Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 205-215.

A Note on Secantoptics

Magdalena Skrzypiec

Institute of Mathematics, UMCS
pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland
e-mail: mskrzypiec@hektor.umcs.lublin.pl

Abstract. In this paper we define a certain generalization of isoptic curves. We call the new curves secantoptics, since we use secants to construct them. We extend to secantoptics some properties of isoptics known from [1] and [6]. At the end we discuss other generalizations of isoptics and relations between them.

MSC 2000: 53A04, 53C44
Keywords: secantoptic, isoptic, secant

1. Introduction

Isoptic curves occur in connection with cam mechanisms and were studied by engineers [11]. Therefore it seems to be justified to study some generalizations of isoptics, which might have applications in machinery. Let us remind that an α-isoptic of a closed, convex curve is composed of those points in the plane from which the curve is seen under a fixed angle $\pi - \alpha$. In [6] authors give the equation of isoptic in terms of a support function. They derive that a mapping associated with isoptics has positive jacobian and that it is a diffeomorphism. They give the formula for curvature of isoptics and also they formulate and derive a relation called the sine theorem for isoptics. Now, we want to extend these results for secantoptics.

2. Definition of a secantoptic

Let C be an oval, that is, a closed convex curve of class C^2 with the nonvanishing curvature. We introduce a coordinate system with origin O in the interior of C

and denote by $p(t)$, $t \in [0, 2\pi]$ the support function of the curve C. Then, as was shown in [9], the support function is differentiable and the curve C can be parametrized by

$$z(t) = p(t)e^{it} + p'(t)ie^{it} \quad \text{for} \quad t \in [0, 2\pi]. \tag{2.1}$$

Note that for ovals $p'''(t)$ exists and $p(t) + p''(t) > 0$ for $t \in [0, 2\pi]$.

Let C be an oval and let $\beta \in [0, \pi)$, $\gamma \in [0, \pi - \beta)$ and $\alpha \in (\beta + \gamma, \pi)$ be fixed angles. We take a tangent line $l_1(t)$ to the oval C at a point $z(t)$. We construct a secant line $s_1(t)$ of C rotating $l_1(t)$ about the point $z(t)$ through angle $-\beta$. Let us take another tangent line $l_2(t) = l_1(t+\alpha-\beta-\gamma)$ at the point $z(t+\alpha-\beta-\gamma)$ and let $s_2(t)$ be a secant obtained by rotating $l_2(t)$ about the tangency point through angle γ. Then $s_1(t)$ and $s_2(t)$ intersect each other forming a fixed angle α.

Definition 2.1. *The set of intersection points $z_{\alpha,\beta,\gamma}(t)$ of $s_1(t)$ and $s_2(t)$ for $t \in [0, 2\pi]$ form a curve which we call a secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C.*

Note that the intersection points of tangents $l_1(t)$ and $l_2(t)$ for $t \in [0, 2\pi]$ form the isoptic $C_{\alpha-\beta-\gamma}$ of an oval C. In the general case we can formulate a definition of secantoptic for a curve which is not convex. Moreover, we can use any other parametrization for curve C, but parametrization with the support function is very convenient. In this paper we want to consider only secantoptics of ovals.

Consider two triangles, T_1 with vertices $z(t)$, $z(t+\alpha-\beta-\gamma)$ and $z_{\alpha-\beta-\gamma}(t)$ and T_2 with vertices $z(t)$, $z(t+\alpha-\beta-\gamma)$ and $z_{\alpha,\beta,\gamma}(t)$. The segment $[z(t), z(t+\alpha-\beta-\gamma)]$ forms a common side of triangles T_1 and T_2. The point $z_{\alpha-\beta-\gamma}(t)$ lies in the interior of T_2 for $t \in [0, 2\pi]$, since $0 < \pi - \alpha \leq \pi - \alpha + \beta + \gamma \leq \pi$. Hence the isoptic $C_{\alpha-\beta-\gamma}$ lies in the interior of the secantoptic $C_{\alpha,\beta,\gamma}$. All isoptics of a curve C lie in the exterior of C, so the secantoptic $C_{\alpha,\beta,\gamma}$ lies in the exterior of C, too.

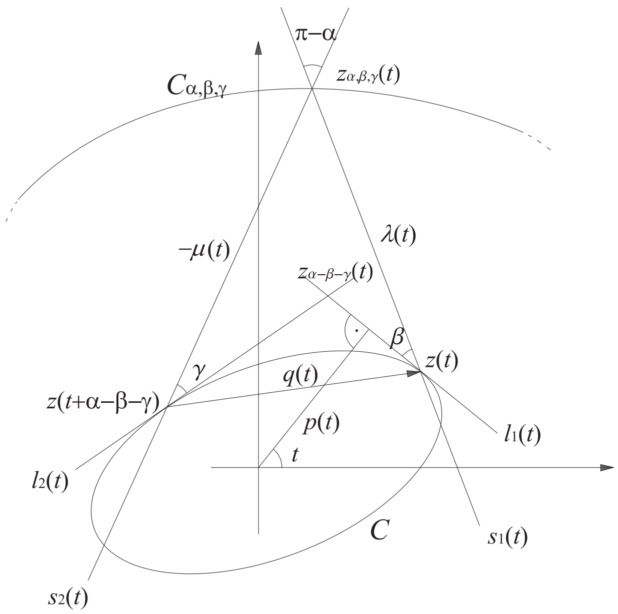

Figure 1.

To obtain an equation of a secantoptic we use the same method as authors [6] for isoptics. Consider the following vector

$$q(t) = z(t) - z(t + \alpha - \beta - \gamma), \tag{2.2}$$

which in terms of the support function may be written as

$$\begin{aligned} q(t) = (p(t) - p(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma) + p'(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma) \\ +i(p'(t) - p(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma) - p'(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma)))e^{it}. \end{aligned}$$

We introduce additional notations to simplify calculations

$$\begin{aligned} b(t) &= p(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma) + p'(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma) - p'(t), \\ B(t) &= p(t) - p(t+\alpha-\beta-\gamma)\cos(\alpha-\beta-\gamma) + p'(t+\alpha-\beta-\gamma)\sin(\alpha-\beta-\gamma), \end{aligned}$$

where $[v, w] = ad - bc$ when $v = a + ib$ and $w = c + id$. Thus we have

$$q(t) = (B(t) - ib(t))e^{it}.$$

The equation of secantoptic $C_{\alpha,\beta,\gamma}$ of the curve C can be derived from the formula

$$z_{\alpha,\beta,\gamma}(t) = z(t) + \lambda(t)ie^{i(t-\beta)} = z(t+\alpha-\beta-\gamma) + \mu(t)ie^{i(t+\alpha-\beta)}, \tag{2.3}$$

where $\lambda(t)$ and $-\mu(t)$ are the segments of secants (Figure 1) and may be written as

$$\lambda(t) = \frac{b(t)\sin(\alpha-\beta) - B(t)\cos(\alpha-\beta)}{\sin\alpha}, \tag{2.4}$$

$$\mu(t) = \frac{-(b(t)\sin\beta + B(t)\cos\beta)}{\sin\alpha}. \tag{2.5}$$

The equation of secantoptic in terms of the support function is then

$$z_{\alpha,\beta,\gamma}(t) = (p(t) + \lambda(t)\sin\beta + i(p'(t) + \lambda(t)\cos\beta))e^{it}, \tag{2.6}$$

where

$$\begin{aligned} \lambda(t) = {} & \frac{1}{\sin\alpha}(p(t+\alpha-\beta-\gamma)\cos\gamma + p'(t+\alpha-\beta-\gamma)\sin\gamma - p'(t)\sin(\alpha-\beta) \\ & -p(t)\cos(\alpha-\beta)). \end{aligned}$$

Note that all secantoptics of a curve C for a fixed $\beta \in [0, \pi)$, fixed $\gamma \in [0, \pi - \beta)$, and various $\alpha \in (\beta + \gamma, \pi)$ form two parameters family of curves $F_{\beta,\gamma}(\alpha, t)$. Note, that if we take $\beta = 0$ and $\gamma = 0$, then we get isoptics. At the Figure 2 one can see geometric shapes of secantoptics for a curve with $p(t) = a + b\cos 3t$, where $b > 0$ and $a > 8b$.

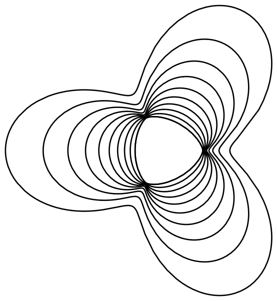

Figure 2.

3. Jacobian and curvature

Let C be a fixed oval, $t \mapsto z(t)$ for $t \in [0, 2\pi]$. We denote by $e(C)$ the exterior of C and by $\zeta(\alpha)$ a set of points $z_{\alpha,\beta,\gamma}(0)$ for $\alpha \in (\beta+\gamma, \pi)$ and a fixed β and γ. We define a mapping

$$F_{\beta,\gamma} : (\beta+\gamma, \pi) \times (0, 2\pi) \mapsto e(C) \setminus \zeta$$

for secantoptics $F_{\beta,\gamma}(\alpha, t)$.

We are going to determine partial derivatives of $F_{\beta,\gamma}$ at (α, t)

$$\frac{\partial F_{\beta,\gamma}}{\partial \alpha} = \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t))(\sin\beta + i\cos\beta)e^{it}, \tag{3.1}$$

$$\begin{aligned}\frac{\partial F_{\beta,\gamma}}{\partial t} = \frac{1}{\sin\alpha}&((R(t+\alpha-\beta-\gamma)\sin\beta\sin\gamma - R(t)\sin(\alpha-\beta)\sin\beta \\ &+ B(t)\cos(\alpha-2\beta) - b(t)\sin(\alpha-2\beta)) + i(R(t)\cos(\alpha-\beta)\sin\beta \\ &+ R(t+\alpha-\beta-\gamma)\sin\gamma\cos\beta + b(t)\cos(\alpha-2\beta) \\ &+ B(t)\sin(\alpha-2\beta)))e^{it}.\end{aligned} \tag{3.2}$$

Now, we can compute jacobian $J(F_{\beta,\gamma})$ of $F_{\beta,\gamma}$ at (α, t)

$$J(F_{\beta,\gamma}) = \frac{1}{\sin\alpha}(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t))(R(t)\sin\beta + \lambda(t)) > 0. \tag{3.3}$$

Expressions $R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t)$ and $R(t)\sin\beta + \lambda(t)$ which we obtained in the jacobian seem to be interesting for us. There is the osculating circle of radius $R(t)$ at every point $z(t)$ of oval C. If we elongate the segment $[z_{\alpha,\beta,\gamma}(t), z(t)]$ at the orthogonal projection of radius of osculating circle at $z(t)$ on line $s_1(t)$ we obtain the segment of secant $s_1(t)$ which length is $R(t)\sin\beta + \lambda(t)$. Similarly by elongation the segment $[z_{\alpha,\beta,\gamma}(t), z(t+\alpha-\beta-\gamma)]$ at the orthogonal projection of radius of osculating circle at $z(t+\alpha-\beta-\gamma)$ on line $s_2(t)$ we get the segment of length $R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t)$. We define a vector

$$\begin{aligned}Q(t) &= -(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t))ie^{i(t+\alpha-\beta)} - (R(t)\sin\beta + \lambda(t))ie^{i(t-\beta)} \\ &= (B(t) + R(t+\alpha-\beta-\gamma)\sin\gamma\sin(\alpha-\beta) - R(t)\sin^2\beta + i(-b(t) \\ &\quad -R(t+\alpha-\beta-\gamma)\sin\gamma\cos(\alpha-\beta) - R(t)\sin\beta\cos\beta))e^{it}\end{aligned} \tag{3.4}$$

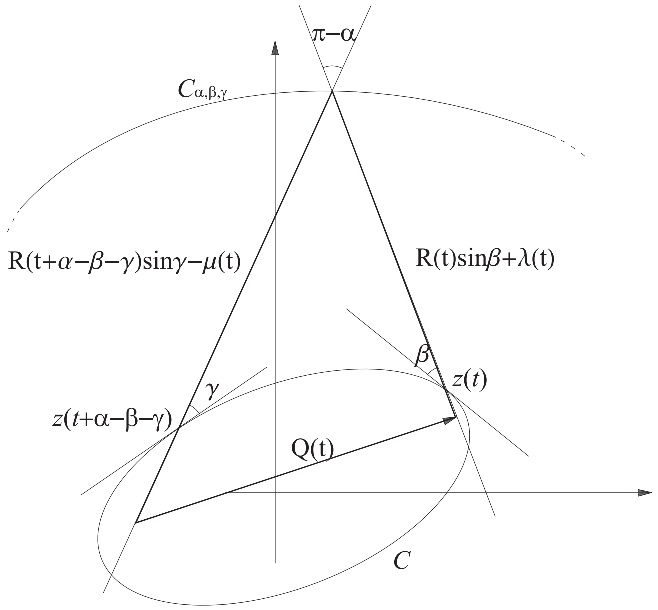

Figure 3.

which will be very useful to simplify calculations.

We are looking for a geometric interpretation of the vector $Q(t)$. Consider the envelope Γ_1 of the set of lines $s_1(t)$, so-called "evolutoid" [3], [5] of C. We want to give an interpretation of Γ_1 in terms of the starting curve. Consider a parametrization

$$\Gamma_1 : \quad z_1(t) = \Psi_1(t)e^{it} + \Psi_1'(t)ie^{it},$$

where

$$\Psi_1(t) = p(t+\beta)\cos\beta - p'(t+\beta)\sin\beta, \quad t \in [0, 2\pi],$$

and $p(t)$ is the support function of C. Note that for $\beta = \frac{\pi}{2}$ this evolutoid is the evolute of C. Similarly we can define Γ_2 as the envelope of the set of lines $s_2(t)$. Then we obtain

$$\Gamma_2 : \quad z_2(t) = \Psi_2(t)e^{it} + \Psi_2'(t)ie^{it},$$

where

$$\Psi_2(t) = p(t-\gamma)\cos\gamma + p'(t-\gamma)\sin\gamma, \quad t \in [0, 2\pi].$$

We have two curves Γ_1 and Γ_2. Now we construct an α-isoptic of a pair of these envelopes. Consider a vector

$$q_1(t) = z_1(t) - z_2(t+\alpha)$$

and notice that for argument $t - \beta$ we obtain

$$q_1(t-\beta) = z_1(t-\beta) - z_2(t+\alpha-\beta) = Q(t).$$

We introduce the following notations:

$$L_1(t-\beta) = \lambda(t) + R(t)\sin\beta,$$
$$M_1(t-\beta) = -(R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t)),$$

where

$$q_1(t-\beta) = M_1(t-\beta)ie^{i(t+\alpha-\beta)} - L_1(t-\beta)ie^{i(t-\beta)}.$$

Therefore the equation of an isoptic of a pair of Γ_1 and Γ_2 has the form

$$\begin{aligned} z_e(t-\beta) &= z(t-\beta) + L_1(t-\beta)ie^{i(t-\beta)} \\ &= e^{it}(p(t) + \lambda(t)\sin\beta + i(p'(t) + \lambda(t)\cos\beta)). \end{aligned}$$

Hence

$$z_e(t-\beta) = z_{\alpha,\beta,\gamma}(t).$$

Therefore we can call a "secantoptic" an "isoptic curve of a pair of, generally different, evolutoids of C". We are going to answer the question, whether the secantoptics are regular curves. Consider a tangent vector to secantoptic

$$\begin{aligned} \left|z'_{\alpha,\beta,\gamma}(t)\right| = \frac{1}{\sin\alpha}(& B^2(t) + b^2(t) + 2R(t+\alpha-\beta-\gamma)\rho(t)\sin\alpha\sin\gamma \\ & +2R(t)\eta(t)\sin\alpha\sin\beta + R^2(t+\alpha-\beta-\gamma)\sin^2\gamma \\ & +2R(t+\alpha-\beta-\gamma)R(t)\sin\beta\sin\gamma\cos\alpha + R^2(t)\sin^2\beta)^{1/2}, \end{aligned}$$

where

$$\rho(t) = \frac{1}{\sin\alpha}(B(t)\sin(\alpha-\beta) + b(t)\cos(\alpha-\beta)), \tag{3.5}$$

$$\eta(t) = \frac{1}{\sin\alpha}(b(t)\cos\beta - B(t)\sin\beta). \tag{3.6}$$

Since

$$\begin{aligned} |Q(t)|^2 = & B^2(t) + b^2(t) + 2R(t+\alpha-\beta-\gamma)\rho(t)\sin\alpha\sin\gamma \\ & +2R(t)\eta(t)\sin\alpha\sin\beta + R^2(t+\alpha-\beta-\gamma)\sin^2\gamma \\ & +2R(t+\alpha-\beta-\gamma)R(t)\sin\beta\sin\gamma\cos\alpha + R^2(t)\sin^2\beta \neq 0, \end{aligned}$$

then we obtain

$$\left|z'_{\alpha\beta}(t)\right| = \frac{|Q(t)|}{\sin\alpha}. \tag{3.7}$$

Corollary 3.1. *Secantoptics $C_{\alpha,\beta,\gamma}$ of an oval C for $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$ and $\alpha \in (\beta+\gamma,\pi)$ are regular curves.*

We are interested in the curvature of a secantoptic. Since we assumed that C is an oval, $R'(t)$ exists for $t \in [0,2\pi]$. We calculate the curvature from the formula

$$\kappa(t) = \frac{[z'_{\alpha,\beta,\gamma}(t), z''_{\alpha,\beta,\gamma}(t)]}{|z'_{\alpha,\beta,\gamma}(t)|^3}.$$

The numerator of this formula has the form

$$\begin{aligned} [z'_{\alpha\beta}(t), z''_{\alpha\beta}(t)] = \frac{1}{\sin^2\alpha}(& 2|q(t)|^2 - [q(t),q'(t)] + \sin\beta\sin\alpha(3R(t)\eta(t) - R'(t)\mu(t)) \\ & + \sin\gamma\sin\alpha(3R(t+\alpha-\beta-\gamma)\rho(t) - R'(t+\alpha-\beta-\gamma)\lambda(t)) \\ & +2R(t)R(t+\alpha-\beta-\gamma)(2\cos\alpha\sin\beta\sin\gamma - \sin\beta\sin(\alpha-\gamma) \\ & - \sin\gamma\sin(\alpha-\beta)) + \sin\alpha\sin\beta\sin\gamma(R(t+\alpha-\beta-\gamma)R'(t) \\ & -R(t)R'(t+\alpha-\beta-\gamma)) + 2(R^2(t+\alpha-\beta-\gamma)\sin^2\gamma \\ & +R(t)R(t+\alpha-\beta-\gamma)\sin\beta\sin\gamma\cos\alpha + R^2(t)\sin^2\beta). \end{aligned}$$

Notice that

$$\begin{aligned}[Q(t),Q'(t)] = {} & [q(t),q'(t)] + R(t)R(t+\alpha-\beta-\gamma)\sin\alpha\sin(\beta+\gamma)\\ & +\sin\beta\sin\alpha(R'(t)\mu(t)+R(t)\eta(t)) + \sin\gamma\sin\alpha(R'(t+\alpha-\beta-\gamma)\lambda(t)\\ & +R(t+\alpha-\beta-\gamma)\rho(t)) + (R(t)R'(t+\alpha-\beta-\gamma)\\ & +R(t+\alpha-\beta-\gamma)R'(t))\sin\alpha\sin\beta\sin\gamma,\end{aligned}$$

therefore the formula for the curvature may be written as

$$\kappa(t) = \frac{\sin\alpha}{|Q(t)|^3}(2|Q(t)|^2 - [Q(t),Q'(t)]).$$

Hence we have a condition for convexity of secantoptics.

Theorem 3.2. *A secantoptic $C_{\alpha,\beta,\gamma}$ of an oval C is convex if and only if*

$$[Q(t),Q'(t)] \le 2|Q(t)|^2 \quad \text{for} \quad t\in[0,2\pi]. \tag{3.8}$$

4. Sine theorem for secantoptics

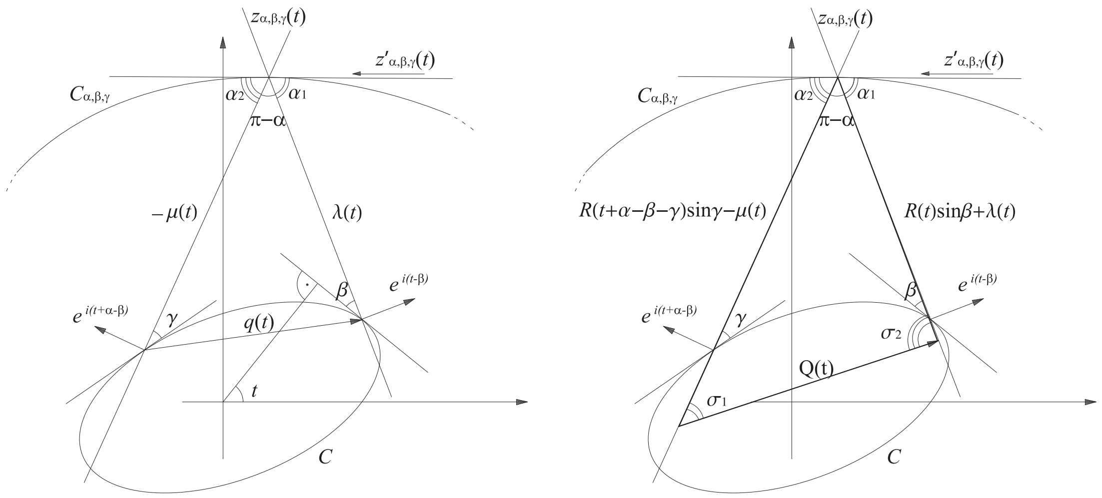

Figure 4.

The sine theorem for isoptics is known from [1] and [6]. We extend it now for secantoptics using the method which is presented in [6]. Consider a tangent line to secantoptic $C_{\alpha,\beta,\gamma}$ at $z_{\alpha,\beta,\gamma}(t)$. Let α_1 and α_2 be angles as at Figure 4. It is clear that

$$\sin\alpha_1 = \frac{-[z'_{\alpha,\beta,\gamma}(t), ie^{i(t-\beta)}]}{|z'_{\alpha,\beta,\gamma}(t)|},$$

where

$$[z'_{\alpha,\beta,\gamma}(t), ie^{i(t-\beta)}] = -(\lambda(t)+R(t)\sin\beta)$$

and

$$\sin\alpha_2 = \frac{[z'_{\alpha,\beta,\gamma}(t), ie^{i(t+\alpha-\beta)}]}{|z'_{\alpha,\beta,\gamma}(t)|},$$

where

$$[z'_{\alpha,\beta,\gamma}(t), ie^{i(t+\alpha-\beta)}] = R(t+\alpha-\beta-\gamma)\sin\gamma - \mu(t).$$

Since we know that

$$|z'_{\alpha,\beta,\gamma}(t)| = \frac{|Q(t)|}{\sin\alpha},$$

then we obtain

$$\frac{|Q(t)|}{\sin\alpha} = \frac{\lambda(t)+R(t)\sin\beta}{\sin\alpha_1} \quad \text{and} \quad \frac{|Q(t)|}{\sin\alpha} = \frac{R(t+\alpha-\beta-\gamma)\sin\gamma-\mu(t)}{\sin\alpha_2}.$$

Thus we have the following theorem.

Theorem 4.1. *Secantoptics have the following property*

$$\frac{|Q(t)|}{\sin\alpha} = \frac{\lambda(t)+R(t)\sin\beta}{\sin\alpha_1} = \frac{R(t+\alpha-\beta-\gamma)\sin\gamma-\mu(t)}{\sin\alpha_2}.$$

Using the classical sine theorem for triangle we obtain

$$\alpha_1 = \sigma_1 \quad \text{and} \quad \alpha_2 = \sigma_2.$$

5. Other properties of secantoptics

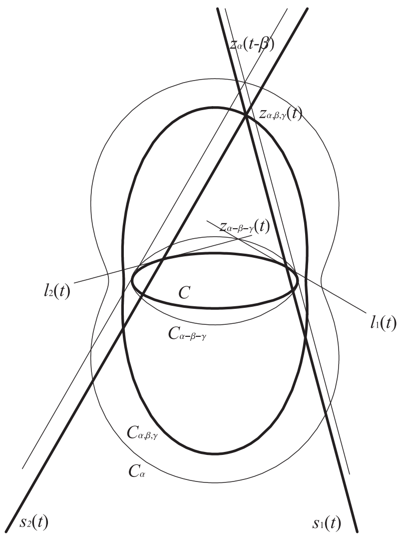

Figure 5.

Let us consider secants $s_1(t)$ and $s_2(t)$ to oval C as in definition of secantoptic. They are parallel to tangent lines to C at $z(t-\beta)$ and $z(t+\alpha-\beta)$, respectively. In this way we obtain a point $z_\alpha(t-\beta)$ on the isoptic C_α (Figure 5). The assumption $\beta \in [0,\pi)$, $\gamma \in [0,\pi-\beta)$ and $\alpha \in (\beta+\gamma,\pi)$ guarantees that the isoptic $C_{\alpha-\beta-\gamma}$ exists.

Corollary 5.1. *There are two isoptic curves C_α and $C_{\alpha-\beta-\gamma}$ connected to each secantoptic $C_{\alpha,\beta,\gamma}$ of oval C.*

Another property of secantoptics presents the following theorem.

Theorem 5.2. *Let $z(t) = p(t)e^{it} + p'(t)ie^{it}$ for $t \in [0, 2\pi]$ be an oval. Let $t \mapsto z_{\alpha,\beta,\gamma}(t)$ be its secantoptic defined in (2.6). Then*

$$\int_0^{2\pi} z(t)dt = \int_0^{2\pi} z_{\alpha,\beta,\gamma}(t)dt. \tag{5.1}$$

Proof. It is known [6] that

$$\int_0^{2\pi} z(t)dt = 2\int_0^{2\pi} p(t)e^{it}dt.$$

Consider the right side of our hypothesis

$$\begin{aligned}\int_0^{2\pi} z_{\alpha,\beta,\gamma}(t)dt = &\int_0^{2\pi} z(t)dt + \frac{\sin\beta}{\sin\alpha}(\cos\gamma\int_0^{2\pi} p(t+\alpha-\beta-\gamma)e^{it}dt\\ &+ \sin\gamma\int_0^{2\pi} p'(t+\alpha-\beta-\gamma)e^{it}dt - \sin(\alpha-\beta)\int_0^{2\pi} p'(t)e^{it}dt\\ &- \cos(\alpha-\beta)\int_0^{2\pi} p(t)e^{it}dt) + i\frac{\cos\beta}{\sin\alpha}(\cos\gamma\int_0^{2\pi} p(t+\alpha-\beta-\gamma)e^{it}dt\\ &+ \sin\gamma\int_0^{2\pi} p'(t+\alpha-\beta-\gamma)e^{it}dt - \sin(\alpha-\beta)\int_0^{2\pi} p'(t)e^{it}dt\\ &- \cos(\alpha-\beta)\int_0^{2\pi} p(t)e^{it}dt).\end{aligned}$$

By calculating the integrals we obtain

$$\begin{aligned}\int_0^{2\pi} p(t+\alpha-\beta-\gamma)e^{it}dt &= (\cos(\alpha-\beta-\gamma) - i\sin(\alpha-\beta-\gamma))\int_0^{2\pi} p(t)e^{it}dt,\\ \int_0^{2\pi} p'(t+\alpha-\beta-\gamma)e^{it}dt &= -(\sin(\alpha-\beta-\gamma) + i\cos(\alpha-\beta-\gamma))\int_0^{2\pi} p(t)e^{it}dt,\\ \int_0^{2\pi} p'(t)e^{it}dt &= -i\int_0^{2\pi} p(t)e^{it}dt.\end{aligned}$$

Finally we have

$$\int_0^{2\pi} z_{\alpha,\beta,\gamma}(t)dt = \int_0^{2\pi} z(t)dt. \qquad \square$$

Let us make a note about other generalizations of isoptics. Consider the tangent $l_1(t)$ to oval C at $z(t)$. We want to construct a secant $s_1(t)$ to C which makes a fixed angle β with $l_1(t)$. We can do it in three ways. First, as we did for secantoptics, we rotate $l_1(t)$ through angle $-\beta$, second, we rotate $l_1(t)$ through angle β and third, we rotate $l_1(t)$ through angle $\pi - \beta$. The secant $s_2(t)$ we obtain by rotation $l_2(t)$ through angle γ, $-\gamma$ or $-(\pi - \gamma)$, respectively. Notice that for $\beta \in [-\pi, \pi]$ and $\gamma \in [-\pi, \pi]$ we get the same family of curves for these three cases. The second and third case we can formulate in terms of secantoptics. Let $F_{\beta,\gamma}(\alpha, t) = F(t, \alpha, \beta, \gamma)$ in this section. Then

$$F_2(t, \alpha, \beta, \gamma) = F(t, \alpha, -\beta, -\gamma) \quad \text{and} \quad F_3(t, \alpha, \beta, \gamma) = F(t, \alpha, \pi - \beta, \pi - \gamma).$$

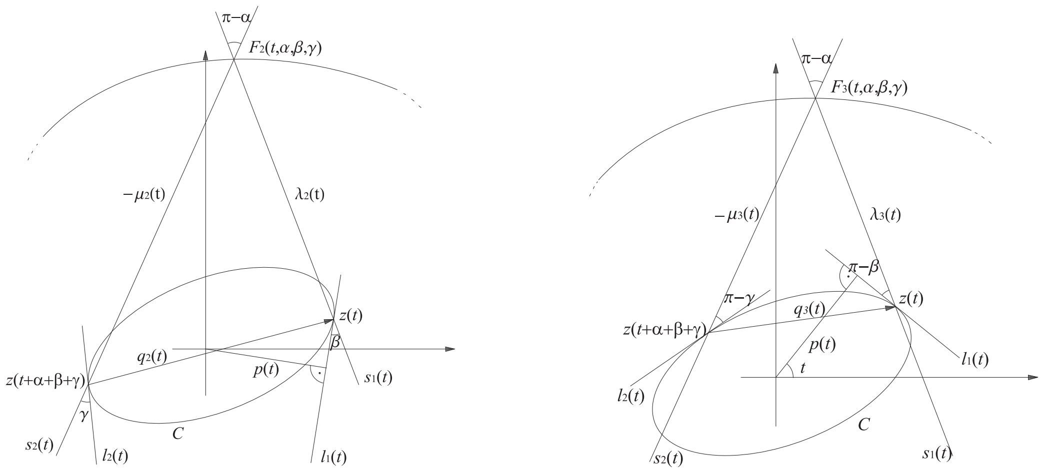

Figure 6. a) A construction of the second case b) a construction of the third case

Unfortunately, not all obtained curves have nice shapes and properties. We want to obtain positive jacobian of mapping $F_{\beta,\gamma}(\alpha, t)$, given by (3.3). Therefore, for the first case we assume that $\beta \in [0, \pi)$ and $\gamma \in [0, \pi - \beta)$, for the second that $\beta \in (-\pi, 0]$ and $\gamma \in (\beta - \pi, 0]$ and for the third that $\beta \in (0, \pi]$ and $\gamma \in (\beta, \pi]$.

The author would like to thank the referee for his remarks which improved essentially this paper.

References

[1] Benko, K.; Cieślak, W.; Góźdź, S.; Mozgawa, W.: *On isoptic curves*. An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **36**(1) (1990), 47–54. Zbl 0725.52002

[2] Braude; L.: *Über einige Transformationen ebener Kurven*. Monatsh. Math. u. Phys. **26**(1) (1915), 199–210. JFM 45.0830.04

[3] Braude, L.: *Über einige Verallgemeinerungen des Begriffes der Evolutoïde*. Archiv der Math. und Physik **20**(3) (1912), 44–52. JFM 43.0653.03

[4] Ferréol, R.: *Courbe isoptique.* From Mathcurve – Encyclopédie des formes mathématiques remarquables.
http://www.mathcurve.com/courbes2d/isoptic/isoptic.shtml

[5] Ferréol, R.: *Développoïde d'une courbe plane.* From Mathcurve–Encyclopédie des formes mathématiques remarquables.
http://www.mathcurve.com/courbes2d/developpee/developpoide.shtml

[6] Cieślak, W.; Miernowski, A.; Mozgawa, W.: *Isoptics of a closed strictly convex curve.* Lect. Notes Math. **1481**, 28–35. Springer, Berlin 1991. Zbl 0739.53001

[7] Hilton, H.; Colomb, R. E.: *On Orthoptic and Isoptic Loci.* American J. **39** (1917), 86–94. JFM 46.0954.02

[8] Inzinger, R.: *Über die Evolutoiden und Zwischenevolutoiden von Raumkurven.* Monatsh. Math. Phys., **42** (1935), 57–84. Zbl 0011.32201

[9] Santalo, L.: *Integral geometry and geometric probability.* Encyclopedia of Mathematics and its Applications **1**, Section: Probability, Reading, Mass. 1976. Zbl 0342.53049

[10] Weisstein, E. W.: *Isoptic Curve.* From MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/IsopticCurve.html

[11] Wunderlich, W.: *Contributions to the geometry of cam mechanisms with oscillating followers,* Journal of Mechanisms **6**(1) (1971), 1–20.

Received December 12, 2006