Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 71-96.

Areas of Certain Polygons in Connection with Determinants of Rectangular Matrices

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Abstract. This article can be considered as an appendix to the article [2]. Here we have some results concerning areas of certain polygons.

MSC 2000: 51E12, 51M044
Keywords: determinant of rectangular matrix, polygon, area of polygon

1. Introduction

In [1] the following definition of determinants of rectangular matrices is given:
The determinant of $m \times n$ matrix A with columns $A_1, \dots, A_n$ and $m \le n$ is the sum

$$\sum_{1 \le j_1 < j_2 < \cdots < j_m \le n} (-1)^{r+s} |A_{j_1}, \dots, A_{j_m}|, \tag{1.1}$$

where $r = 1 + \cdots + m$, $s = j_1 + \cdots + j_m$.

It is clear that every real $m \times n$ matrix $A = [A_1, \dots, A_n]$ determines a polygon in $\mathbb{R}^m$ (the columns of the matrix correspond to the vertices of the polygon) and vice versa. The polygon which corresponds to the matrix $[A_1, \dots, A_n]$ will be denoted by $A_1 \dots A_n$.

In the following we shall restrict ourselves to the case when $m = 2$ (polygons in $\mathbb{R}^2$). Here we list some results given in [2] which will be used.

Theorem 3. *Let $A_1 \dots A_n$ be a polgon in $\mathbb{R}^2$. Then*

$$2 \text{ area of } A_1 \dots A_n = |A_1 + A_2, A_2 + A_3, \dots, A_{n-1} + A_n, A_n + A_1|. \tag{1.2}$$

Corollary 6.1. *If n is odd, then for every point X in $\mathbb{R}^2$ it holds*

$$|A_1 + X, \ldots, A_n + X| = |A_1, \ldots, A_n|. \tag{1.3}$$

Theorem 7. *Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ and let n be an even integer. Then for every point X in $\mathbb{R}^2$ it holds*

$$|A_1 + X, A_2 + X, \ldots, A_n + X| = |A_1, A_2, \ldots, A_n| \tag{1.4}$$

only if $\sum_{i=1}^{n}(-1)^i A_i = 0$.

Corollary 7.1. *It holds*

$$|A_1 + A_2 + X, \ldots, A_n + A_1 + X| = |A_1 + A_2, \ldots, A_n + A_1|, \tag{1.5}$$

where $\sum_{i=1}^{n}(-1)^i A_i = 0$ need not be fulfilled. (It is enough for n to be even.)

Namely, if n is even, then $\sum_{i=1}^{n}(-1)^i (A_i + A_{i+1}) = 0$.

Of course, (1.5) also holds if n is odd since holds (1.3) where now instead of A_i we have $A_i + A_{i+1}$.

Theorem 8. *Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ and let $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then*

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_{n-1}|. \tag{1.6}$$

Theorem 9. *Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ with even n and let $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then*

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_k| + |A_{k+1}, \ldots, A_n|, \tag{1.7}$$

where k may be any integer such that $1 < k < n$.

Corollary 10.1. *If n is odd then for each $i \in \{1, \ldots, n\}$ holds (cyclic)*

$$|A_i, \ldots, A_n, A_1, \ldots, A_{i-1}| = |A_1, \ldots, A_n|. \tag{1.8}$$

If n is even then above relation holds if $\sum_{i=1}^{n}(-1)^i A_i = 0$.
Thus, in both cases when n is odd or when n is even, it holds

$$|A_i + A_{i+1}, \ldots, A_n + A_1, A_1 + A_2, \ldots, A_{i-1} + A_i| = |A_1 + A_2, \ldots, A_n + A_1|. \tag{1.9}$$

Here in this article, we shall mostly deal with k-outscribed polygons where we have the following definition D_k (definition for k-outscribed). This is the fundamental definition for this article.

Let $A_1 \dots A_n$ be any given polygon in $\mathbb{R}^2$ and let k be any given positive integer such that $k < n$. Then polygon $P_1 \dots P_n$, if such exists, will be called k-outscribed polygon to the polygon $A_1 \dots A_n$ if

$$\left.\begin{aligned} P_1 + P_2 + \cdots + P_k &= kA_1, \\ P_2 + P_3 + \cdots + P_{k+1} &= kA_2, \\ &\vdots \\ P_n + P_1 + \cdots + P_{k-1} &= kA_n. \end{aligned}\right\} \qquad (D_k)$$

In other words, if the center of gravity of the vertices $P_i, P_{i+1}, \dots, P_{i+k-1}$ is the vertex A_i.

Remark 1. In the proofs of some of the following theorems we shall for convenience (simplicity writing and complete analogy) first consider an example and then state the main general facts from which will be clear that considered theorem holds good. In this way we also attain the proof be relatively easy to read.

2. Areas of k-outscribed polygons

First we prove the following theorem.

Theorem 1. *Let $A_1 \dots A_n$ be any given polygon in $\mathbb{R}^2$ and let k be any given positive integer such that $k < n$ and $GCD(k, n) = 1$. Then there exists unique k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ and it holds*

$$2 \text{ area of } P_1 \dots P_n = k^2 |B_1 + B_2, B_2 + B_3, \dots, B_n + B_1|, \tag{2.1}$$

where

$$B_i = A_i + A_{i+k} + \cdots + A_{i+(x_k-1)k}, \ i = 1, \dots, n \tag{2.2}$$

and x_k is the least positive integer x such that

$$kx = 1 \pmod{n}. \tag{2.3}$$

(Of course, indices are calculated modulo n.)

Proof. First let us remark that from the equations (D_k) follows

$$P_1 + \cdots + P_n = A_1 + \cdots + A_n. \tag{2.4}$$

It is convenient to consider one example first, say, that where $n = 11$ and $k = 5$. In this case we have the following nine equations

$$P_1 + P_2 + P_3 + P_4 + P_5 = 5A_1, \tag{2.5}$$

$$\begin{aligned}
P_6 + P_7 + P_8 + P_9 + P_{10} &= 5A_6,\\
P_{11} + P_1 + P_2 + P_3 + P_4 &= 5A_{11},\\
P_5 + P_6 + P_7 + P_8 + P_9 &= 5A_5,\\
P_{10} + P_{11} + P_1 + P_2 + P_3 &= 5A_{10},\\
P_4 + P_5 + P_6 + P_7 + P_8 &= 5A_4,\\
P_9 + P_{10} + P_{11} + P_1 + P_2 &= 5A_9,\\
P_3 + P_4 + P_5 + P_6 + P_7 &= 5A_3,
\end{aligned}$$

$$P_8 + P_9 + P_{10} + P_{11} + P_1 = 5A_8, \tag{2.6}$$

or briefly written

$$\sum_{i=1}^{5} P_i = 5A_1, \quad \sum_{i=1+5}^{2\cdot 5} P_i = 5A_6, \quad \ldots, \quad \sum_{i=1+8\cdot 5}^{9\cdot 5} P_i = 5A_8 \tag{2.7}$$

where the sums $1+5$, $1+2\cdot 5$, …, $1+8\cdot 5$ and the products $2\cdot 5$, $3\cdot 5$,…, $9\cdot 5$ are calculated modulo 11. By adding up of the above nine equations we get

$$4(P_1 + P_2 + \cdots + P_{11}) + P_1 = 5(A_1 + A_6 + A_{11} + A_5 + A_{10} + A_4 + A_9 + A_3 + A_8),$$

from which follows

$$P_1 = -4S + 5(A_1 + A_6 + A_{11} + A_5 + A_{10} + A_4 + A_9 + A_3 + A_8),$$

where

$$S = A_1 + A_2 + \cdots + A_{11} \tag{2.8}$$

since holds $P_1 + \cdots + P_{11} = A_1 + \cdots + A_{11}$.

In this connection let us remark that the left-hand side of the equation (2.5) begins with P_1 and that left-hand side of the equation (2.6) ends with P_1.

If we begin with the equation

$$P_2 + P_3 + P_4 + P_5 + P_6 = 5A_2,$$

where now P_2 is the first, in the end the ninth equation will be

$$P_9 + P_{10} + P_{11} + P_1 + P_2 = 5A_9.$$

From so obtained nine equations

$$\sum_{i=2}^{1+5} P_i = 5A_2, \quad \sum_{i=2+5}^{1+2\cdot 5} P_i = 5A_7, \quad \ldots, \quad \sum_{i=2+8\cdot 5}^{1+9\cdot 5} P_i = 5A_9$$

by adding up, we get

$$P_2 = -4S + 5(A_2 + A_7 + A_1 + A_6 + A_{11} + A_5 + A_{10} + A_4 + A_9),$$

where S is given by (2.8).

In the same way can be get P_3, P_4 and so on. Thus, we have

$$P_i = -4S + 5(A_i + A_{i+5} + A_{i+2\cdot 5} + \cdots + A_{i+8\cdot 5}),\ i = 1, \ldots, 11. \tag{2.9}$$

It can be seen that $x_5 = 9$ is the least positive solution of the equation

$$5x = 1 \,(\mathrm{mod}\ 11). \tag{2.10}$$

So for each $P_i, i = 1, \ldots, 11$, there are 9 equations which completely determine P_i.

In this connection let us remark that for the least positive solution x_5 of the equation (2.10) there is positive integer y_5 such that

$$5x_5 - 11y_5 = 1. \tag{2.11}$$

It is easy to see that instead of $-4S$ in the relation (2.9) can be written $-y_5S$. Thus, the equation (2.11) is closely connected with P_i given by (2.9).

It is not difficult to see that analogously holds generally for the case when $GCD(k, n) = 1$. Namely, we have Diofant's equation

$$kx - ny = 1, \tag{2.12}$$

and it holds: If x_k is the least positive integer such that holds (2.3) and y_k is such that

$$kx_k - ny_k = 1, \tag{2.13}$$

then the solution of the system

$$P_i + P_{i+1} + \cdots + P_{i+k-1} = kA_i, \quad i = 1, \ldots, n \tag{2.14}$$

is given by

$$P_i = -y_kS + k(A_i + A_{i+k} + A_{i+2k} + \cdots + A_{i+(x_k-1)k}), \quad i = 1, \ldots, n \tag{2.15}$$

where

$$S = A_1 + \cdots + A_n. \tag{2.16}$$

In this connection let us remark that, for example, instead of equations given by (2.7) we have the following equations

$$\sum_{i=1}^{k} P_i = k \cdot A_1, \quad \sum_{i=1+k}^{2k} P_i = k \cdot A_{1+k}, \ldots, \quad \sum_{i=1+(x_k-1)k}^{k\cdot x_k} P_i = k \cdot A_{1+(x_k-1)k}.$$

Thus, the solving of the system (2.14) reduces in fact to the solving of the Diofant's equation (2.12). So, if $n = 11$, k=5, we have Diofant's equation (2.11) whose solution is given by

$$x = 11u - 2, \qquad y = 5u - 1$$

where $u \in Z$. For $u = 0$ we have $x_5 = 9$, $y_5 = 4$.

Now, when we have $P_1, \dots, P_n$ given by (2.15), we can use property (1.3) or property (1.5). Namely, if we take $X = 2y_k S$ we can write

$$\begin{aligned}|P_1 + P_2, \dots, P_n + P_1| &= |P_1 + P_2 + 2y_k S, \dots, P_n + P_1 + 2y_k S| \\ &= |kB_1 + kB_2, \dots, kB_n + kB_1| \\ &= k^2 |B_1 + B_2, \dots, B_n + B_1|,\end{aligned}$$

where $B_1, \dots, B_n$ are given by (2.2).

In this connection let us remark that

$$|k(B_1 + B_2), \dots, k(B_n + B_1)| = k^2 |B_1 + B_2, \dots, B_n + B_1|$$

since determinant has two rows.

This completes the proof of Theorem 1. □

Corollary 1.1. *Let $A_1 \dots A_n$ and k be as in Theorem 1. Then there is a unique $(n-k)$-outscribed polygon $Q_1 \dots Q_n$ to the polygon $A_1 \dots A_n$ for which*

$$2 \text{ area of } Q_1 \dots Q_n = (n-k)^2 |C_1 + C_2, \dots, C_n + C_1|, \tag{2.17}$$

where

$$C_i = A_i + A_{i+n-k} + A_{i+2(n-k)} + \dots + A_{i+(x_{n-k}-1)(n-k)}, \quad i = 1, \dots, n$$

and x_{n-k} is the least positive solution of the equation

$$(n-k)x = 1 \,(\text{mod } n). \tag{2.18}$$

Proof. It is easy to see that $GCD(n,k) = 1 \Leftrightarrow GCD(n, n-k) = 1$. □

Theorem 2. *Let x_k be as in Theorem 1 and x_{n-k} as in Corollary 1.1. Then*

$$x_k + x_{n-k} = n. \tag{2.19}$$

Proof. From the known fact number theory it follows

$$x_k < n, \quad x_{n-k} < n. \tag{2.20}$$

Namely,

$$\begin{aligned}&\{k \cdot 1, k \cdot 2, \dots, k(n-1)\}_{\text{mod } n} = \{1, 2, \dots, n-1\}, \\ &\{(n-k) \cdot 1, (n-k) \cdot 2, \dots, (n-k)(n-1)\}_{\text{mod } n} = \{1, 2, \dots, n-1\}.\end{aligned}$$

Now, from

$$kx_k = 1(\text{mod } n), \quad (n-k)x_{n-k} = 1 \,(\text{mod } n)$$

or

$$kx_k = pn + 1, \quad (n-k)x_{n-k} = qn + 1$$

where p and q are some integers, follows

$$k(x_k + x_{n-k}) = (p - q + x_{n-k})n.$$

Thus, $x_k + x_{n-k}$ must be divisible by n, since holds (2.20). □

Theorem 3. *Let $P_1 \dots P_n$ be as in Theorem* 1 *and $Q_1 \dots Q_n$ as in Corollary* 1.1. *Then*

$$(1/k^2) \text{ area of } P_1 \dots P_n = (1/(n-k)^2) \text{ area of } Q_1 \dots Q_n. \tag{2.21}$$

In other words, it holds

$$|B_1 + B_2, \dots, B_n + B_1| = |C_1 + C_2, \dots, C_n + C_1|, \tag{2.22}$$

where

$$B_i = A_i + A_{i+k} + \dots + A_{i+(x_k-1)k}, \quad i = 1, \dots, n \tag{2.23}$$

$$C_i = A_i + A_{i+(n-k)} + \dots + A_{i+(x_{n-k}-1)(n-k)}, \quad i = 1, \dots, n. \tag{2.24}$$

Proof. First we show that there is similarity from $P_1 \dots P_n$ to $Q_1 \dots Q_n$ given by

$$Q_{n-k+i} = -\frac{n-k}{k} P_i + \frac{S}{k}, \quad i = 1, \dots, n \tag{2.25}$$

where S is given by (2.16). For this purpose it is enough to show that for each $i = 1, \dots, n$ it holds

$$Q_{n-k+i} + Q_{n-k+i+1} + \dots + Q_{n-k+i+(n-k)-1} = (n-k)A_{n-k+i}.$$

For simplicity, taking $i = 1$, we can write

$$\begin{aligned}
\sum_{i=1}^{n-k} Q_{n-k+i} &= -\frac{n-k}{k}(P_1 + P_2 + \dots + P_{n-k}) + \frac{(n-k)S}{k} \\
&= -\frac{n-k}{k}[S - (P_{n-k+1} + P_{n-k+2} + \dots + P_{n-k+k})] + \frac{(n-k)S}{k} \\
&= -\frac{n-k}{k}[S - kA_{n-k+1}] + \frac{(n-k)S}{k} \\
&= (n-k)A_{n-k+1}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&|Q_{n-k+1} + Q_{n-k+2}, \dots, Q_{n-k+n} + Q_{n-k+1}| \\
&= |-\frac{n-k}{k}(P_1 + P_2) + \frac{2S}{k}, \dots, -\frac{n-k}{k}(P_n + P_1) + \frac{2S}{k}| \\
&= (\frac{n-k}{k})^2 |P_1 + P_2, \dots, P_n + P_1|,
\end{aligned}$$

which according to (1.5) can be written as

$$k^2 |Q_1 + Q_2, \dots, Q_n + Q_1| = (n-k)^2 |P_1 + P_2, \dots, P_n + P_1|.$$

This proves Theorem 3. $\square$

Example 1. Let $A_1 \ldots A_{11}$ and k be as in Theorem 1. Since in this case $x_5 = 9, y_5 = 4, x_6 = 2, y_6 = 1$, we have

$$\begin{aligned}
P_1 &= -4S + 5(A_1 + A_6 + A_{11} + A_5 + A_{10} + A_4 + A_9 + A_3 + A_8) \\
&= -5S + 5(A_1 + A_6 + A_{11} + A_5 + A_{10} + A_4 + A_9 + A_3 + A_8) + S \\
&= -5(A_2 + A_7) + S, \\
Q_1 &= 6(A_1 + A_7) - S.
\end{aligned}$$

Generally it holds

$$P_i = -5(A_{i+1} + A_{(i+1)+5}) + S, \quad i = 1, \ldots, 11 \tag{2.26}$$

$$Q_i = 6(A_i + A_{i+6}) - S, \quad i = 1, \ldots, 11 \tag{2.27}$$

so that we have

$$\begin{aligned}
|P_1 + P_2, \ldots, P_{11} + P_1| &= 25|A_2 + A_7 + A_3 + A_8, \ldots, A_{11} + A_6 + A_2 + A_7|, \\
|Q_1 + Q_2, \ldots, Q_{11} + Q_1| &= 36|A_2 + A_7 + A_3 + A_8, \ldots, A_{11} + A_6 + A_2 + A_7|,
\end{aligned}$$

where properties (1.5) and (1.9) are used.

In this connection let us remark that each column of determinants

$$|P_1 + P_2, \ldots, P_n + P_1|, \ |Q_1 + Q_2, \ldots, Q_n + Q_1|$$

can be expressed as sum of four corresponding vertices since from $x_5 + x_6 = 11$ follows $2x_6 = 2(11 - x_5) = 4$.

Analogously holds generally since from $x_k + x_{n-k} = n$ follows $2x_{n-k} = 2(n - x_k)$.

Concerning relation (2.27), we can, using (2.26), write

$$\begin{aligned}
Q_{6+i} &= -\frac{6}{5}(-5(A_{i+1} + A_{i+6} + S)) + \frac{S}{5} \\
&= 6(A_{i+1} + A_{i+6}) - S.
\end{aligned}$$

So, if $i = 6$, then $Q_1 = 6(A_7 + A_1) - S$.

Now we shall consider the case where $GCD(k, n) > 1$. The following theorem is easy to prove.

Theorem 4. *Let $A_1 \ldots A_n$ be a polygon in $\mathbb{R}^2$ and let k be an integer such that $1 < k < n$ and $GCD(k, n) = d > 1$. Then only one of the following two assertions is true:*

(i) *There is no k-outscribed polygon to the polygon $A_1 \ldots A_n$*

(ii) *There are infinitely many k-outscribed polygons to the polygon $A_1 \ldots A_n$.*

The second appears only if

$$\left.\begin{aligned} &A_1 + A_{1+k} + A_{1+2k} + \cdots + A_{1+(\hat{x}-1)k} \\ =&A_2 + A_{2+k} + A_{2+2k} + \cdots + A_{2+(\hat{x}-1)k} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ =&A_d + A_{d+k} + A_{d+2k} + \cdots + A_{d+(\hat{x}-1)k}, \end{aligned}\right\} \qquad (E_k)$$

where $\hat{x}$ is the least positive integer which is a solution of the equation

$$kx = 0\,(\mathrm{mod} n). \tag{2.28}$$

In other words, the relations given by (E_k) (existence for k-outscribed) are necessary and sufficient conditions for existence. In particular, if $GCD(k, n) = 2$, then (E_k) can be written as $\sum_{i=1}^{n}(-1)A_i = 0$.

Proof. First let us remark that the relations (conditions) (E_k) express that the center of gravity of the vertices

$$A_i,\ A_{i+k},\ A_{i+2k}, \ldots,\ A_{i+(\hat{x}-1)k}$$

is the same for each $i = 1, \ldots, d$. Alike it is easy to see that this center is also the center of gravity of the vertices $A_1, A_2, \ldots, A_n$. Namely, if c is such that

$$\frac{A_i + A_{i+k} + A_{i+2k} + \cdots + A_{i+(\hat{x}-1)k}}{\hat{x}} = c, \quad i = 1, \ldots, d$$

then

$$\hat{x}cd = \sum_{i=1}^{n} A_i \quad \text{or} \quad c = \frac{1}{n}\sum_{i=1}^{n} A_i, \quad \text{since } \hat{x}d = n.$$

To prove Theorem 4 we shall first consider one example, say, that where $n = 15$ and $k = 6$. In this case we have

$$\begin{aligned} &P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 6A_1, \\ &P_7 + P_8 + P_9 + P_{10} + P_{11} + P_{12} = 6A_7, \\ &P_{13} + P_{14} + P_{15} + P_1 + P_2 + P_3 = 6A_{13}, \\ &P_4 + P_5 + P_6 + P_7 + P_8 + P_9 = 6A_4, \\ &P_{10} + P_{11} + P_{12} + P_{13} + P_{14} + P_{15} = 6A_{10}, \end{aligned}$$

$$\begin{aligned} &P_2 + \cdots P_7 = 6A_2, &\qquad &P_3 + \cdots + P_8 = 6A_3 \\ &P_8 + \cdots P_{13} = 6A_8, &&P_9 + \cdots + P_{14} = 6A_9 \\ &P_{14} + \cdots P_4 = 6A_{14}, &&P_{15} + \cdots + P_5 = 6A_{15} \\ &P_5 + \cdots P_{10} = 6A_5, &&P_6 + \cdots + P_{11} = 6A_6 \\ &P_{11} + \cdots P_1 = 6A_{11}, &&P_{12} + \cdots + P_2 = 6A_{12} \end{aligned}$$

Since $2(P_1 + \cdots + P_{15}) = 2S$, where $S = A_1 + \cdots + A_{15}$, it holds

$$\begin{aligned} 2S &= 6(A_1 + A_7 + A_{13} + A_4 + A_{10}) \\ &= 6(A_2 + A_8 + A_{14} + A_5 + A_{11}) \\ &= 6(A_3 + A_9 + A_{15} + A_6 + A_{12}). \end{aligned}$$

The above relations can be written as

$$\left.\begin{aligned} &A_1 + A_{1+6} + A_{1+2\cdot 6} + A_{1+3\cdot 6} + A_{1+(\hat{x}-1)6} \\ &= A_2 + A_{2+6} + A_{2+2\cdot 6} + A_{2+3\cdot 6} + A_{2+(\hat{x}-1)6} \\ &= A_3 + A_{3+6} + A_{3+2\cdot 6} + A_{3+3\cdot 6} + A_{3+(\hat{x}-1)6}. \end{aligned}\right\} \qquad (E_6)$$

since $x = 5$ is the least positive integer such that $6x = 0\,(\mathrm{mod}\ 15)$.

From this it is clear that the system

$$P_i + P_{i+1} + \cdots + P_{i+5} = 6A_i, \quad i = 1, \ldots, 15$$

has a solution if and only if holds (E_6).

In the same way can be seen that the system

$$P_i + P_{i+1} + \cdots + P_{i+k-1} = kA_i, \quad i = 1, \ldots, n$$

where $GCD(k, n) = d > 1$, has a solution if and only if holds (E_k). Here instead of the equations stated in example where $n = 15$ and $k = 6$, we have the equations

$$\sum_{i=1}^{k} P_i = kA_1, \ \sum_{i=1+k}^{2k} P_i = kA_{1+k}, \ \ldots, \ \sum_{i=1+(\hat{x}-1)k}^{k\hat{x}} P_i = kA_{1+(\hat{x}-1)k} \qquad (\mathrm{i}_1)$$

....

$$\sum_{i=d}^{d-1+k} P_i = kA_d, \ \sum_{i=d+k}^{d-1+2k} P_i = kA_{d+k}, \ldots, \ \sum_{i=d+(\hat{x}-1)k}^{d-1+\hat{x}k} P_i = kA_{d+(\hat{x}-1)k} \qquad (\mathrm{i}_d)$$

where $x = \hat{x}$ is the least positive integer such that $kx = 0(\mathrm{mod}\ n)$.

In this connection let us remark that from the first $\hat{x}$ equations given by (i_1), by adding up, follows

$$\frac{k}{d}S = k(A_1 + A_{1+k} + A_{1+2k} + \cdots + A_{1+(\hat{x}-1)k}),$$

where $S = A_1 + \cdots + A_n$. Also let us remark that from $\frac{k}{d}n = k\frac{n}{d}$ follows that $\hat{x} = \frac{n}{d}$.

Concerning term $\frac{k}{d}S$ on the left-hand side of the above equation let us remark that $\frac{k}{d}n = k\hat{x}$.

Thus, supposing that holds (E_k), it remains to prove that matrix of the system has rank equal $n - d + 1$. First about the matrix of the system where $n = 15$ and

$k=6$, that is, about the matrix

$$K=\begin{bmatrix}
1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&1&1&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\
1&1&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&1&1&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&1&1&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}.$$

Consider the 15 lines in K as vectors x_i. We have

$$x_1=(1111110\ldots0),\ldots,x_{15}=(111110\ldots01).$$

$\{x_i\}$ is linear dependent if

$$\sum_{i=1}^{15}\lambda_i x_i=0 \text{ with } \sum_{i=1}^{15}\lambda_i^2>0. \qquad \text{(R)}$$

There is

$$\left.\begin{aligned}
x_1+x_7+x_{13}+x_4+x_{10}&=(11\ldots1)\\
x_2+x_8+x_{14}+x_5+x_{11}&=(11\ldots1)\\
x_3+x_9+x_{15}+x_6+x_{12}&=(11\ldots1).
\end{aligned}\right\} \qquad (2.29)$$

Choose

$$\begin{gathered}
\lambda_1=\lambda_7=\lambda_{13}=\lambda_4=\lambda_{10}\\
\lambda_2=\lambda_8=\lambda_{14}=\lambda_5=\lambda_{11}\\
\lambda_3=\lambda_9=\lambda_{15}=\lambda_6=\lambda_{12}.
\end{gathered}$$

Then

$$\sum_{i=1}^{15}\lambda_i x_i=2\cdot(\lambda_1+\lambda_2+\lambda_3)\cdot(11\ldots1)=0 \text{ for } \lambda_3=-\lambda_1-\lambda_2.$$

Thus, there exist at least two parameters λ_1,λ_2 for (R). From this it is clear that the rank of K is $\le 15-2=13$.

On the other hand there is

$$\sum_{i=1}^{13} \lambda_i x_i = 0 \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{13} = 0.$$

Therefore, the rank of K is exactly 13.

The general proof is completely analogous to the proof where $n = 15$ and $k = 6$. Instead of the equations given by (2.29) we have the equations

$$\begin{aligned} &x_1 + x_{1+k} + x_{1+2k} + \cdots + x_{1+(\hat{x}-1)k} = (11\ldots 1),\\ &x_2 + x_{2+k} + x_{2+2k} + \cdots + x_{2+(\hat{x}-1)k} = (11\ldots 1),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &x_d + x_{d+k} + x_{d+2k} + \cdots + x_{d+(\hat{x}-1)k} = (11\ldots 1). \end{aligned} \tag{2.30}$$

Let $\lambda_1, \ldots, \lambda_n$ be chosen such that

$$\begin{gathered} \lambda_1 = \lambda_{1+k} = \lambda_{1+2k} = \cdots = \lambda_{1+(\hat{x}-1)k}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \lambda_d = \lambda_{d+k} = \lambda_{d+2k} = \cdots = \lambda_{d+(\hat{x}-1)k}. \end{gathered}$$

Then

$$\sum_{i=1}^{n} \lambda_i x_i = \frac{k}{d}(\lambda_1 + \lambda_2 + \cdots + \lambda_d)(11\ldots 1) = 0 \ \text{ for } \ \lambda_d = -\lambda_1 - \cdots - \lambda_{d-1}.$$

This means that there exist at least $d-1$ parameters $\lambda_1, \ldots, \lambda_{d-1}$ such that

$$\sum_{i=1}^{n} \lambda_i x_i = 0 \text{ with } \sum_{i=1}^{n} \lambda_i^2 > 0.$$

Thus, if the matrix of the system is denoted by L, it holds

$$\text{rank of } L \le n - d + 1.$$

On the other hand there is

$$\sum_{i=1}^{n} \lambda_i x_i = 0 \Longrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{n-d+1} = 0.$$

Therefore the rank of L is exactly $n-d+1$. This completes the proof of Theorem 4. $\square$

Corollary 4.1. *If matrix of the system*

$$P_i - P_{i+k} = k(A_i - A_{i+1}), \quad i = 1, \ldots, n \tag{2.31}$$

has rank r, then matrix of the system

$$P_i - P_{i+n-k} = (n-k)(A_i - A_{i+1}), \quad i = 1, \ldots, n \tag{2.32}$$

has also rank r.

Proof. It holds $GCD(k,n) = GCD(n-k,n)$. □

In this connection let us remark that, if M is matrix of the system (2.31), then M^T is matrix of the system (2.32).

Theorem 5. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ and let k be an integer such that $1 < k < n$ and $GCD(k,n) = d > 1$. Then the following two assertions are valid:*

(i) *The condition for being k-outscribed polygon to the polygon $A_1 \dots A_n$ is the same as the condition for being $(n-k)$-outscribed polygon to the polygon $A_1 \dots A_n$.*

(ii) *For every polygon $P_1 \dots P_n$ which is k-outscribed to the polygon $A_1 \dots A_n$ there exists the $(n-k)$-outscribed polygon $Q_1 \dots Q_n$ to the polygon $A_1 \dots A_n$ which is similar to the polygon $P_1 \dots P_n$ and it holds*

$$\text{area of } Q_1 \dots Q_n = \left(\frac{n-k}{k}\right)^2 \text{ area of } P_1 \dots P_n. \tag{2.33}$$

Proof. To prove (i), we shall first, for simplicity writing, consider one example, say, that considered in Theorem 4, where $n = 15$, $k = 6$. In this case we have found that the condition for being 6-outscribed polygon to the polygon $A_1 \dots A_{15}$ is given by

$$A_1 + A_7 + A_{13} + A_4 + A_{10} = A_2 + A_8 + A_{14} + A_5 + A_{11} = A_3 + A_9 + A_{15} + A_6 + A_{12}. \tag{2.34}$$

Now from Theorem 4 and from

$$\begin{aligned}
&Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 = 9A_1\\
&Q_{10} + Q_{11} + Q_{12} + Q_{13} + Q_{14} + Q_{15} + Q_1 + Q_2 + Q_3 = 9A_{10}\\
&Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10} + Q_{11} + Q_{12} = 9A_4\\
&Q_{13} + Q_{14} + Q_{15} + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 = 9A_{13}\\
&Q_7 + Q_8 + Q_9 + Q_{10} + Q_{11} + Q_{12} + Q_{13} + Q_{14} + Q_{15} = 9A_7
\end{aligned} \tag{2.35}$$

$$\begin{array}{ll}
Q_2 + \cdots + Q_{10} = 9A_2, & Q_3 + \cdots + Q_{11} = 9A_3\\
Q_{11} + \cdots + Q_4 = 9A_{11}, & Q_{12} + \cdots + Q_5 = 9A_{12}\\
Q_5 + \cdots + Q_{13} = 9A_5, & Q_6 + \cdots + Q_{14} = 9A_6\\
Q_{14} + \cdots + Q_7 = 9A_{14}, & Q_{15} + \cdots + Q_9 = 9A_{15}\\
Q_8 + \cdots + Q_1 = 9A_8, & Q_9 + \cdots + Q_2 = 9A_9
\end{array}$$

we see that the condition for being 9-outscribed polygon to the polygon $A_1 \dots A_{15}$ is the same as that for being 6-outscribed polygon to the polygon $A_1 \dots A_{15}$.

Here it is important to see that $\overline{x} = 5$ is the least positive integer which is a solution of the equation

$$6x = 0 \,(\mathrm{mod} 15) \tag{2.36}$$

and that also $\overline{x} = 5$ is the least positive integer which is a solution of the equation

$$9x = 0 \,(\mathrm{mod} 15). \tag{2.37}$$

In this connection let us remark that, for example, the first of the five equations given by (2.35) begins with Q_1 and the fifth ends with Q_{15}. Similar holds when instead of Q_1 there is Q_2 or Q_3.

Also let us remark that for $i = 1, 2, 3$ and $\overline{x} - 1 = 4$, $k = 6$, $n - k = 9$ it holds

$$\begin{aligned}&\{A_i, A_{i+k}, A_{i+2k}, A_{i+3k}, A_{i+4k}\}\\ &\qquad = \{A_i, A_{i+n-k}, A_{i+2(n-k)}, A_{i+3(n-k)}, A_{i+4(n-k)}\}\end{aligned} \tag{2.38}$$

or

$$\begin{aligned}&\{i, i+k, i+2k, i+3k, i+4k\}_{\text{mod } 15}\\ &\qquad = \{i, i+n-k, i+2(n-k), i+3(n-k), i+4(n-k)\}_{\text{mod } 15}\end{aligned}$$

since

$$\begin{aligned}i+k &= i+4(n-k) \pmod{15},\\ i+2k &= i+3(n-k) \pmod{15},\\ i+3k &= i+2(n-k) \pmod{15},\\ i+4k &= i+n-k \pmod{15}.\end{aligned}$$

Namely it holds

$$5k = jn \pmod{15}, \quad j = 4, 3, 2, 1.$$

In this connection let us remark that each of the integers k, $2k$, $3k$, $4k \,(\text{mod } 15)$ is less then 15 since $\overline{x} = 5$ is the least positive integer which is a solution of the equation (2.36).

It is not difficult to see that analogously holds generally for the case when $GCD(k, n) = d > 1$. Instead of equations (2.36) and (2.37) we have equations

$$kx = 0 \,(\text{mod } n), \qquad (n-k)x = 0 \,(\text{mod } n) \tag{2.39}$$

with property that $\hat{x} = \frac{n}{d}$ is the least positive integer such that hold both of the equations (2.39). Also let us remark that instead of relation (2.38) we have relation

$$\{A_i, A_{i+k}, \ldots, A_{i+(\hat{x}-1)k}\} = \{A_i, A_{i+n-k}, \ldots, A_{i+(\hat{x}-1)(n-k)}\},$$

where $i = 1, \ldots, d$.

Thus assertion (i) is proved.

To prove (ii) let $P_1 \ldots P_n$ be any given polygon which is k-outscribed to the polygon $A_1 \ldots A_n$ and let Q_{n-k+i} be given by

$$Q_{n-k+i} = -\frac{n-k}{k} P_i + \frac{S}{k}, \quad i = 1, \ldots, n$$

as in Theorem 3 where $GCD(k, n) = 1$. In the same way as in Theorem 3 can be shown that

$$\sum_{i=1}^{n-k} Q_{n-k+i} = (n-k) A_{n-k+1}.$$

Also in the same way as in Theorem 3 we have

$$\begin{aligned}&|Q_{n-k+1}+Q_{n-k+2},\dots,Q_{n-k+n}+Q_{n-k+1}|\\ =&|-\frac{n-k}{k}(P_1+P_2)+\frac{2S}{k},\cdots,-\frac{n-k}{k}(P_n+P_1)+\frac{2S}{k}|\\ =&\left(\frac{n-k}{k}\right)^2|P_1+P_2,\dots,P_n+P_1|,\end{aligned}$$

which according to (1.5) can be written as

$$|Q_1+Q_2,\dots,Q_n+Q_1|=\left(\frac{n-k}{k}\right)^2|P_1+P_2,\dots,P_n+P_1|.$$

This completes the proof of Theorem 5. □

Corollary 5.1. *Let $P_1\dots P_n$ be any given polygon which is k-outscribed to the polygon $A_1\dots A_n$. Then there exists unique $(n-k)$-outscribed polygon $Q_1\dots Q_n$ to the polygon $A_1\dots A_n$ such that*

$$Q_{n-k+i}=-\frac{n-k}{k}P_i+\frac{S}{k},\ i=1,\dots,n.$$

Such two polygons, $P_1\dots P_n$ and $Q_1\dots Q_n$, can be called conjugate polygons. They are similar.

Corollary 5.2. *Let by $\bar{P}_1\dots\bar{P}_n$ be denoted the k-outscribed polygon to the polygon $A_1\dots A_n$ with property that it can be k-outscribed, and let by $\bar{Q}_1\dots\bar{Q}_n$ be denoted the $(n-k)$-outscribed polygon to the polygon $A_1\dots A_n$ with the property that it can be $(n-k)$-outscribed. Then polygons $\bar{P}_1\dots\bar{P}_n$ and $\bar{Q}_1\dots\bar{Q}_n$ are conjugate, that is, from*

$$Q_{n-k+i}=-\frac{n-k}{k}\bar{P}_i+\frac{S}{k},\ i=1,\dots,n$$

follows $Q_{n-k+i}=\bar{Q}_{n-k+i}$, $i=1,\dots,n$.

Proof. From

$$\sum_{i=1}^{n}(-1)^iQ_i=\sum_{i=1}^{n}(-1)^i\left[-\frac{n-k}{k}\bar{P}_i+\frac{S}{k}\right]=0$$

follows $\sum_{i=1}^{n}(-1)^iQ_i=0$. Thus, $Q_1\dots Q_n$ is polygon which is $(n-k)$-outscribed to the polygon $A_1\dots A_n$ and has the property that it can be $(n-k)$-outscribed. Such polygon is unique, since equation $\sum_{i=1}^{n}(-1)^iQ_i=0$ completely determines Q_1. Thus, $Q_1\dots Q_n=\bar{Q}_1\dots\bar{Q}_n$. □

Theorem 6. *Let $A_1\dots A_n$ be a polygon in $\mathbb{R}^2$, where n is even, and let k be a positive integer such that $k<n$ and $GCD(k,n)=2$. Then every polygon which is k-outscribed to the polygon $A_1\dots A_n$ has the same area. In other words, for every two polygons $P_1\dots P_n$ and $\hat{P}_1\dots\hat{P}_n$ which are k-outscribed to the polygon $A_1\dots A_n$ it holds*

$$|P_1+P_2,\dots,P_n+P_1|=|\hat{P}_1+\hat{P}_2,\dots,\hat{P}_n+\hat{P}_1|. \tag{2.40}$$

Proof. We suppose that (E_k) is fulfilled, that is, $\sum_{i=1}^{n}(-1)^i A_i = 0$. The system

$$P_i + P_{i+1} + \cdots + P_{i+k-1} = kA_i,\ i = 1,\ldots,n \tag{2.41}$$

can be written as

$$P_i - P_{i+k} = k(A_i - A_{i+1}),\ i = 1,\ldots,n. \tag{2.42}$$

It is easy to see that from the above equations follows

$$P_{1+ki} = P_1 + S_{1+ki},\ i = 1,2,\ldots,\frac{n}{2} - 1, \tag{2.43}$$

$$P_{2+ki} = P_2 + S_{2+ki},\ i = 1,2,\ldots,\frac{n}{2} - 1 \tag{2.44}$$

where S_{1+ki} and S_{2+ki} are sums of certain vertices $A_1 \ldots A_n$.

For example, if $n = 10$ and $k = 6$, then

$$\begin{aligned}
&P_7 = P_{1+6} = P_1 - 6(A_1 - A_2),\\
&P_3 = P_{1+12} = P_7 - 6(A_7 - A_8) = P_1 - 6(A_1 - A_2 + A_7 - A_8),\\
&P_9 = P_{1+18} = P_3 - 6(A_3 - A_4) = P_1 - 6(A_1 - A_2 + A_7 - A_8 + A_3 - A_4),\\
&P_5 = P_{1+24} = P_1 + 6(A_5 - A_6)
\end{aligned}$$

where $S_{1+6} = -6(A_1 - A_2)$, $S_{1+12} = -6(A_1 - A_2 + A_7 - A_8)$ and so on.

Analogously holds for S_{2+6i}. Here we have

$$\{P_{2+6}, P_{2+12}, P_{2+18}, P_{2+24}\} = \{P_8, P_4, P_{10}, P_6\}.$$

Since in the case when $GCD(k,n) = 2$, the integer k must be even. Thus, both of the indices i and $i+1$ in $P_i + P_{i+1}$ are odd or both even. Therefore in the expression of each of $P_3, P_5, \ldots, P_{n-1}$ appears P_1 and in the expression of each of $P_4, P_6, \ldots, P_n$ appears P_2, so that determinant

$$|P_1 + P_2, P_2 + P_3, \ldots, P_n + P_1|$$

can be written as

$$|P_1 + P_2, P_1 + P_2 + K_1, \ldots, P_1 + P_2 + K_{n-1}|, \tag{2.45}$$

where each K_i is expressed by $A_1, \ldots, A_n$ and also $P_1 + P_2$ are expressed by $A_1, \ldots, A_n$.

Of course, that above determinant is a constant can also be seen using property (1.5). Namely, since $\sum_{i=i}^{n}(-1)^i(P_i + P_{i+1}) = 0$, the above determinant can be written as

$$|O, K_1, \ldots, K_{n-1}| \quad \text{or} \quad |K_1, \ldots, K_{n-1}|. \tag{2.46}$$

This proves Theorem 6. □

Corollary 6.1. *Let $A_1 \ldots A_n$ and k be as in Theorem 6. Then all the polygons which are $(n-k)$-outscribed to the polygon $A_1 \ldots A_n$ have the same area.*

The proof is quite analogous to the proof of Theorem 6.

Of course, that Corollary 6.1 is valid can also be concluded from the fact that for every k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ there exists $(n-k)$-outscribed polygon $Q_1 \dots Q_n$ to the polygon $A_1 \dots A_n$ which is similar to the polygon $P_1 \dots P_n$.

Here let us remark that it is not always easy to express corresponding determinants in the most simple form for every k. The case when $k = 4$ may be interesting and it will be considered in the following theorem.

Theorem 7. *Let $k = 4$ and $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ where $GCD(k,n) = 2$ and $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then for every 4-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ it holds*

$$P_i + P_{i+1} = 2(A_i - A_{i+2} + A_{i+4} - \cdots + A_{i+n-2}), \quad i = 1, \dots n. \tag{2.47}$$

Proof. We have to prove that

$$P_i + P_{i+1} + P_{i+2} + P_{i+3} = 4A_i, \quad i = 1, \dots, n. \tag{2.48}$$

Since $GCD(k,n) = 2$, P_1 can be arbitrary. Let $P_2, \dots, P_n$ be given (inductively) by

$$P_{i+1} = -P_i + 2(A_i - A_{i+2} + A_{i+4} - \cdots + A_{i+n-2}),\ i = 1, \dots, n-1. \tag{2.49}$$

It is easy to see that from

$$\begin{aligned} P_i + P_{i+1} &= 2(A_i - A_{i+2} + A_{i+4} - \cdots + A_{i+n-2}), \\ P_{i+2} + P_{i+3} &= 2(A_{i+2} - A_{i+4} + \cdots - A_{i+n-2} + A_{i+2+n-2}) \end{aligned}$$

by adding up, we get $P_i + P_{i+1} + P_{i+2} + P_{i+3} = 4A_i$. This also can be proved in the following way.

The equation $P_1 + P_2 + \sum_{i=3}^{6} P_i + \cdots + \sum_{i=n-3}^{n} P_i = \sum_{i=1}^{n} A_i$ can be written as

$$P_1 + P_2 = \sum_{i=1}^{n} A_i - 4(A_3 + A_7 + \cdots + A_{n-3})$$

or, since $\sum_{i=1}^{n} A_i = 2(A_1 + A_3 + \cdots + A_{n-1}) = 2(A_2 + A_4 + \cdots + A_n)$,

$$P_1 + P_2 = 2(A_1 - A_3 + A_5 - A_7 + \cdots - A_{n-3} + A_{n-1}).$$

In the same way can be seen that holds (2.47). This proves Theorem 7. □

Corollary 7.1. *It holds*

$$2 \text{ area of } P_1 \dots P_n = 16|B_1, \dots, B_n|, \tag{2.50}$$

where

$$B_i = A_{i+2} + A_{i+6} + A_{i+10} + \cdots + A_{i+n-4}, \quad i = 1, \dots, n. \tag{2.51}$$

(Of course, the property given by (1.4) *is used.)*

Proof. The sum $P_i + P_{i+1}$ can also be written as

$$P_i + P_{i+1} = 2T - 4(A_{i+2} + A_{i+6} + A_{i+10} + \cdots + A_{i+n-4}),\ i = 1, \ldots, n$$

where $T = A_1 + A_3 + A_5 + \cdots + A_{n-1} = A_2 + A_4 + A_6 + \cdots + A_n$. □

Corollary 7.2. *Let $Q_1 \ldots Q_n$ be an $(n-4)$-outscribed polygon to the polygon $A_1 \ldots A_n$. Then*

$$2 \text{ area of } Q_1 \ldots Q_n = (n-4)^2 |B_1, \ldots, B_n|, \tag{2.52}$$

where $B_1, \ldots, B_n$ are given by (2.51).

Proof. According to Theorem 3 and Theorem 6 it holds

$$Q_{n-4+i} = -\frac{n-4}{4} P_i + \frac{S}{4}, \quad i = 1, \ldots, n.$$

From (2.47) follows

$$\begin{aligned} &P_2 = -P_1 + F_1, \\ &P_3 = P_1 - F_1 + F_2, \\ &P_4 = -P_1 + F_1 - F_2 + F_3, \\ &\text{and so on,} \end{aligned}$$

where $F_1, \ldots, F_n$ are expressions which depend only of $A_1, \ldots, A_n$ and can be written as

$$F_i = 2T - 4(A_{i+2} + A_{i+6} + A_{i+10} + \cdots + A_{i+n-4}),\ i = 1, \ldots, n$$

where $T = A_1 + A_3 + A_5 + \cdots + A_{n-1} = A_2 + A_4 + A_6 + \cdots + A_n$.

Since

$$\begin{aligned} &Q_{n-4+1} = -\frac{n-4}{4} P_1 + \frac{S}{4}, \\ &Q_{n-4+2} = -\frac{n-4}{4}(-P_1 + F_1) + \frac{S}{4}, \\ &Q_{n-4+3} = -\frac{n-4}{4}(P_1 - F_1 + F_2) + \frac{S}{4}, \\ &\text{and so on,} \end{aligned}$$

it holds

$$\begin{aligned} &Q_{n-4+1} + Q_{n-4+2} = -\frac{n-4}{n} F_1 + \frac{S}{2}, \\ &Q_{n-4+2} + Q_{n-4+3} = -\frac{n-4}{n} F_2 + \frac{S}{2}, \\ &\text{and so on.} \end{aligned}$$

Using properties given by (1.5) and (1.8) we get (2.52). □

In the following theorem will be considered one more case when it is relatively easy to express corresponding determinants in a simple form.

Theorem 8. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ with property that $\sum_{i=1}^{n}(-1)^i A_i = 0$ and let $k = \frac{n}{2} - 1$, where n is divisible by 2 but not by 4. Then for every polygon $P_1 \dots P_n$ which is k-outscribed to the polygon $A_1 \dots A_n$ it holds*

$$2 \text{ area of } P_1 \dots P_n = k^2 |C_1, \dots, C_n|, \tag{2.53}$$

where

$$C_i = A_i + A_{i+k}, i = 1, \dots, n. \tag{2.54}$$

Proof. First it is clear that $GCD(n, \frac{n}{2} - 1) = 2$ since n is even integer but not divisible by 4. So we can write

$$P_1 + P_2 + (P_3 + \dots + P_{3+k-1}) + (P_{3+k} + \dots + P_{3+2k-1}) = \sum_{i=1}^{n} A_i$$

or

$$P_1 + P_2 = S - k(A_3 + A_{3+k}), \text{ where } S = \sum_{i=1}^{n} A_i.$$

In the same way can be seen that

$$P_i + P_{i+1} = S - k(A_{i+2} + A_{i+2+k}), \ i = 1, \dots, n.$$

Thus, according to the properties (1.3) and (1.8) we have

$$\begin{aligned} \text{area of } P_1 \dots P_n &= |A_3 + A_{3+k}, \dots, A_1 + A_{1+k}, A_2 + A_{2+k}| \\ &= |A_1 + A_{1+k}, \dots, A_n + A_{n+k}|. \end{aligned}$$

This proves Theorem 8. □

Here may be interesting that for each $k = 2, 4, \dots, \frac{n}{2} - 1, \frac{n}{2} + 1, \dots, n - 2$ we get relatively simple expressions for $P_i + P_{i+1}$. For example, let $n = 14$. If $k = 2$, then

$$P_1 + P_2 + (P_3 + P_4) + \dots + (P_{13} + P_{14}) = S, \text{ where } S = \sum_{i=1}^{14} A_i,$$

from which follows

$$\begin{aligned} P_1 + P_2 &= S - 2(A_3 + A_5 + \dots + A_{13}) \\ &= 2A_1 + \sum_{i=1}^{14} (-1)^i A_i = 2A_1. \end{aligned}$$

Thus, in this case

$$\text{area of } P_1 \dots P_{14} = 4|A_1, \dots, A_{14}|.$$

If $k = 4$, then $P_i + P_{i+1} = S - 4(A_{i+2} + A_{i+2+4} + A_{i+2+8}), \ i = 1, \dots, 14$.

If $k = 6$, then $P_i + P_{i+1} = S - 6(A_{i+2} + A_{i+2+6}), \ i = 1, \dots, 14$.

In the case when $k = 8, 10, 12$ holds Theorem 5.

Theorem 9. *Let j, k, n be positive integers such that*

$$n = jk + 2 \text{ and } GCD(k, n) = 2.$$

Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ which can be k-outscribed, that is $\sum_{i=1}^{n}(-1)^i A_i = 0$. Then for every k-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ it holds

$$\text{area of } P_1 \dots P_n = k^2 |B_1, \dots, B_n|,$$

where $B_i = A_i + A_{i+k} + \cdots + A_{i+(j-1)k}$, $i = 1, \dots, n$.

Proof. The equality

$$P_1 + P_2 + \sum_{i=3}^{2+k} P_i + \sum_{i=3+k}^{2+2k} P_i + \cdots + \sum_{i=3+(j-1)k}^{2+jk} P_i = S,$$

where $S = \sum_{i=1}^{n} A_i$, can be written as

$$P_1 + P_2 = S - k(A_3 + A_{3+k} + \cdots + A_{3+(j-1)k}).$$

In the same way can be seen that

$$P_i + P_{i+1} = S - k(A_{i+2} + A_{i+2+k} + \cdots + A_{i+2+(j-1)k}),\ i = 1, \dots, n.$$

In the expression for area of $P_1 \dots P_n$ the properties (1.4) and (1.7) are used. This proves Theorem 9. □

It is easy to see that analogously holds in the case when there are positive integers $a_1, \dots, a_j$ and $b_1, \dots, b_j$ such that

$$\begin{aligned} n &= a_1 k + b_1, \\ n &= a_2 b_1 + b_2, \\ &\dots\dots\dots\dots\dots \\ n &= a_j b_j + b_{j+1}, \end{aligned}$$

where $b_{j+1} = 2$. For example, if $n = 22$, $k = 6$ then can be written

$$\begin{aligned} 22 &= 3 \cdot 6 + 4, \\ 22 &= 5 \cdot 4 + 2. \end{aligned}$$

Concerning areas of the polygon in $\mathbb{R}^2$ in connection with g-determinant, the following theorem can be interesting.

Theorem 10. *Let $A_1, \dots, A_n$ be any given real $2 \times n$ matrix. Then*

$$\begin{aligned} |A_1, \dots, A_n| = {} & |A_1, A_2, A_3| + |A_1 - A_2 + A_3, A_4, A_5| + \\ & |A_1 - A_2 + A_3 - A_4 + A_5, A_6, A_7| + \cdots + L, \end{aligned} \tag{2.55}$$

where

$$L = |\sum_{i=1}^{n-2}(-1)^{i+1}A_i, A_{n-1}, A_n| \text{ if } n \text{ is odd,} \tag{2.56}$$

$$L = |\sum_{i=1}^{n-1}(-1)^{i+1}A_i, A_n| \text{ if } n \text{ is even.} \tag{2.57}$$

Proof. The proof follows directly from definition of g-determinant of $2 \times n$ matrix $[A_1, \ldots, A_n]$. Namely, it is easy to see that relation (2.55) can be written as

$$|A_1, \ldots, A_n| = \sum_{i \le j_1 < j_2 \le n} (-1)^{1+2+j_1+j_2}|A_{j_1}, A_{j_2}|. \tag{2.58}$$

(See relation (1.1).)
So, for example, if $n = 5$ then

$$\begin{aligned} |A_1, \ldots, A_5| &= |A_1, A_2, A_3| + |A_1 - A_2 + A_3, A_4, A_5| \\ &= |A_1, A_2| - |A_1, A_3| + |A_2, A_3| + |A_1, A_4| + |A_1, A_5| \\ &\quad -|A_2, A_4| - |A_2, A_5| + |A_3, A_4| + |A_3, A_5| + |A_4, A_5|. \end{aligned}$$

Here let us remark that in (2.58) instead of $(-1)^{1+2+j_1+j_2}$ can be written $(-1)^{1+j_1+j_2}$.

In the case when n is odd, then in the sum on the right-hand side of (2.55) there are $\frac{n-1}{2}$ determinants whose type is 2×3. If n is even then there are $\frac{n}{2}$ such determinants or $\frac{n-1}{2}$ in the case when $\sum_{i=1}^{n}(-1)^i A_i = 0$, namely, then L given by (2.57) is zero. □

Corollary 10.1. *If n is even and $\sum_{i=1}^{n}(-1)^i A_i = 0$, then*

$$|A_1, \ldots, A_n| = |A_1, \ldots, A_{n-1}|. \tag{2.59}$$

Proof. In this case the term L is zero. (Cf. with (1.6).) □

Corollary 10.2. *If n is even and $\sum_{i=1}^{n}(-1)^i A_i = 0$ then for every 2-outscribed polygon $P_1 \ldots P_n$ to the polygon $A_1 \ldots A_n$ it holds*

$$\text{area of } P_1 \ldots P_n = 2(|A_1, A_2, A_3| + |S_1, A_4, A_5| + \cdots + |S_u, A_{n-2}, A_{n-1}|)$$

or

$$\begin{aligned} \text{area of } P_1 \ldots P_n = 2(&\text{area of parallelogram } A_1A_2A_3S_1 + \\ &\text{area of parallelogram } S_1A_4A_5S_2 + \cdots + \\ &\text{area of parallelogram } S_uA_{n-2}A_{n-1}S_v), \end{aligned}$$

where

$$S_1 = \sum_{i=1}^{3}(-1)^{i+1}A_i, \quad S_2 = \sum_{i=1}^{5}(-1)^{i+1}A_i, \ldots,$$

$$S_u = \sum_{i=1}^{n-3}(-1)^{i+1}A_i, \quad S_v = A_n.$$

Proof. Since by Theorem 1 in [2] it holds

$$2 \text{ area of } P_1 \dots P_n = |P_1 + P_2, \dots, P_n + P_1|,$$

and $P_1, \dots, P_n$ are such that $P_i + P_{i+1} = 2A_i,\ i = 1, \dots, n$ we can write

$$2 \text{ area of } P_1 \dots P_n = |2A_1, 2A_2, \dots, 2A_n|$$

or

$$\text{area of } P_1 \dots P_n = 2|A_1, A_2, \dots, A_n|. \tag{2.60}$$

In this connection let us remark that, for example, $|A_1, A_2, A_3| =$ area of parallelogram $A_1A_2A_3S_1$ since by (2.60) it holds

$$|A_1, A_2A_3| = \frac{1}{2} \text{ area of 2-outscribed triangle to the triangle } A_1A_2A_3.$$

As an illustration can be seen Figure 1, where $P_1, \dots P_6$ is a 2-outscribed to the hexagon $A_1 \dots A_6$. In this case it holds

$$\begin{aligned}\text{area of } P_1 \dots P_6 = 2(&\text{area of parallelogram } A_1A_2A_3S_1 + \\ &\text{area of parallelogram } S_1A_4A_5A_6).\end{aligned}$$

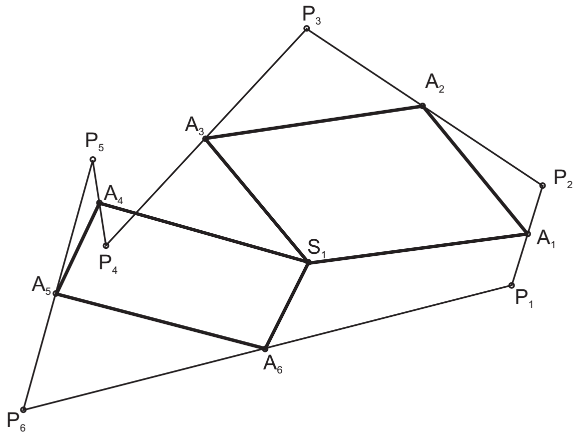

Figure 1

□

Corollary 10.3. *If n is even and $\sum_{i=1}^{n}(-1)^i A_i = 0$, then for every 2-outscribed polygon $P_1 \dots P_n$ to the polygon $A_1 \dots A_n$ and for (unique) 2-outscribed polygon $Q_1 \dots Q_{n-1}$ to the polygon $A_1 \dots A_{n-1}$ it holds*

$$\textit{area of } P_1 \dots P_n = \textit{ area of } Q_1 \dots Q_{n-1}.$$

Proof. Since $Q_i + Q_{i+1} = 2A_i,\ i = 1, \dots, n-1$, by Theorem 1 in [2] it holds

$$\text{area of } Q_1 \dots Q_{n-1} = 2|A_1, \dots A_{n-1}|.$$

In this connection let us remark that $Q_1 = A_n$, that is, it holds

$$Q_1 = \sum_{i=1}^{n-1} (-1)^{i+1} A_i.$$

Geometrical illustration for $n = 6$ can be seen in Figure 2.

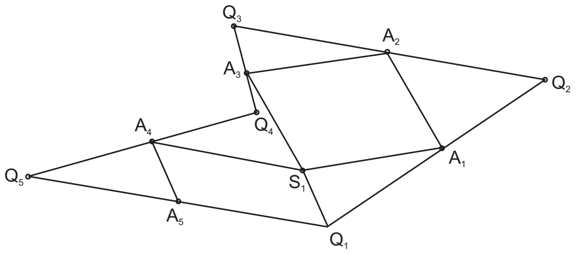

Figure 2

Here let us remark that also in the case when $GCD(k, n) = 2$ and $k > 2$, using Theorem 10, can be obtained interesting relations. □

Till now we have dealt with areas of the k-outscribed n-gons in $\mathbb{R}^2$ where $GCD(k, n) = 2$. Now the following question arises: What is the situation with areas of the k-outscribed n-gons in the case where $GCD(k, n) > 2$? The following theorem gives answer to the question and, as will be seen, can be very interesting.

Theorem 11. *Let $A_1 \dots A_n$ be a polygon in $\mathbb{R}^2$ and let k be an integer such that $1 < k < n$, $GCD(k, n) = d > 2$ and that holds (E_k). Then the k-outscribed polygons to the polygon $A_1 \dots A_n$ have different areas.*

Proof. For simplicity writing and complete analogy let $n = 3r$, where $r > 1$ is an integer, and let $k = 3$. Then $GCD(k, n) = 3$. In this case we have equalities(conditions)

$$\left.\begin{array}{l} A_1 + A_4 + A_7 + \cdots + A_{3r-2} = L \\ A_2 + A_5 + A_8 + \cdots + A_{3r-1} = L \\ A_3 + A_6 + A_9 + \cdots + A_{3r} = L \end{array}\right\} \quad (E_3)$$

where $L = \frac{1}{3} \sum_{i=1}^{3r} A_i$. Since $k = 3$ we have the equations

$$\left.\begin{array}{l} P_1 + P_2 + P_3 = 3A_1 \\ P_2 + P_3 + P_4 = 3A_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P_{3r} + P_1 + P_2 = 3A_{3r} \end{array}\right\} \quad (D_3)$$

from which follows

$$\begin{aligned}
&P_1 - P_4 = 3(A_1 - A_2)\\
&P_2 - P_5 = 3(A_2 - A_3)\\
&\dots\dots\dots\dots\dots\dots\\
&P_{3r-2} - P_1 = 3(A_{3r-2} - A_{3r-1})\\
&P_{3r-1} - P_2 = 3(A_{3r-1} - A_{3r})\\
&P_{3r} - P_3 = 3(A_{3r-A_1}).
\end{aligned}$$

Now using the above equations, we can write

$$\begin{aligned}
&P_1\\
&P_4 = P_1 - 3(A_1 - A_2)\\
&P_7 = P_4 - 3(A_4 - A_5) = P_1 - 3(A_1 - A_2) - 3(A_4 - A_5)\\
&P_{10} = P_7 - 3(A_7 - A_8) = P_1 - 3(A_1 - A_2) - 3(A_4 - A_5) - 3(A_7 - A_8)\\
&\qquad\vdots\\
&P_{3r-2} = P_{3r-5} - 3(A_{3r-5} - A_{3r-4}) = P_1 - 3(A_1 - A_2) - \cdots - 3(A_{3r-5} - A_{3r-4})
\end{aligned}$$

$$\begin{aligned}
&P_2\\
&P_5 = P_2 - 3(A_2 - A_3)\\
&P_8 = P_5 - 3(A_5 - A_6) = P_2 - 3(A_2 - A_3) - 3(A_5 - A_6)\\
&P_{11} = P_8 - 3(A_8 - A_9) = P_2 - 3(A_2 - A_3) - 3(A_5 - A_6) - 3(A_8 - A_9)\\
&\qquad\vdots\\
&P_{3r-1} = P_{3r-4} - 3(A_{3r-4} - A_{3r-3}) = P_2 - 3(A_2 - A_3) - \cdots - 3(A_{3r-4} - A_{3r-3})
\end{aligned}$$

$$\begin{aligned}
&P_3\\
&P_6 = P_3 - 3(A_3 - A_4)\\
&P_9 = P_6 - 3(A_6 - A_7) = P_3 - 3(A_3 - A_4) - 3(A_6 - A_7)\\
&P_{12} = P_9 - 3(A_9 - A_{10}) = P_3 - 3(A_3 - A_4) - 3(A_6 - A_7) - 3(A_9 - A_{10})\\
&\qquad\vdots\\
&P_{3r} = P_{3r-3} - 3(A_{3r-3} - A_{3r-2}) = P_3 - 3(A_3 - A_4) - \cdots - 3(A_{3r-3} - A_{3r-2}).
\end{aligned}$$

Since $P_1 + P_2 + P_3 = 3A_1$ follows

$$\begin{aligned}
&P_3 = -P_1 - P_2 + 3A_1\\
&P_6 = -P_1 - P_2 + 3A_1 - 3(A_3 - A_4)\\
&P_9 = P_6 - 3(A_6 - A_7) = -P_1 - P_2 + 3A_1 - 3(A_3 - A_4) - 3(A_6 - A_7)\\
&P_{12} = P_9 - 3(A_9 - A_{10}) = -P_1 - P_2 + 3A_1 - 3(A_3 - A_4) - 3(A_6 - A_7) - 3(A_9 - A_{10})\\
&\qquad\vdots\\
&P_{3r} = -P_1 - P_2 + 3A_1 - 3(A_3 - A_4) - 3(A_6 - A_7) - \cdots - 3(A_{3r-3} - A_{3r-2}).
\end{aligned}$$

From the above relations it is clear that the set $P^{(3)}$ of 3-outscribed polygons to the polygon $A = A_1 \dots A_n$ is 4-parametric with the parameters $P_1 = (\lambda_1, \kappa_1)$ and $P_2 = (\lambda_2, \kappa_2)$. Therefore, these polygons have different areas. $\square$

Example 2. Let $n = 6$.

$\underline{k=1}$: $\mathbf{P}^{(1)} = \mathbf{A}$:

$$\begin{array}{cccccc} 2 & 2,25 & 1,75 & 1 & 0,75 & 1,25 \\ 1 & 1,75 & 2,25 & 2 & 1,25 & 0,75 \end{array} \qquad \textstyle\sum_{i=1}^{6} A_i = \binom{9}{9}, \text{ area:} \frac{3}{2}$$

(E_3) is fulfilled.

$\underline{k=3}$: $\mathbf{P}^{(3)}$:

$$\begin{array}{cccccc} \lambda_1 & \lambda_2 & 6-\lambda_1-\lambda_2 & 0,75+\lambda_1 & -1,5+\lambda_2 & 3,75-\lambda_1-\lambda_2 \\ \kappa_1 & \kappa_2 & 3-\kappa_1-\kappa_2 & 2,25+\kappa_1 & 1,5+\kappa_2 & 2,25-\kappa_1-\kappa_2 \end{array}$$

$$2 \cdot \text{area } : 6\lambda_1\kappa_2 - 6\lambda_2\kappa_1 - 4,5\lambda_1 + 2,25\lambda_2 + 13,5\kappa_1 - 6,75\kappa_2 + 6,75$$

<u>Special case 1</u>: $\lambda_1 = 1,25$; $\lambda_2 = 2$; $\kappa_1 = 0,5$; $\kappa_2 = 0,5$:

$$\begin{array}{cccccc} 1,25 & 2 & 2,75 & 2 & 0,5 & 0,5 \\ 0,5 & 0,5 & 2 & 2,75 & 2 & 1,25 \end{array} \qquad \text{area:} \frac{27}{8} = \frac{9}{4} \cdot \frac{3}{2} = 3,375$$

<u>Special case 2</u>: $\lambda_1 = 1,2$; $\lambda_2 = 2$; $\kappa_1 = 0,5$; $\kappa_2 = 0,5$:

$$\begin{array}{cccccc} 1,2 & 2 & 2,8 & 1,95 & 0,5 & 0,55 \\ 0,5 & 0,5 & 2 & 2,75 & 2 & 1,25 \end{array} \qquad \text{area:} \frac{273}{80} = 3,4125$$

Figure 3

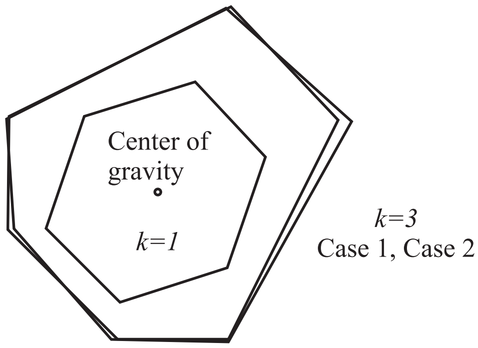

Figure 4

Now in the end of the article it can be said the following.

1. The k-outscribed polygons can be constructed and calculated not only in the case when $GCD(k,n) \le 2$ but also in the case when $GCD(k,n) > 2$. Of course, in both cases the condition (E_k) must be fulfilled.

2. The solution is $2(d-1)$ parametric. For example, $P_1, \ldots, P_{d-1}$ can be arbitrary.
3. In the case when $d > 2$, the areas of k-outscribed polygons are different.

Acknowledgements. The author of the article express thankful to Prof. Dr. Johannes Böhm for many, very useful proposals and information.

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Received May 20, 2005