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The Elementary Geometry of a Triangular World with Hexagonal Circles

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Abstract. We provide a collinearity based elementary axiomatics of optimal quantifier complexity $\forall\exists\forall\exists$ for the geometry inside a triangle and reprove that collinearity cannot be defined in terms of segment congruence, the metric being Hilbert's projective metric.

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1. Introduction

Hilbert [7, Anh. 1] introduced a metric inside a bounded convex domain of the Euclidean plane by defining the length of a segment AB – the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ intersecting the convex domain in the points B' and A' – as the logarithm of the cross-ratio $[A, B, B', A']$ of A', A, B, B', in case $A \neq B$, and 0 otherwise. In case the boundary of the convex domain is an ellipse, the resulting geometry is Klein's model of plane hyperbolic geometry.

The geometry one obtains in case the boundary of the convex domain is a triangle was first studied in [20], and then in [5] and [6]. Circles in this geometry being hexagons, one may call it, with [2], Zenonian. In all of these papers, the

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triangle is embedded in the Euclidean plane over the real numbers and the metric is defined in terms of the logarithm function. The resulting geometry is thus not an elementary one, in the sense that there can be no axiom system expressed in first-order logic having only this standard model, nor do logarithms make sense in arbitrary ordered fields, which would be the most natural candidate to replace the field of real numbers. Axiomatizations in first-order logic produce what is called an *elementary* version of the geometry to be axiomatized, and in which fewer theorems are true than in the standard versions over the real field (e.g. the Archimedean behavior of any pair of segments can no longer hold in such an elementary version). A comprehensive survey of axiomatizations of elementary geometries can be found in the second part of [21].

Elementary hyperbolic geometry was born in 1903 when Hilbert [8], [7] provided, using the end-calculus to introduce coordinates, a first-order axiomatization for it. By providing pure incidence-based definitions for the usual notions of hyperbolic geometry (such as Tarski's betweenness and equidistance, see [21]), Menger [13] showed that elementary hyperbolic geometry can be expressed only in terms of the notions of point, line, and incidence (or points and the ternary relation of collinearity). He and his students have provided in [14]–[17], [1], [3], [10], [11] readable incidence-based axiom systems for elementary hyperbolic geometry, culminating in the purely first-order logic axiom system of H. Skala [22]. For Menger [17, p. 91], the results obtained by Jenks [11] and by DeBaggis [3] rank "among the most remarkable achievements in the theory of the hyperbolic plane and in all of postulational geometry".

The purpose of this paper is: (i) to axiomatize the geometry inside a triangle in the manner of Menger-Skala, in a first-order language based on the notions of point and the ternary relation L of collinearity; (ii) to prove that its quantifier complexity is the lowest possible one; (iii) to define the notion of equidistance $\equiv$ of two point-pairs in terms of L in a manner that by-passes the notion of logarithm; and (iv) to reprove (as this already follows from [5, Proposition 8]) that L is not definable in terms of $\equiv$.

The axiomatization is not meant to be ideal; its purpose is rather to show that an elementary version of this geometry is possible and that it can be expressed in terms of collinearity alone.

2. The elementary incidence geometry of the interior of a triangle

2.1. The informal axiom system

To describe in terms of incidence or collinearity what it is like to be inside a triangle, one needs to first describe what it is like to be inside a convex curve. In terms of the notion of betweenness, this has been first done by Sperner [23], without a precise description of the class of models of the axiom system. A precise description of the models of a like-minded axiom system was provided by Szczerba [25]. In [1], [3], [10], [11] we find the axioms for the elementary incidence geometry of the interior of a convex set in a projective plane. Here the term 'convex' has the meaning assigned to it by Steinitz [24, p. 34], a set with the property that any two of its points can be joined by a line segment and for which there exists a projective line it is disjoint from. To these axioms, H. Skala [22] adds the projective forms of the Desargues and Pappus axioms, in which all the points are interior points[1] (she allows for rimpoints (boundary points) as well, although, by the results of [23] and [25], assuming these axioms for interior points only suffices), as well as an axiom that would imply that the boundary is an ellipse in the projective plane (Pascal's theorem on hexagons inscribed in conics). What we need in our case is to drop one of the axioms, which excludes weakly convex domains, having straight segments in their rim (Axiom 7 in [22]), and to replace the Pascal axiom with one implying that the boundary of the convex domain is a triangle, while keeping all the other axioms.

The axiom system can be formulated either in a two-sorted first-order language, with individual variables for *points* (upper-case) and *lines* (lower-case), and a single binary relation $|$ as primitive notion, with $P|l$ to be read 'point P is incident with line l', or in a one-sorted language, with *points* as the only variables and the ternary relation of collinearity L, with $L(abc)$ to be read as 'a, b, c are collinear points (not necessarily different)'. We shall first formulate the axioms in an informal language paraphrasing the formal axioms in terms of the two-sorted language, and then present the formal axioms in terms of L alone.

To shorten the statement of some of the axioms we define:

(1) the notion of betweenness β, with $\beta(A,B,C)$ ('B lies between A and C') to denote 'the points A, B, and C are three distinct points and every line through B intersects at least one line of each pair of intersecting lines which pass through A and C;

(2) the notions of ray and segment in the usual way, i.e. a point X is on (incident with) a ray $\overrightarrow{AB}$ (with $A \neq B$) if and only if $X = A$ or $X = B$ or $\beta(A,X,B)$ or $\beta(A,B,X)$, and a point X is incident with the segment AB if and only if $X = A$ or $X = B$ or $\beta(A,X,B)$;

(3) the notion of ray parallelism, for two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, not part of the same line – to be denoted by $\overrightarrow{AB} \upharpoonleft\!\upharpoonright \overrightarrow{CD}$ – by the condition that every line that meets one of the two rays meets the other ray or the segment AC. Two lines or a line and a ray are said to be parallel if they contain parallel rays.

The axioms, which we present in informal language, their formalization being straightforward, are:

A 1. *Any two distinct points are on exactly one line.*

A 2. *Each line is on at least one point.*

A 3. *There exist three collinear and three non-collinear points.*

[1]L. W. Szczerba told me in December 2003 that he had proved that Pappus for interior points does imply Desargues for interior points, but the result was never published and there is no manuscript containing the proof either. Since I do not know how to prove the implication, I have decided to assume both axioms in this paper.

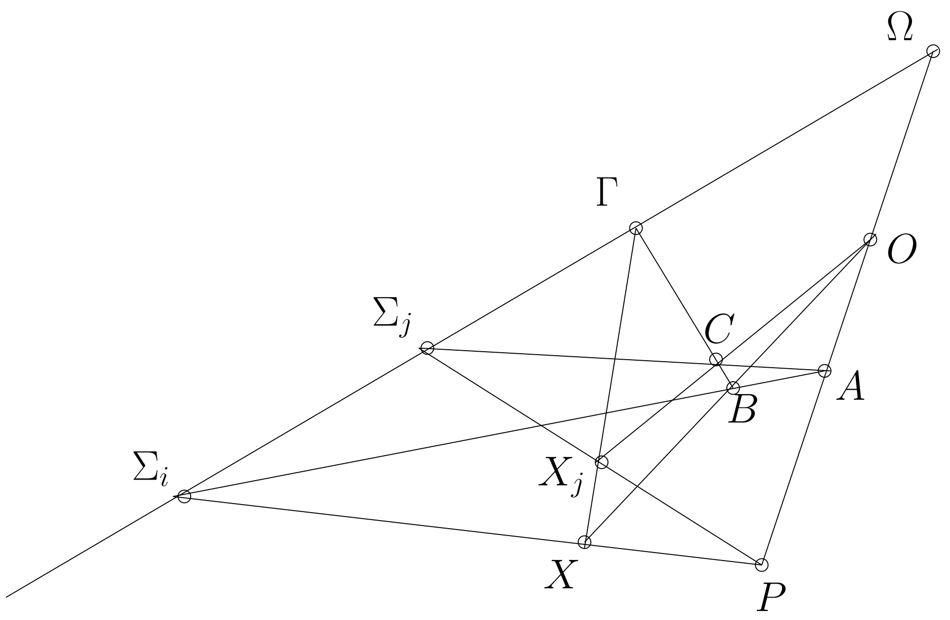

Figure 1. The rim must be a triangle

A 4. *Of three collinear points, at least one has the property that every line through it intersects at least one of each pair of intersecting lines through the other two.*

A 5. *If P is not on l, then there exist two distinct lines on P not meeting l and such that each line meeting l meets at least one of those two lines.*

A 6. *(Desargues) Let a, b, c be three different lines, O a point incident with each of them, each containing pairs of distinct points (A_1, A_2), (B_1, B_2), and (C_1, C_2) respectively. If M lies on the lines A_1B_1 and A_2B_2, N lies on the lines A_1C_1 and A_2C_2, and P lies on the lines B_1C_1 and B_2C_2, then M, N, and P are collinear.*

A 7. *(Pappus) Let a and b be different lines containing points A_1, A_2, A_3 and B_1, B_2, B_3 respectively, with $A_i \neq B_j$ for all $i, j \in \{1, 2, 3\}$ and $A_i \neq A_j$, $B_i \neq B_j$ for $i \neq j$. If M lies on the lines A_1B_2 and A_2B_1, N lies on the lines A_1B_3 and A_3B_1, and P lies on the lines A_2B_3 and A_3B_2, then M, N, and P are collinear.*

A 8. *If P, X_1, X_2, X_3, X_4 are five points, with $P \neq X_k$, for all $k = 1, 2, 3, 4$, and such that, for all $m \neq n$, X_m does not belong to the ray $\overrightarrow{PX_n}$, then there are $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, and there are points X, A, B, C, O such that X is on the ray $\overrightarrow{PX_i}$, $X \neq P$, the triples (P, A, O), (X, B, O), (X_j, C, O) consist of different collinear points, and $\overrightarrow{PX} \upharpoonleft\upharpoonright \overrightarrow{AB}$, $\overrightarrow{PX_j} \upharpoonleft\upharpoonright \overrightarrow{AC}$, $\overrightarrow{XX_j} \upharpoonleft\upharpoonright \overrightarrow{BC}$.*

That the axioms A1–A6 characterize open convex domains in an ordered Pappian projective plane, can be seen by noticing that, according to [1], with the betweenness relation defined as above, the axioms A1–A5 imply both the linear betweenness axioms and the Pasch axiom. By A5, a model of A1–A5 cannot contain a projective line, and thus, by [4], would have to be disjoint from a projective line as well. Thus, a model of A1–A7 satisfies all the axioms from [25], so it must be a convex open subset of an ordered Pappian projective plane. All that's left to show is the fact that the boundary of that open subset is a triangle.

The reason axiom A8 determines the shape of the boundary can be seen by noticing that if the boundary is indeed a triangle with vertices $\Delta_1, \Delta_2, \Delta_3$ (we will denote all rimpoints by capital Greek letters), with P a point in its interior, and the four points X_i for $i = 1, 2, 3, 4$ are such that all the rays $\overrightarrow{PX_i}$ are different, then by the pigeonhole principle, at least two of the rays $\overrightarrow{PX_i}$ must lie inside the same closed triangles $P\Delta_k\Delta_l$ for some k and l in $\{1, 2, 3\}$. Let's denote two such rays by $\overrightarrow{PX_i}$ and $\overrightarrow{PX_j}$ and let Σ_i and Σ_j denote the points in which they intersect the sides of the triangle $\Delta_1\Delta_2\Delta_3$ (see Figure 1). One of Σ_i and Σ_j must be different from Δ_k and Δ_l, and so there must be points on the side $\Delta_k\Delta_l$ of the rim triangle that lie outside the segment $\Sigma_i\Sigma_j$. W. l. o. g. me may assume that there is such a point Γ on the open segment $\Delta_k\Delta_l$ with Σ_j strictly between Γ and Σ_i. Let Ω be a point on the segment $\Delta_k\Delta_l$, such that Γ lies strictly between Ω and Σ_j. Let O be a point on the ray $\overrightarrow{P\Omega}$, and let A be a point on the open segment OP. By Pasch, the ray $\overrightarrow{\Gamma X_j}$ must intersect the segment $P\Sigma_i$, an intersection we denote by X; the segments OX and $A\Sigma_i$ must also intersect, an intersection we denote by B, and the segments $B\Gamma$ and OX_j must intersect as well, an intersection we denote by C (these two pairs of segments must intersect by Pasch as well). Given that the triangles ABC and PXX_j are perspective from a point (namely from O), they must be, by Desargues – which holds, as first shown in [23], in the extended plane as well – perspective from a line as well, and thus the sides AC and PX_j must meet in a point that is collinear with Γ and Σ_i, i.e. A, C, Σ_j must be collinear. This proves that A8 holds in case the rim is a triangle.

To show that interiors of convex domains in Pappian projective planes which satisfy A8 must have triangular rims, we first show that any point of the rim must lie on a segment that belongs to the rim. Let P be an interior point and let Λ_1 be an arbitrary point of the rim. Let X_1 be a point on the open segment $P\Lambda_1$, let X_2 and X_3 be two points not on the line $P\Lambda_1$, such that X_1 lies between X_2 and X_3, and let X_4 be a point between X_1 and X_2. Let Λ_i denote the intersection of the rays PX_i, for $i = 2, 3, 4$, with the rim. By A8, at least one of the segments $\Lambda_i\Lambda_j$, with $i \neq j$ must belong to the rim. Let us denote by i_0 and j_0 the pair of indices for which $\Lambda_{i_0}\Lambda_{j_0}$ belongs to the rim. If one of i_0 or j_0 is 1, we are done. If $\{i_0, j_0\}$ is $\{2, 3\}$ or $\{3, 4\}$, then we are also done, for then Λ_1 would have to belong to the segment $\Lambda_2\Lambda_3$ or $\Lambda_3\Lambda_4$. The only situation left, is that in which the indices $\{i_0, j_0\}$ is $\{2, 4\}$. Now keep the points X_1, X_2, X_3 fixed and let X_4 vary on the open segment X_2X_1. If there is a position of X_4, for which the pair of indices i_0, j_0 A8 ensures to exist is other than $\{2, 4\}$, then we are done. Suppose that, for all values of X strictly between X_1 and X_2, we have that $\Lambda_2\Lambda$ belongs to the rim, were by Λ we have denoted the intersection of the ray PX with the rim. Denote by A, a point in the extended plane, the intersection of the line $P\Lambda_1$ with the line $\Lambda_2\Lambda_4$. Then all the points on the open segment Λ_2A must be rimpoints. To see this, notice that, for any point Z on the open segment Λ_2A, the segment PZ will intersect the open segment X_1X_2 in a point X, and $\Lambda_2\Lambda$, where Λ is the point of intersection of the ray PX with the rim, must belong to the rim. If Λ were not collinear with Λ_2 and Λ_4, then we would have rays emanating from P and intersecting the rim in two distinct points (ray $P\Lambda$ intersects segment $\Lambda_2\Lambda_4$ or ray $P\Lambda_4$ intersects segment $\Lambda_2\Lambda$), contradicting the convexity of the domain. If $A = \Lambda_1$, then we are done, for then $\Lambda_1\Lambda_2$ belongs to the rim. If $A \neq \Lambda_1$, then A lies on the ray $P\Lambda_1$. Thus the three points of the extended plane P, Λ_1, A are either in the order (i) $P\Lambda_1 A$ or in the order (ii) $PA\Lambda_1$. Let R be a point inside or domain such that P lies strictly between X_4 and R. By Pasch, in case (i), the line $R\Lambda_1$ intersects the open segment $\Lambda_4 A$ in what must be a rimpoint, say Δ. Now, on the segment joining R with Δ, we find another rimpoint, namely Λ_1, contradicting the convexity of our domain. In case (ii), the segment $X_4\Lambda_1$ must intersect the open segment $\Lambda_4 A$ in what must be a rimpoint, say Δ. Thus on the segment joining X_4 with the rimpoint Λ_1 we find another rimpoint, namely Δ, again contradicting the convexity of our domain.

The rim is thus composed of a union of segments. There cannot be fewer than three such segments, and there cannot be more either, for if there were four or more segments belonging to the rim, which are part of different lines, then we could choose four points Λ_i, with $i \in \{1, 2, 3, 4\}$, each lying inside a different segment, and we could pick any interior point P and any points X_i on the segments $P\Lambda_i$, making it impossible to find X, A, B, C, O as required by A8.

2.2. The formal axiom system

We will now express the axiom system presented earlier in terms of points (which will now be denoted by lower-case letters) and collinearity (the relation symbol L) alone.

We first define $\upharpoonleft|$, with $ab \upharpoonleft| cd$ to be read as 'the ray $\vec{ab}$ is parallel to a line cd', as in [19], by

$$\begin{aligned} ab \upharpoonleft| cd \quad :\Leftrightarrow \quad & (\forall uv)(\exists t)\, \neg(L(abu) \wedge L(cdu)) \\ & \wedge(L(cdu) \rightarrow (L(bvt) \wedge (L(aut) \vee L(cdt)))), \end{aligned}$$

and then define

$$ab \upharpoonleft\upharpoonright cd :\Leftrightarrow ab \upharpoonleft| cd \wedge cd \upharpoonleft| ab.$$

Here $ab \upharpoonleft\upharpoonright cd$ stands for 'the ray $\overrightarrow{ab}$ is parallel to the ray $\overrightarrow{cd}$'. The reason why we use a different definition from that given in Section 2.1 will become apparent when we will discuss the quantifier complexity of the axiom system. The quantifier complexity of the old definition is $\forall\exists\forall\exists$ whereas that of the one above is $\forall\exists$. We also define the betweenness relation ($B(abc)$ stands for 'b lies between a and c') as

$$B(abc) :\Leftrightarrow (\forall uv)(\exists w)\, L(bvw) \wedge (L(auw) \vee L(cuw)).$$

and the relation λ, with $\lambda(abc)$ to be read as 'points a, b, c are collinear and different', by

$$\lambda(abc) :\Leftrightarrow L(abc) \wedge a \neq b \wedge b \neq c \wedge c \neq a.$$

The axioms are:

L 1. $L(aba)$,

L 2. $L(abc) \to L(cba) \wedge L(bac)$,

L 3. $a \neq b \wedge L(abc) \wedge L(abd) \to L(acd)$,

L 4. $(\exists abc)\, L(abc)$,

L 5. $(\exists abc)\, \neg L(abc)$,

L 6. $(\forall a_0a_1a_2uv\exists w)\, L(a_0a_1a_2) \to [\bigvee_{i=0}^{2} L(a_ivw) \wedge (L(a_{i+1}uw) \vee L(a_{i-1}uw))]$,

L 7. $(\forall a_1a_2p\exists q_1q_2\forall uv\exists w)\, \neg L(a_1a_2p) \to [\neg L(pq_1q_2) \wedge \neg(L(a_1a_2u) \wedge (L(pq_1u)$ $\vee L(pq_2u))) \wedge (L(a_1a_2u) \wedge \neg L(a_1a_2v) \to (L(uvw) \wedge (L(pq_1w) \vee L(pq_2w)))]$,

L 8. $L(oa_1a_2) \wedge L(ob_1b_2) \wedge L(oc_1c_2) \wedge L(a_1b_1m) \wedge L(a_2b_2m) \wedge L(a_1c_1n)$ $\wedge L(a_2c_2n) \wedge L(b_1c_1p) \wedge L(b_2c_2p) \wedge \neg L(oa_1b_1) \wedge \neg L(ob_1c_1)$ $\wedge \neg L(oa_1c_1) \wedge a_1 \neq a_2 \wedge o \neq a_2 \wedge o \neq b_2 \wedge o \neq c_2 \to L(mnp)$,

L 9. $L(a_1a_2a_3) \wedge L(b_1b_2b_3) \wedge \bigwedge_{i\neq j}(a_i \neq a_j \wedge b_i \neq b_j) \wedge \bigwedge_{1\leq i,j\leq 3} a_i \neq b_j$ $\wedge L(a_1b_2m) \wedge L(a_2b_1m) \wedge L(a_1b_3n) \wedge L(a_3b_1n) \wedge L(a_2b_3p) \wedge L(a_3b_2p)$ $\to L(mnp)$,

L 10. $(\forall px_1x_2x_3x_4)(\exists xoabc)\, (\bigwedge_{i\neq j} \neg B(px_ix_j)) \to \lambda(oap)$ $\wedge[\bigvee_{i\neq j}(\lambda(ocx_j) \wedge (B(pxx_i) \vee B(px_ix)) \wedge bc \upharpoonleft\upharpoonright xx_j \wedge ac \upharpoonleft\upharpoonright px_j)]$ $\wedge \lambda(obx) \wedge ab \upharpoonleft\upharpoonright px$.

Here L1–L3 correspond to the axioms A1, A2; L4, L5 to A3; L6 to A4; L7 to A5; L8 and L9 are the Desargues and Pappus axioms; L10 corresponds to A8. The quantifier complexity of this axiom system is $\forall\exists\forall\exists$, as both axioms L7 and L10 have this quantifier complexity. We will now prove that this complexity is minimal, i.e. that there is no axiom system in the same language for the theory axiomatized by L1–L10 all of whose axioms have lower quantifier complexity.

2.3. Optimal quantifier complexity

The proof that this is so, i.e. that there is no $\forall\exists\forall$-axiom system for our theory, is based on the idea used in the proof in [19] that plane hyperbolic geometry axiomatized with L alone has complexity $\forall\exists\forall\exists$. Let $\mathcal{D}$ denote the theory axiomatized by L1–L10.

Lemma 1. *There is no $\forall\exists\forall$-axiom system for $\mathcal{D}$.*

Proof. According to [12] (cf. also [9, p. 299]), a theory $\mathcal{T}$ is axiomatizable by means of $\forall\exists\forall$-sentences if the union of any ascending $\leq_1$-chain of models of $\mathcal{T}$ is a model of $\mathcal{T}$. Two models $\mathfrak{A}$ and $\mathfrak{B}$ are such that $\mathfrak{A}\leq_1\mathfrak{B}$ if and only if $\mathfrak{A}\subseteq\mathfrak{B}$ and for any purely existential formula $\varphi(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_k)$ are free variables occurring in φ, and for any k-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ with a_i elements of the universe of $\mathfrak{A}$, we have

$$\mathfrak{B}\models \varphi(\mathbf{a}) \Rightarrow \mathfrak{A}\models \varphi(\mathbf{a}). \tag{1}$$

Thus $\mathcal{T}$ is axiomatizable by means of $\forall\exists\forall$-sentences if and only if the union of any sequence of models $\mathfrak{A}_n$, with $n \in \mathbb{N}$ and $\mathfrak{A}_n \leq_1 \mathfrak{A}_{n+1}$, is a model of $\mathcal{T}$. We shall construct such an ascending chain of models of $\mathcal{D}$, whose union is not a model of $\mathcal{D}$. Let $\mathfrak{K}(K,r)$ denote the model of $\mathcal{D}$ in the Euclidean coordinate plane over an ordered field K, whose point-set is the interior of an equilateral triangle with center at the origin $(0,0)$ with circumradius r, $r \in K$, $r > 0$. Let $\overline{K}$ denote the real closure of the ordered field K. Let, for all $n \geq 1$, $\mathfrak{A}_n := \mathfrak{K}(K_n, r_n)$. Let $K_1 = \overline{\mathbb{Q}}$, $r_1 = 1$, and let, for all $n \geq 1$, $K_{n+1} = \overline{K_n(t_n)}$, $\epsilon_n = \frac{1}{t_n}$, and $r_{n+1} = r_n - \epsilon_n$, where $K_n(t_n)$ denotes the field of fractions of the ring of polynomials in t_n, an indeterminate, and the order of K_n is extended to $K_n(t_n)$ by defining $(\sum_{i=0}^k a_i t^i)(\sum_{i=0}^m b_j t^j)^{-1}$ – where $a_i, b_j \in K$, with $a_k \neq 0$ and $b_m \neq 0$ – to be > 0 if and only if $a_k b_m > 0$. Under this ordering $x < t_n$ for all $x \in K_n$, and thus ϵ_n is infinitely small with regard to the elements of K_n. That (1) is satisfied with $\mathfrak{A}= \mathfrak{A}_n$ and $\mathfrak{B}= \mathfrak{A}_{n+1}$ can be seen by noticing that the validity in $\mathfrak{A}$ of any existential formula $\varphi(\mathbf{x})$, in which the free variables $\mathbf{x} = (x_1, \dots x_k)$ are interpreted as elements in $\mathfrak{A}$, translates into the validity in the underlying field K of $\mathfrak{A}$ of a system of equations, negated equations, and inequalities. It follows from Tarski's elimination of quantifiers for real closed fields that if such a system is not solvable in K (which is a real closed field), then it cannot be solvable, with the same interpretation of $\mathbf{x}$ in any real closed field which is an extension of K either.

Now $\mathfrak{U}:= \cup_{n\geq 1}\mathfrak{A}_n$ is not a model of $\mathcal{D}$ as there are no (limiting) parallels (see [19] for a proof). □

We have thus shown that

Theorem 1. L1–L10 *is a* $\forall\exists\forall\exists$*-axiom system of minimal quantifier complexity for the collinearity theory of the interior of triangles in Pappian ordered projective planes.*

3. Defining the metric

To define the congruence of two segments in this geometry one proceeds exactly like in the case of hyperbolic geometry, as done in [17]. Given two segments P_1P_2 and $P_1'P_2'$ on two lines l and l' which are parallel (i.e. which meet in a rimpoint Π), we can assume that we have renamed the endpoints of the two segments such that $\overrightarrow{P_2P_1} \upharpoonleft\!\!\upharpoonright \overrightarrow{P_2'P_1'}$. Let m_i be the line through P_i which is parallel to l' and different from l (such a line exists by A5), and let m_i' be the line through P_i' which is parallel to l and different from l'. Let A_i be the point of intersection of m_i and m_i'. The two segments are congruent precisely if A_1, A_2, Π are collinear, or, put otherwise, if $\overrightarrow{A_2A_1} \upharpoonleft\!\!\upharpoonright \overrightarrow{P_2P_1}$. If the two segments P_1P_2 and $P_1'P_2'$ lie on two lines l and l' which are not parallel, then there is a line m which is a common parallel to l and l', which can chosen to be the line joining two 'ends' of l and l' which do not lie on the same side of the triangle forming the rim. We then say that P_1P_2 and $P_1'P_2'$ are congruent if there is a segment MN on m to which they are both congruent.

To see that this notion coincides with the notion of segment congruence in terms of a Hilbert metric, notice that the latter amounts to the equality of two cross-ratios, and that our definition is saying precisely the same thing, given that cross-ratios are preserved under perspectivities. If we denote, in the case in which the two segments lie on parallel lines, the other end of l with Δ and the other end of l' with Γ (thus l intersects the rim in the rimpoints Π and Δ, and l' intersects it in Π and Γ), and we denote by P the intersection of the line A_1A_2 with $\Gamma\Delta$ (which is either a line or a side of the rim, making P a point in the former case, and a rimpoint in the latter) then the perspectivity with center Δ maps P, A_1, A_2, Π into Γ, P_1, P_2, Π, whereas the perspectivity with center Γ maps P, A_1, A_2, Π into Δ, P_1', P_2', Π. The cross ratios $[P_1, P_2, \Pi, \Delta]$ and $[P_1', P_2', \Pi, \Gamma]$ are both equal to $[A_1, A_2, \Pi, P]$.

Unlike in hyperbolic geometry, it is not possible to define collinearity in terms of segment congruence. To see this, let $\mathfrak{P}$ be Phadke's [20] model for this geometry, which consists of the first quadrant of the affine plane over $\mathbb{R}$, with the distance $\varrho(a, b)$ between two points $a = (\alpha, \beta)$ and $b = (\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta$ positive real numbers, defined as $|\log(\alpha/\gamma)|$ if the line $L(a, b)$ joining a and b does not make a positive intercept on the x axis; $|\log(\beta/\delta)|$ if $L(a, b)$ does not make a positive intercept on the y axis; $|\log((\alpha\delta)/(\beta\gamma))|$ if $L(a, b)$ makes positive intercepts on both axes. The mapping $\varphi : \mathfrak{P} \to \mathfrak{P}$, defined by $\varphi(x, y) = (x^2, y^2)$ preserves segment congruence, but not collinearity. By Padoa's method, collinearity is not definable in terms of segment congruence, even if we allow for logical means beyond first-order logic that would capture more – or all the – aspects of $\mathfrak{P}$. This fact also follows from [5, Proposition 4 (iii), Proposition 8].

Given that Proposition 7 of [5], which states that the pure segment congruence theory of the triangular world is precisely that of a two-dimensional Minkowski geometry (normed 2-dimensional space), remains valid when $\mathbb{R}$ is replaced by an Archimedean ordered Euclidean field (the norm will then take its values in the positive cone of that field), we can provide an infinitary $\mathcal{L}_{\omega_1\omega}$ axiomatization for the pure segment congruence theory by using the result in [18], which states that the betweenness relation of the Minkowski geometry can be defined in terms of segment congruence. One simply needs to add to the axiom system of a two-dimensional Minkowski geometry, rephrased in terms of segment congruence alone, an axiom stating that the set of all points equidistant from a fixed point is a hexagon. Note that in [18], the definition of φ_n on page 8 should be changed to

$$\begin{aligned} \varphi_{n+1}(a, b, x) \quad :\Leftrightarrow \quad & \varphi_n(a, b, x) \wedge [(\forall x_3)(\exists x_1 x_2 y)\, \varphi_0(a, b, x_3) \\ & \to \bigwedge_{i=1}^{2} \varphi_0(a, b, x_i) \wedge xy \equiv_2 x_3 x \wedge xy \leq x_1 x_2]. \end{aligned}$$

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