathfrak{P}_2$, and $\mathrm{Res}^{\mathfrak{S}_6}_{\mathfrak{P}_4 \times \mathfrak{P}_2}(D^{(5,1)})$ is indecomposable, by [20]. The $F\mathfrak{S}_6$-module $D^{(4,2)}$ has been treated with the computer. It turns out that $D^{(4,2)}$ is not relatively projective to any maximal subgroup of $\mathfrak{P}_4 \times \mathfrak{P}_2$, and its vertices are thus precisely the Sylow 2-subgroups of $\mathfrak{S}_6$. Its restriction to any vertex is indecomposable. Moreover, we have $(4,2,1) \longleftrightarrow_0 (4,2)$, so that $D^{(4,2,1)}$ has vertex $\mathfrak{P}_4 \times \mathfrak{P}_2$ as well, and common sources with $D^{(4,2)}$. Finally, $D^{(5,2)} \cong S^{(5,2)} \cong Y^{(5,2)}$, and has thus trivial source and vertex $(\mathfrak{P}_2)^3$, by Theorem 2.5.

Now, let $n = 9$ and $x = 2$. Then we have

$$(x+6, x-1) = (8,1) \longleftrightarrow_1 (7) \quad \text{and} \quad (x+4, x+1) = (6,3) \longleftrightarrow_1 (5,2).$$

Therefore we obtain the claimed assertion about the vertices and sources of the corresponding simple modules. Furthermore, computer calculations show that $D^{(4,3,2)} | \mathrm{Ind}^{\mathfrak{S}_9}_{\mathfrak{S}_5 \times \mathfrak{S}_2}(D^{(3,2)} \boxtimes D^{(2)})$ and $D^{(3,2)} \boxtimes D^{(2)} | \mathrm{Res}^{\mathfrak{S}_9}_{\mathfrak{S}_5 \times \mathfrak{S}_2}(D^{(4,3,2)})$. Hence $D^{(x+2,x+1,x)} = D^{(4,3,2)}$ and $D^{(3,2)} \boxtimes D^{(2)}$ have $\mathfrak{Q}_4 \times \mathfrak{P}_2$ as a common vertex, and source $\mathrm{Res}^{\mathfrak{A}_5 \times \mathfrak{S}_2}_{\mathfrak{Q}_4 \times \mathfrak{P}_2}(E_+^{(3,2)} \boxtimes D^{(2)})$. This follows from Lemma 2.2 and Theorem 6.3 (ii). The assertion for $x > 2$ then again follows from Lemma 6.1 and Theorem 4.3. $\square$

Theorem 6.5. *Let* $p = 2$, $n \in \mathbb{N}$, *and let* B *be a* 2*-block of* $F\mathfrak{S}_n$ *of* 2*-weight* $w = 4$, *and with* 2*-core* κ_x. *For the simple modules belonging to* B *we then obtain the following:*

(i) *in case* $n = 8$

- $D^{(8)}$ *has vertex* $\mathfrak{P}_8$,
- $D^{(7,1)}$ *has vertex* $\mathfrak{P}_8$,

- $D^{(6,2)}$ *has vertex* $\mathfrak{P}_8$,
- $D^{(5,3)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_8}_{\mathfrak{Q}_8}(E_+^{(5,3)})$,
- $D^{(4,3,1)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_8}_{\mathfrak{Q}_8}(E_+^{(4,3,1)})$.

The restrictions of the first three modules to $\mathfrak{P}_8$ *are also sources of these.*

(ii) *in case* $n = 9$
- $D^{(9)}$ *has vertex* $\mathfrak{P}_8$ *and trivial source,*
- $D^{(7,2)}$ *has vertex* $\mathfrak{P}_8$ *and source* $\mathrm{Res}^{\mathfrak{S}_9}_{\mathfrak{P}_8}(D^{(7,2)})$,
- $D^{(6,2,1)}$ *has vertex* $\mathfrak{P}_8$ *and source* $\mathrm{Res}^{\mathfrak{S}_8}_{\mathfrak{P}_8}(D^{(6,2)})$,
- $D^{(5,4)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_9}_{\mathfrak{Q}_8}(E_+^{(5,4)})$,
- $D^{(5,3,1)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_8}_{\mathfrak{Q}_8}(E_+^{(4,3,1)})$.

(iii) *in case* $n = 11$
- $D^{(10,1)}$ *has vertex* $\mathfrak{P}_8$ *and trivial source,*
- $D^{(8,3)}$ *has vertex* $\mathfrak{P}_8$ *and source* $\mathrm{Res}^{\mathfrak{S}_9}_{\mathfrak{P}_8}(D^{(7,2)})$,
- $D^{(6,5)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_9}_{\mathfrak{Q}_8}(E_+^{(5,4)})$,
- $D^{(6,3,2)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{Q}_4$ *and source* $\mathrm{Res}^{\mathfrak{S}_4\times\mathfrak{A}_5}_{\mathfrak{P}_4\times\mathfrak{Q}_4}(D^{(4)} \boxtimes E_+^{(3,2)})$,
- $D^{(5,3,2,1)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_8}_{\mathfrak{Q}_8}(E_+^{(4,3,1)})$.

(iv) *in case* $n \geq 14$
- $D^{(x+8,x-1,x-2,\ldots,0)}$ *has vertex* $\mathfrak{P}_8$ *and trivial source,*
- $D^{(x+6,x+1,x-2,x-3,\ldots,0)}$ *has vertex* $\mathfrak{P}_8$ *and source* $\mathrm{Res}^{\mathfrak{S}_9}_{\mathfrak{P}_8}(D^{(7,2)})$,
- $D^{(x+4,x+3,x-2,x-3,\ldots,0)}$ *has vertex* $\mathfrak{Q}_8$ *and source* $\mathrm{Res}^{\mathfrak{A}_9}_{\mathfrak{Q}_8}(E_+^{(5,4)})$,
- $D^{(x+4,x+1,x,x-3,x-4,\ldots,0)}$ *has vertex* $\mathfrak{P}_4 \times \mathfrak{Q}_4$ *and source* $\mathrm{Res}^{\mathfrak{S}_4\times\mathfrak{A}_5}_{\mathfrak{P}_4\times\mathfrak{Q}_4}(D^{(4)} \boxtimes E_+^{(3,2)})$,
- $D^{(x+2,x+1,x,x-1,x-4,x-5\ldots,0)}$ *has vertex* $\mathfrak{E}_8$ *and* 4*-dimensional sources. Here* $\mathfrak{E}_8 \leq \mathfrak{S}_8$ *denotes an elementary abelian group of order* 8*, acting regularly on* $\{1, \ldots, 8\}$.

Proof. Again the assertions about $D^{(8)}$ and $D^{(9)}$ clearly hold, and $D^{(7,1)}$ and $D^{(10,1)}$ have vertex $\mathfrak{P}_8$, by [20]. Furthermore, $\mathrm{Res}^{\mathfrak{S}_8}_{\mathfrak{P}_8}(D^{(7,1)})$ is a source of $D^{(7,1)}$, by [20], and $D^{(10,1)}$ has trivial source, since $D^{(10,1)} \cong S^{(10,1)} \cong Y^{(10,1)}$. The remaining modules for $n = 8$ have been treated with the computer: $D^{(6,2)}$ restricts indecomposably to $\mathfrak{P}_8$, and is not relatively projective to any maximal subgroup of $\mathfrak{P}_8$. Moreover, the modules $D^{(5,3)}$ and $D^{(4,3,1)}$ are relatively $\mathfrak{A}_8$-projective, by Theorem 2.6, and neither of them is relatively projective to any maximal subgroup of $\mathfrak{Q}_8$. Both $E_+^{(5,3)}$ and $E_+^{(4,3,1)}$ restrict indecomposably to $\mathfrak{Q}_8$. In case $n = 9$ we have

$$(6,2,1) \longleftrightarrow_0 (6,2) \quad \text{and} \quad (5,3,1) \longleftrightarrow_0 (4,3,1).$$

This yields the vertices and sources of the corresponding simple modules, as stated. The vertices and sources of $D^{(7,2)}$ and $D^{(5,4)}$ have been determined with the computer as well as those of the simple $F\mathfrak{S}_{11}$-module $D^{(6,3,2)}$. It turns out that $D^{(7,2)}$ is not relatively projective to any maximal subgroup of $\mathfrak{P}_8$, and that the restriction of $D^{(7,2)}$ to $\mathfrak{P}_8$ is indecomposable. Furthermore, $D^{(5,4)}$ is relatively $\mathfrak{A}_8$-projective, by Theorem 2.6. Our computations show that it is not relatively projective to any maximal subgroup of $\mathfrak{Q}_8$, and that the restriction of $E_+^{(5,4)}$ to $\mathfrak{Q}_8$ is indecomposable. In the case of the module $D^{(6,3,2)}$ we indeed have $D^{(6,3,2)}|\operatorname{Ind}_{\mathfrak{S}_4\times\mathfrak{S}_5}^{\mathfrak{S}_{11}}(D^{(4)}\boxtimes D^{(3,2)})$ and $D^{(4)}\boxtimes D^{(3,2)}|\operatorname{Res}_{\mathfrak{S}_4\times\mathfrak{S}_5}^{\mathfrak{S}_{11}}(D^{(6,3,2)})$ so that $D^{(4)}\boxtimes D^{(3,2)}$ and $D^{(6,3,2)}$ have common vertex $\mathfrak{P}_4\times\mathfrak{Q}_4$, and common source $\operatorname{Res}_{\mathfrak{P}_4\times\mathfrak{Q}_4}^{\mathfrak{S}_4\times\mathfrak{A}_5}(D^{(4)}\boxtimes E_+^{(3,2)})$, by Theorem 6.3 and Lemma 2.2. Furthermore, for $n=11$, we get

$$(8,3)\longleftrightarrow_1(7,2),\quad (6,5)\longleftrightarrow_1(5,4)\quad\text{and}\quad (5,3,2,1)\longleftrightarrow_1(5,3,1),$$

and hence obtain the vertices and sources of the corresponding modules, as claimed.

Finally, let $n=14$ and $x=3$. Then

$$\begin{aligned}(x+8,x-1,x-2)&=(11,2,1)\longleftrightarrow_0(10,1),\\ (x+6,x+1,x-2)&=(9,4,1)\quad\longleftrightarrow_0(8,3),\\ (x+4,x+3,x-2)&=(7,6,1)\quad\longleftrightarrow_0(6,5),\\ (x+4,x+1,x)&=(7,4,3)\quad\longleftrightarrow_0(6,3,2).\end{aligned}$$

This proves the assertion about the vertices and sources of the corresponding modules. It remains to consider the simple module $D^{(5,4,3,2)}$ of dimension 35840. Using the fact that $D^{(5,4,3,2)}\cong D^{(8,6)}\otimes D^{(9,5)}$ (cf. Corollary 3.21 in [11]) this module could be treated with the computer. By Theorem 2.6, $D^{(5,4,3,2)}$ is relatively $\mathfrak{A}_8$-projective, and the computations show that the vertices of $D^{(5,4,3,2)}$ are conjugate to an elementary abelian group $\mathfrak{E}_8$ of order 8, which acts regularly on $\{1,\ldots,8\}$. To be a bit more precise,

$$\operatorname{Res}_{\mathfrak{P}_8}^{\mathfrak{S}_{14}}(D^{(5,4,3,2)})\cong 16M_1\oplus 16M_2\oplus 16M_3\oplus\text{ proj.},$$

where $M_1,\ M_2,\ M_3$ are pairwise non-isomorphic, indecomposable of dimension 64, and have vertex $\mathfrak{E}_8$ (cf. [7], Example 3.2 (b)). Moreover, $D^{(5,4,3,2)}$ has sources of dimension 4. This completes the case $x=3$. Theorem 4.3 and Lemma 6.1 now yield the assertions, for $x\geq 4$. $\square$

Remark 6.6. (a) For the sources of the simple $F\mathfrak{S}_{14}$-module $D^{(5,4,3,2)}$ we do not have a similar description as for the sources of the other simple modules considered above. Nevertheless, we can provide some information about them. Therefore we fix an $\mathfrak{E}_8$-source L of $D^{(5,4,3,2)}$. Then L is a selfdual $F\mathfrak{E}_8$-module with Loewy series

$$\begin{bmatrix} F\\ F\oplus F\\ F\end{bmatrix}.$$

The radical of a module over a group algebra can be computed via the MAGMA function `JacobsonRadical`. Thus, in particular, L is not isomorphic to the 4-dimensional $F\mathfrak{E}_8$-module $(F\mathfrak{E}_8)/(\mathbf{J}(F\mathfrak{E}_8))^2$ where $\mathbf{J}(F\mathfrak{E}_8)$ denotes the Jacobson radical of the group algebra $F\mathfrak{E}_8$. Furthermore, L is faithful so that it cannot be isomorphic to an outer tensor product of two indecomposable modules for proper subgroups of $\mathfrak{E}_8$. Otherwise, one of the groups would be the cyclic group C_2 of order 2, and the respective tensor factor would have to be the regular FC_2-module. But then the vertices of L would be strictly smaller than $\mathfrak{E}_8$, by Lemma 2.2.

(b) So far, we have determined the vertices of all but two simple $F\mathfrak{S}_n$-modules, for $n \leq 14$ and $p = 2$, and for $n \leq 15$ and $p = 3$. Our results are summarized in [7]. Among these vertices, the elementary abelian group $\mathfrak{E}_8$ is the only group which is not conjugate to some Sylow p-subgroup of a group

$$\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k} \times \mathfrak{A}_{n_{k+1}} \times \cdots \times \mathfrak{A}_{n_l} \leq \mathfrak{S}_n,$$

for appropriate $k, l, n_1, \ldots, n_l \in \mathbb{N}_0$.

The case $p = 3$

We now consider the case $p = 3$. By Theorem 2.3, the simple $F\mathfrak{S}_n$-modules belonging to a block B of 3-weight $w \leq 2$ have exactly the defect groups of B as vertices. Therefore we now investigate the simple $F\mathfrak{S}_n$-modules in 3-blocks of 3-weight 3. First of all, in [9] M. Fayers determines the following transversal for the Scopes classes of 3-blocks of 3-weight 3:

<table>
<tr><th>n</th><th colspan="2">blocks</th><th colspan="2">blocks per Scopes class</th></tr>
<tr><td>9</td><td colspan="2">$\emptyset$</td><td colspan="2">1</td></tr>
<tr><td>10</td><td colspan="2">(1)</td><td colspan="2">1</td></tr>
<tr><td>11</td><td>(2)</td><td>(1^2)</td><td>1</td><td>1</td></tr>
<tr><td>13</td><td>$(3,1)$</td><td>$(2,1^2)$</td><td>1</td><td>1</td></tr>
<tr><td>14</td><td colspan="2">$(3,1^2)$</td><td colspan="2">2</td></tr>
<tr><td>15</td><td>$(4,2)$</td><td>$(2^2,1^2)$</td><td>∞</td><td>∞</td></tr>
<tr><td>19</td><td>$(5,3,1^2)$</td><td>$(4,2^2,1^2)$</td><td>∞</td><td>∞</td></tr>
<tr><td>25</td><td colspan="2">$(6,4,2^2,1^2)$</td><td colspan="2">∞</td></tr>
</table>

The last columns record the numbers of 3-blocks for symmetric groups belonging to the respective Scopes classes. These can be read from Figure 1 in [9]. Hence, in order to determine the vertices of all simple modules for symmetric groups belonging to blocks of 3-weight 3, it suffices to consider the simple modules in the blocks listed above. For $n \in \{11, 13, 15, 19\}$, the respective blocks of $F\mathfrak{S}_n$ are conjugate, so that we only need to determine the vertices of the simple modules belonging to one of the two blocks. In fact, applying the modular branching rules repeatedly, we even obtain the following:

Proposition 6.7. *Let $n > 15$, and let D^λ be a simple $F\mathfrak{S}_n$-module belonging to a block of weight* 3*. Then there are an $m \leq 15$ and a simple $F\mathfrak{S}_m$-module D^μ belonging to a block of weight* 3 *such that $D^\mu | \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_m}(D^\lambda)$ and $D^\lambda | \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_m}(D^\mu)$. Thus D^λ and D^μ have common vertices and common sources.*

Proof. It suffices to prove the assertion for the block of $F\mathfrak{S}_{19}$ with 3-core $(5,3,1^2)$ and for the block of $F\mathfrak{S}_{25}$ with 3-core $(6,4,2^2,1^2)$. It has already been shown in [9], Proposition 2 that the only simple $F\mathfrak{S}_{19}$-module belonging to the block with 3-core $(5,3,1^2)$ which is not in relation "$\longleftrightarrow_1$" to a simple $F\mathfrak{S}_{17}$-module belonging to the block with 3-core $(4,2,1^2)$ is $D^{(8,6,4,1)}$. But we have

$$(8,6,4,1) \longleftrightarrow_0 (8,6,4) \longleftrightarrow_1 (7,5,3),$$

and hence $D^{(7,5,3)} | \operatorname{Res}^{\mathfrak{S}_{19}}_{\mathfrak{S}_{15}}(D^{(8,6,4,1)})$ and $D^{(8,6,4,1)} | \operatorname{Ind}^{\mathfrak{S}_{19}}_{\mathfrak{S}_{15}}(D^{(7,5,3)})$. Since the block of $F\mathfrak{S}_{17}$ with 3-core $(4,2,1^2)$ is Scopes equivalent to the block of $F\mathfrak{S}_{14}$ with 3-core $(3,1^2)$ the assertion follows, for $n = 19$.

Now let $n = 25$. Again, by [9], Proposition 2, $D^{(9,7,5,2,1^2)}$ is the only simple $F\mathfrak{S}_{25}$-module in the block with 3-core $(6,4,2^2,1^2)$ which is not in relation "$\longleftrightarrow_2$" to one of the simple $F\mathfrak{S}_{23}$-modules in the block with 3-core $(5,3,2^2,1^2)$ which is Scopes equivalent to the block of $F\mathfrak{S}_{19}$ with 3-core $(4,2^2,1^2)$. Here we have

$$(9,7,5,2,1^2) \longleftrightarrow_1 (9,7,5,1^2) \longleftrightarrow_2 (8,6,4,1) \longleftrightarrow_0 (8,6,4) \longleftrightarrow_1 (7,5,3).$$

Thus $D^{(7,5,3)} | \operatorname{Res}^{\mathfrak{S}_{25}}_{\mathfrak{S}_{15}}(D^{(9,7,5,2,1^2)})$ and $D^{(9,7,5,2,1^2)} | \operatorname{Ind}^{\mathfrak{S}_{25}}_{\mathfrak{S}_{15}}(D^{(7,5,3)})$. This proves the proposition. $\square$

Remark 6.8. (a) Due to the previous proposition, as far as the vertices of simple $F\mathfrak{S}_n$-modules in 3-blocks of 3-weight 3 are concerned, it is sufficient to determine the vertices of simple $F\mathfrak{S}_n$-modules belonging to the respective blocks, for $n \leq 15$. Again, combining branching rules and computational methods, we have been able to do this with one exception, namely the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ of dimension 43497. As a matter of memory request, we have not even been able to construct this module so far. The "smallest" Specht module having a composition factor isomorphic to $D^{(7,5,3)}$ is $S^{(7,5,3)}$ of dimension 45045. Furthermore, the modules $D^{(7,5,2)}$ of dimension 13012 and $D^{(7,5,1^2)}$ of dimension 15444 are the only simple $F\mathfrak{S}_{14}$-modules whose inductions to $\mathfrak{S}_{15}$ have composition factors isomorphic to $D^{(7,5,3)}$. Therefore also the technique of constructing simple modules as constituents of induced simple modules fails here.

(b) Since the simple $F\mathfrak{S}_{15}$-module $D^{(7,5,3)}$ has dimension 43497, the highest 3-power dividing $\dim(D^{(7,5,3)})$ is 3^5. Moreover, $D^{(7,5,3)}$ cannot have cyclic vertices, by Theorem 2.4. Thus we get $|V| \in \{9, 27, 81\}$, for a vertex V of $D^{(7,5,3)}$. For all prime numbers p and $n \in \mathbb{N}$, the $\mathfrak{S}_n$-conjugacy classes of subgroups of $\mathfrak{S}_n$ of order at most p^3 which can possibly occur as vertices of simple $F\mathfrak{S}_n$-modules belonging to a block of given weight $w \geq 0$ have been determined in [6], Proposition 6.1. From those results we deduce that a vertex $V \leq \mathfrak{S}_9$ of $D^{(7,5,3)}$ has to be conjugate to one of the following groups, each of which is unique up to conjugation in $\mathfrak{S}_9$:

- $\mathfrak{P}_9$,
- $(\mathfrak{P}_3)^3$,
- the extraspecial group of order 27 and exponent 3,
- the extraspecial group of order 27 and exponent 9,
- the elementary abelian group $\mathfrak{E}_9$ of order 9, acting regularly on $\{1, \ldots, 9\}$.

(c) The tables below contain our results on simple $F\mathfrak{S}_n$-modules belonging to 3-blocks of weight 3. Here blocks are labelled by their 3-cores and simple modules by the corresponding partitions. Neighbouring partitions are Mullineux conjugate, and the corresponding simple modules have thus the same vertices and sources. In case that a simple $F\mathfrak{S}_n$-module is also a Specht module, the column "Specht" contains the partition of that Specht module. The column "dim." contains the dimensions of the simple modules. We have also included the dimensions of the sources in column "sce". In fact, all sources are determined by their dimensions. Besides the trivial sources and the indecomposable restrictions to $\mathfrak{P}_9$ of the simple modules belonging to the principal block of $F\mathfrak{S}_9$, there are $(\mathfrak{P}_3)^3$-sources of dimension 4 and $\mathfrak{P}_9$-sources of dimension 28. The 4-dimensional sources are isomorphic and in $N_{\mathfrak{S}_n}((\mathfrak{P}_3)^3)$ conjugate to $\mathrm{Res}^{\mathfrak{S}_3 \times \mathfrak{S}_6}_{\mathfrak{P}_3 \times \mathfrak{P}_6}(D^{(3)} \boxtimes D^{(3,2,1)})$. For the sources of dimension 28, we do not have a concrete description.

(d) Looking at the computational data, one observes that vertices of simple $F\mathfrak{S}_n$-modules belonging to blocks of a fixed weight $w \geq p$ tend to become "smaller" as n is growing. Namely, in all our explicit examples, the situation has been as follows: we choose a transversal for the Scopes classes of blocks of weight w such that the representatives are blocks of $F\mathfrak{S}_n$ where n is the least integer for which $F\mathfrak{S}_n$ possesses a block in the considered Scopes class. Then simple modules belonging to a block in the Scopes class represented by the principal block of $F\mathfrak{S}_{pw}$ have vertex $\mathfrak{P}_{pw}$ or $\mathfrak{Q}_{pw}$, whereas the orders of the vertices of simple modules belonging to blocks in the higher Scopes classes become successively smaller.

n	blocks	partitions		dim.	vtx	sce	Specht
9	$\emptyset$	(9)	$(5,4)$	1	$\mathfrak{P}_9$	1	(9) (1^9)
		$(8,1)$	$(4^2,1)$	7	$\mathfrak{P}_9$	7	-
		$(7,1^2)$	$(4,3,2)$	21	$\mathfrak{P}_9$	21	-
		$(6,3)$	$(3^2,2,1)$	41	$\mathfrak{P}_9$	41	-
		$(6,2,1)$	$(5,2^2)$	35	$\mathfrak{P}_9$	35	-

n	blocks		partitions		dim.	vtx	sce	Specht	
10	(1)		(10)	(5^2)	1	P_9	1	(10)	(1^{10})
			$(8,2)$	$(4^2,1^2)$	34	P_9	7	-	
			$(7,3)$	$(4,3,2,1)$	41	P_9	41	-	
			$(7,2,1)$	$(4,3^2)$	84	P_3^3	1	$(7,1^3)$	$(4,1^6)$
			$(6,2,1^2)$	$(5,2^2,1)$	224	P_9	35	-	
11	(2)	(1^2)	(11)	$(6,5)$	1	P_9	1	(11)	(1^{11})
			$(8,3)$	$(4^2,2,1)$	109	P_9	28	-	-
			$(8,2,1)$	$(4^2,3)$	120	P_3^3	1	$(8,1^3)$	$(4,1^7)$
			$(7,3,1)$	$(4,3,2,1^2)$	320	P_9	41	-	-
			$(6,3,1^2)$	$(5,2^2,1^2)$	791	P_9	35	-	-
			$(5^2,1)$	$(10,1)$	10	P_9	1	$(2,1^9)$	$(10,1)$
			$(5,4,1^2)$	$(9,2)$	34	P_9	7	-	-
			$(5,3^2)$	$(7,2^2)$	210	P_3^3	1	$(5,1^6)$	$(7,1^4)$
			$(5,3,2,1)$	$(6,2^2,1)$	714	P_3^3	4	-	-
			$(4,3,2^2)$	$(7,4)$	131	P_9	41	-	-
13	$(3,1)$	$(2,1^2)$	$(12,1)$	$(6^2,1)$	12	P_9	1	$(12,1)$	$(2,1^{11})$
			$(9,4)$	$(5,4,2^2)$	417	P_9	28	-	-
			$(9,2^2)$	$(5,4^2)$	495	P_3^3	1	$(9,1^4)$	$(5,1^8)$
			$(7,4,2)$	$(4,3,2^2,1^2)$	5082	P_9	41	-	-
			$(6,5,2)$	$(11,1^2)$	66	P_9	1	$(3,1^{10})$	$(11,1^2)$
			$(6,4,3)$	$(8,3,2)$	792	P_3^3	1	$(6,1^7)$	$(8,1^5)$
			$(6,4,2,1)$	$(5,3,2^2,1)$	10296	P_3^3	1	$(6,4,1^3)$	$(5,2^3,1^2)$
			$(6,3,2,1^2)$	$(6,3,2,2)$	8568	P_3^3	4	-	-
			$(4^2,2^2,1)$	$(8,4,1)$	1275	P_9	41	-	-
			$(5,4,2,1^2)$	$(9,3,1)$	1299	P_9	7	-	-
14	$(3,1^2)$		$(12,1^2)$	$(6^2,2)$	78	P_9	1	$(12,1^2)$	$(3,1^{11})$
			$(9,4,1)$	$(5,4,2^2,1)$	4290	P_9	28	-	
			$(8,4,2)$	$(4^2,2^2,1^2)$	6357	P_9	41	-	
			$(6,4^2)$	$(9,3,2)$	1287	P_3^3	1	$(6,1^8)$	$(9,1^5)$
			$(6,4,2^2)$	$(6,3,2^2,1)$	28665	P_3^3	1	$(6,4,1^4)$	$(6,2^3,1^2)$
15	$(4,2)$	$(2^2,1^2)$	$(13,2)$	$(7,6,1^2)$	90	P_9	1	$(13,2)$	$(2^2,1^{11})$
			$(10,5)$	$(5^2,3,2)$	1548	P_9	28	-	-
			$(10,2^2,1)$	$(5^2,4,1)$	7722	P_3^3	1	$(10,2,1^3)$	$(5,2,1^8)$
			$(7,5,3)$	$(4,3^2,2^2,1)$	43497			-	-
			$(7,5,2,1)$	$(5,3^2,2^2)$	61425	P_3^3	1	$(7,5,1^3)$	$(5,2^4,1^2)$
			$(7,4,3,1)$	$(8,3^2,1)$	19305	P_3^3	1	$(7,2,1^6)$	$(8,2,1^5)$
			$(7,3^2,1^2)$	$(7,3^2,2)$	32670	P_3^3	4	-	-
			$(6,5,3,1)$	$(11,2,1^2)$	2925	P_9	1	$(4,2,1^9)$	$(11,2,1^2)$
			$(5^2,3,1^2)$	$(10,3,1^2)$	8163	P_9	7	-	-
			$(4^2,3,2^2)$	$(8,5,1^2)$	26937	P_9	41	-	-

$\boxed{n = 19}$

blocks		partitions		dim.	vtx	sce	Specht	
$(5,3,1^2)$	$(4,2^2,1^2)$	$(14,3,1^2)$	$(7^2,3,1^2)$	49248	$\mathfrak{P}_9$	1	$(14,3,1^2)$	$(4,2^2,1^{11})$
		$(11,6,1^2)$	$(6,5,3^2,2)$	630990	$\mathfrak{P}_9$	28	-	-
		$(11,3^2,2)$	$(7,5^2,1^2)$	840294	$\mathfrak{P}_3^3$	1	$(11,3,1^5)$	$(7,2^2,1^8)$
		$(8,6,4,1)$	$(4^2,3^2,2^2,1)$	8892342			-	-
		$(8,6,3,2)$	$(7,4,3^2,2)$	4081428	$\mathfrak{P}_3^3$	1	$(8,6,1^5)$	$(7,2^5,1^2)$
		$(8,5,4,2)$	$(10,4,3,1^2)$	1200420	$\mathfrak{P}_3^3$	1	$(8,3,1^8)$	$(10,2^2,1^5)$
		$(8,3^2,2^2,1)$	$(7,5,3^2,1)$	16325712	$\mathfrak{P}_3^3$	1	$(8,3,2^3,1^2)$	$(7,5,2,1^5)$
		$(7,6,4,2)$	$(13,2^2,1^2)$	160056	$\mathfrak{P}_9$	1	$(5,3,1^{11})$	$(13,2^2,1^2)$
		$(5^2,4,2^2,1)$	$(10,5,2,1^2)$	6642324	$\mathfrak{P}_9$	28	-	-
		$(5,4^2,2^2,1^2)$	$(9,5,3,1^2)$	8113608	$\mathfrak{P}_9$	41	-	-

$\boxed{n = 25}$

blocks	partitions		dim.	vtx	sce	Specht	
$(6,4,2^2,1^2)$	$(15,4,2^2,1^2)$	$(8,7,5,3,1^2)$	981204840	$\mathtt{P}_9$	1	$(15,4,2^2,1^2)$	$(6,4,2^2,1^{11})$
	$(12,7,2^2,1^2)$	$(6^2,5,3^2,2)$	10614852360	$\mathtt{P}_9$	28	-	
	$(12,4^2,3,1^2)$	$(9,6,5,3,1^2)$	8448555960	$\mathtt{P}_3^3$	1	$(12,4,2^2,1^5)$	$(9,4,2^2,1^8)$
	$(9,7,5,2,1^2)$	$(6,4^2,3^2,2^2,1)$	112593255390			-	
	$(9,7,4,3,1^2)$	$(9,4^2,3^2,2)$	34166954250	$\mathtt{P}_3^3$	1	$(9,7,2^2,1^5)$	$(9,4,2^5,1^2)$

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