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The Geometric Structure of the Inverse Gamma Distribution*

Tongzhu Li Linyu Peng Huafei Sun

Department of Mathematics, Beijing Institute of Technology
Beijing, 100081 China
e-mail: litz@pku.org.cn bityuquansuperman@126.com sunhuafei@263.net

Abstract. In the present paper we study the geometric structure of the inverse Gamma manifold from the viewpoint of information geometry and give the Kullback divergence, the J-divergence and the geodesic equations. Also, some applications of the inverse Gamma distribution are provided.

Keywords: the inverse Gamma distribution, Ricci curvature, Gaussian curvature, divergence

1. Introduction

As we all know that information geometry has been applied into many fields, such as statistical inference, system control, neural network and quantum theory. Some scholars studied the statistical manifolds from the viewpoint of information geometry. In [1], Amari proposed the concept of information geometry and studied the exponential distribution families. Dodson ([4]) and his colleagues investigated the bivariate normal distribution, the Gamma distribution, the McKay distribution and the Frund distribution and gave the geometric structures of these distributions.

In the present paper, we obtain the Ricci curvatures, the Gaussian curvature of the inverse Gamma manifold and give the Kullback divergence, the J-divergence and the geodesic equations. In Section 5, some applications of the inverse Gamma distribution are provided.

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2. Preliminaries

Definition 2.1. *For a density function $p(x,\theta)$, where $\theta = (\theta^1, \theta^2, \ldots, \theta^n)$, the function $l(x,\theta)$ is defined as*

$$l(x,\theta) = \ln p(x,\theta). \tag{2.1}$$

Definition 2.2. *We call $M = \{l(x,\theta) | (\theta^1, \theta^2, \ldots, \theta^n) \in R^n\}$ an n-dimensional distribution manifold, where $(\theta^1, \theta^2, \ldots, \theta^n)$ plays the role of the coordinate system.*

Definition 2.3. *The Fisher information matrix (g_{ij}) is defined by*

$$(g_{ij}) = (E_\theta[\partial_i l \ \partial_j l]), \ i, \ j = 1, 2, \ldots, n, \tag{2.2}$$

where

$$\partial_i l = \frac{\partial}{\partial \theta^i} l(x,\theta), \ i = 1, 2, \ldots, n.$$

The inverse of (g_{ij}) is denoted by

$$(g^{ij}) = (g_{ij})^{-1}, \ i, \ j = 1, 2, \ldots, n. \tag{2.3}$$

Definition 2.4. *The Riemannian connection Γ_{ijk} is defined by*

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}), \ i, \ j, \ k = 1, 2, \ldots, n \tag{2.4}$$

and the α-connections are defined by

$$\Gamma^{(\alpha)}_{ijk} = \Gamma_{ijk} - \frac{\alpha}{2} T_{ijk}, \ i, \ j, \ k = 1, 2, \ldots, n, \tag{2.5}$$

where

$$T_{ijk} = E[\partial_i l \ \partial_j l \ \partial_k l], \ i, \ j, \ k = 1, 2, \ldots, n. \tag{2.6}$$

Definition 2.5. *In the θ coordinate system, the α-curvature tensors $R^{(\alpha)}_{ijkl}$ are defined by*

$$\begin{gathered} R^{(\alpha)}_{ijkl} = (\partial_j \Gamma^{s(\alpha)}_{ik} - \partial_i \Gamma^{s(\alpha)}_{jk}) g_{sl} + (\Gamma^{(\alpha)}_{jtl} \Gamma^{t(\alpha)}_{ik} - \Gamma^{(\alpha)}_{itl} \Gamma^{t(\alpha)}_{jk}), \\ i, \ j, \ k, \ l, \ s, \ t = 1, 2, \ldots, n, \end{gathered} \tag{2.7}$$

where

$$\Gamma^{k(\alpha)}_{ij} = \Gamma^{(\alpha)}_{ijs} g^{sk}, \ i, \ j, \ k, \ s = 1, 2, \ldots, n. \tag{2.8}$$

Definition 2.6. *The Ricci curvatures $R^{(\alpha)}_{ik}$ are defined by*

$$R^{(\alpha)}_{ik} = R^{(\alpha)}_{ijkl} g^{jl}, \ i, \ j, \ k, \ l = 1, 2, \ldots, n. \tag{2.9}$$

Definition 2.7. *For $n = 2$, the Gaussian curvature $K^{(\alpha)}$ is defined by*

$$K^{(\alpha)} = \frac{R^{(\alpha)}_{1212}}{\det(g_{ij})}. \tag{2.10}$$

Definition 2.8. *Assume $P(x,\theta_p)$ and $P(x,\theta_q)$ are two points of the manifold M, then the Kullback divergence $K(P_p,P_q)$ is defined by*

$$\begin{aligned} K(P_p,P_q) &= E_{\theta_p}\left[\ln\frac{P(x,\theta_p)}{P(x,\theta_q)}\right] \\ &= \int P(x,\theta_p)\ln\frac{P(x,\theta_p)}{P(x,\theta_q)}dx \end{aligned} \tag{2.11}$$

and the J-divergence $J(P_p,P_q)$ is defined by

$$J(P_p,P_q)) = \int (P(x,\theta_p)-P(x,\theta_q))\ln\frac{P(x,\theta_p)}{P(x,\theta_q)}dx. \tag{2.12}$$

When the two points $P(x,\theta_p)$ and $P(x,\theta_q)$ are close enough, from the Taylor's formula, one can see that

$$K(\theta,\theta+d\theta) = \frac{1}{2}ds^2$$

and

$$J(\theta,\theta+d\theta) = ds^2.$$

Definition 2.9. *The geodesic equations of the manifold M with coordinate $\theta = (\theta^1,\theta^2,\dots,\theta^n)$ are characterized by*

$$\frac{d^2\theta^k}{dt^2} + \Gamma^k_{ij}\frac{d\theta^i}{dt}\frac{d\theta^j}{dt} = 0. \tag{2.13}$$

3. The inverse Gamma manifold

The set

$$\{p(x,\theta) = \frac{\mu^\lambda}{x^{\lambda+1}\Gamma(\lambda)}\exp\left\{-\frac{\mu}{x}\right\},\ \theta=(\theta^1,\theta^2)=(\mu,\lambda),\ x>0,\lambda>0,\mu>0\}$$

is called the *inverse Gamma manifold*, where

$$p(x,\theta) = \frac{\mu^\lambda}{x^{\lambda+1}\Gamma(\lambda)}\exp\left\{-\frac{\mu}{x}\right\},\ x>0,\lambda>0,\mu>0$$

is the probability density function of the inverse Gamma distribution.

Theorem 3.1. *The α-Gaussian curvature of the inverse Gamma manifold is given by*

$$\begin{aligned} K^{(\alpha)} &= \frac{1-\alpha^2}{4(\lambda\Gamma(\lambda)\Gamma''(\lambda)-\lambda(\Gamma'(\lambda))^2-\Gamma^2(\lambda))^2}(\Gamma^3(\lambda)\Gamma''(\lambda) \\ &\quad - \Gamma^2(\lambda)(\Gamma'(\lambda))^2 + \lambda\Gamma^3(\lambda)\Gamma^{(3)}(\lambda) - 3\lambda\Gamma^2(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2\lambda\Gamma(\lambda)(\Gamma'(\lambda))^3). \end{aligned}$$

Proof. Defining

$$\begin{aligned} l(x;\theta) &= \ln p(x,\theta) \\ &= -\frac{\mu}{x} + \lambda \ln\mu - (\lambda+1)\ln x - \ln\Gamma(\lambda), \end{aligned}$$

then we can see that

$$\partial_1 l = -\frac{1}{x} + \frac{\lambda}{\mu},\ \partial_2 l = \ln\mu - \ln x - \frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$$

and

$$\partial_1\partial_1 l = -\frac{\lambda}{\mu^2},\ \partial_1\partial_2 l = \partial_2\partial_1 l = \frac{1}{\mu},\ \partial_2\partial_2 l = \frac{(\Gamma'(\lambda))^2 - \Gamma(\lambda)\Gamma''(\lambda)}{\Gamma^2(\lambda)}.$$

From (2.2), we get the Fisher information matrix

$$(g_{ij}) = \begin{pmatrix} \frac{\lambda}{\mu^2} & -\frac{1}{\mu} \\ -\frac{1}{\mu} & \frac{\Gamma(\lambda)\Gamma''(\lambda)-(\Gamma'(\lambda))^2}{\Gamma^2(\lambda)} \end{pmatrix}.$$

Thus the square of the arc length is given by

$$(ds)^2 = \frac{\lambda}{\mu^2}(d\lambda)^2 - \frac{2}{\mu}d\lambda d\mu + \frac{\Gamma(\lambda)\Gamma''(\lambda) - (\Gamma'(\lambda))^2}{\Gamma^2(\lambda)}(d\mu)^2.$$

The inverse matrix of (g_{ij}) is given by

$$(g^{ij}) = \begin{pmatrix} \frac{\mu^2\Gamma(\lambda)\Gamma''(\lambda)-\mu^2(\Gamma'(\lambda))^2}{\lambda\Gamma(\lambda)\Gamma''(\lambda)-\lambda(\Gamma'(\lambda))^2-\Gamma^2(\lambda)} & \frac{\mu\Gamma^2(\lambda)}{\lambda\Gamma(\lambda)\Gamma''(\lambda)-\lambda(\Gamma'(\lambda))^2-\Gamma^2(\lambda)} \\ \frac{\mu\Gamma^2(\lambda)}{\lambda\Gamma(\lambda)\Gamma''(\lambda)-\lambda(\Gamma'(\lambda))^2-\Gamma^2(\lambda)} & \frac{\lambda\Gamma^2(\lambda)}{\lambda\Gamma(\lambda)\Gamma''(\lambda)-\lambda(\Gamma'(\lambda))^2-\Gamma^2(\lambda)} \end{pmatrix}.$$

From (2.6), we can get

$$T_{111} = -\frac{2\lambda}{\mu^3},\ T_{121} = T_{211} = T_{112} = \frac{1}{\mu^2},\ T_{221} = T_{212} = T_{122} = 0, \tag{3.1}$$

$$T_{222} = \frac{\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3}{\Gamma^3(\lambda)}. \tag{3.2}$$

From (2.4), we can get

$$\Gamma_{111} = -\frac{1}{\mu^3},\ \Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \frac{1}{2\mu^2},\ \Gamma_{122} = \Gamma_{212} = \Gamma_{221} = 0, \tag{3.3}$$

$$\Gamma_{222} = \frac{\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3}{2\Gamma^3(\lambda)}. \tag{3.4}$$

Then from (2.5) and (3.1) $\rightarrow$ (3.4), we can get

$$\Gamma^{(\alpha)}_{111} = \frac{\lambda(\alpha-1)}{\mu^3},\ \Gamma^{(\alpha)}_{112} = \Gamma^{(\alpha)}_{121} = \Gamma^{(\alpha)}_{211} = \frac{1-\alpha}{2\mu^2},\ \Gamma^{(\alpha)}_{212} = \Gamma^{(\alpha)}_{122} = \Gamma^{(\alpha)}_{221} = 0, \tag{3.5}$$

$$\Gamma_{222}^{(\alpha)} = \frac{\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3}{2\Gamma^3(\lambda)}(1-\alpha). \tag{3.6}$$

From (2.8), (3.5) and (3.6), we get

$$\Gamma_{11}^{1(\alpha)} = -\frac{2\lambda\Gamma(\lambda)\Gamma''(\lambda) - 2\lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda)}{2\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.7}$$

$$\Gamma_{11}^{2(\alpha)} = -\frac{\lambda\Gamma^2(\lambda)}{2\mu^2(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.8}$$

$$\Gamma_{12}^{1(\alpha)} = \frac{\Gamma(\lambda)\Gamma''(\lambda) - (\Gamma'(\lambda))^2}{2(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.9}$$

$$\Gamma_{12}^{2(\alpha)} = \frac{\Gamma^2(\lambda)}{2\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.10}$$

$$\Gamma_{21}^{1(\alpha)} = \frac{\Gamma(\lambda)\Gamma''(\lambda) - (\Gamma'(\lambda))^2}{2(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.11}$$

$$\Gamma_{21}^{2(\alpha)} = \frac{\Gamma^2(\lambda)}{2\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha), \tag{3.12}$$

$$\Gamma_{22}^{1(\alpha)} = \frac{\mu(\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3)}{2\Gamma(\lambda)(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha) \tag{3.13}$$

and

$$\Gamma_{22}^{2(\alpha)} = \frac{\lambda(\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3)}{2\Gamma(\lambda)(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(1-\alpha). \tag{3.14}$$

From (2.7) and (3.5) $\to$ (3.14), we can get

$$\begin{aligned} R_{1212}^{(\alpha)} = {} & \frac{1-\alpha^2}{4\mu^2\Gamma(\lambda)(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(\Gamma^2(\lambda)\Gamma''(\lambda) \\ & - \Gamma(\lambda)(\Gamma'(\lambda))^2 + \lambda\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\lambda\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2\lambda(\Gamma'(\lambda))^3). \end{aligned}$$

Then from (2.10), by a direct calculation, we can obtain

$$\begin{aligned} K^{(\alpha)} = {} & \frac{1-\alpha^2}{4(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))^2}(\Gamma^3(\lambda)\Gamma''(\lambda) \\ & - \Gamma^2(\lambda)(\Gamma'(\lambda))^2 + \lambda\Gamma^3(\lambda)\Gamma^{(3)}(\lambda) - 3\lambda\Gamma^2(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2\lambda\Gamma(\lambda)(\Gamma'(\lambda))^3). \end{aligned}$$

This completes the proof of Theorem 3.1. □

From Theorem 3.1 we get the following

Corollary 3.1.

(1) $R_{1212}^{(0)} = 0$ *and* $\Gamma_{ij}^{k(1)} = 0$, *namely, the inverse Gamma manifolds are* ± 1*-flat and the coordinate system is* 1*-affine.*

(2) *When $\alpha = 0$, the Gaussian curvature satisfies*

$$K^{(0)} = \frac{1}{4(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))^2}(\Gamma^3(\lambda)\Gamma''(\lambda) - \Gamma^2(\lambda)(\Gamma'(\lambda))^2 + \lambda\Gamma^3(\lambda)\Gamma^{(3)}(\lambda) - 3\lambda\Gamma^2(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2\lambda\Gamma(\lambda)(\Gamma'(\lambda))^3).$$

Theorem 3.2. *The Kullback divergence of the inverse Gamma manifold is given by*

$$K(P_p, P_q) = \lambda_p \ln \mu_p - \lambda_q \ln \mu_q + \ln \frac{\Gamma(\lambda_q)}{\Gamma(\lambda_p)} - \frac{\lambda_p}{\mu_p}(\mu_p - \mu_q) - (\lambda_p - \lambda_q)(\ln \mu_p - \frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)})$$

and the J-divergence is given by

$$J(P_p, P_q) = (\mu_p - \mu_q)(\frac{\lambda_q}{\mu_q} - \frac{\lambda_p}{\mu_p}) + (\lambda_p - \lambda_q)(\ln \frac{\mu_q}{\mu_p} + \frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)} - \frac{\Gamma'(\lambda_q)}{\Gamma(\lambda_q)}).$$

Proof. From (2.11), we can get

$$\begin{aligned} K(P_p, P_q) &= E_{\theta_p}[\ln \frac{P(x,\theta_p)}{P(x,\theta_q)}] = \int P(x,\theta_p) \ln \frac{P(x,\theta_p)}{P(x,\theta_q)} dx \\ &= \lambda_p \ln \mu_p - \lambda_q \ln \mu_q + \ln \frac{\Gamma(\lambda_q)}{\Gamma(\lambda_p)} - \frac{\lambda_p}{\mu_p}(\mu_p - \mu_q) \\ &\quad - (\lambda_p - \lambda_q)(\ln \mu_p - \frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)}). \end{aligned}$$

Then from (2.12) we can get

$$\begin{aligned} J(P_p, P_q) &= \int (P(x,\theta_p) - P(x,\theta_q)) \ln \frac{P(x,\theta_p)}{P(x,\theta_q)} dx \\ &= K(P_p, P_q) + K(P_q, P_p) \\ &= (\mu_p - \mu_q)(\frac{\lambda_q}{\mu_q} - \frac{\lambda_p}{\mu_p}) + (\lambda_p - \lambda_q)(\ln \frac{\mu_q}{\mu_p} + \frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)} - \frac{\Gamma'(\lambda_q)}{\Gamma(\lambda_q)}). \end{aligned}$$

Corollary 3.2. *When $\mu_p = \mu_q$, then*

$$K(P_p, P_q) = \ln \frac{\Gamma(\lambda_q)}{\Gamma(\lambda_p)} + (\lambda_p - \lambda_q)\frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)}, \ J(P_p, P_q) = (\lambda_p - \lambda_q)(\frac{\Gamma'(\lambda_p)}{\Gamma(\lambda_p)} - \frac{\Gamma'(\lambda_q)}{\Gamma(\lambda_q)}).$$

When $\lambda_p = \lambda_q = \lambda$, then

$$K(P_p, P_q) = \lambda \ln \frac{\mu_p}{\mu_q} - (\mu_p - \mu_q)\frac{\lambda}{\mu_p}, \ J(P_p, P_q) = (\mu_p - \mu_q)(\frac{\lambda}{\mu_q} - \frac{\lambda}{\mu_p}).$$

Theorem 3.3. *The geodesic equations are given by*

$$\begin{aligned}
&\frac{d^2\lambda}{dt^2} - \frac{2(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda)) + \Gamma^2(\lambda)}{2\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(\frac{d\lambda}{dt})^2 \\
&- \frac{\Gamma(\lambda)\Gamma''(\lambda) - (\Gamma'(\lambda))^2}{\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda)}\frac{d\lambda}{dt}\frac{d\mu}{dt} \\
&- \frac{\mu(\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3)}{2\Gamma(\lambda)(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(\frac{d\mu}{dt})^2 = 0,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\frac{d^2\mu}{dt^2} - \frac{\lambda\Gamma^2(\lambda)}{2\mu^2(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(\frac{d\lambda}{dt})^2 \\
&+ \frac{\Gamma^2(\lambda)}{\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}\frac{d\lambda}{dt}\frac{d\mu}{dt} \\
&- \frac{\lambda(\Gamma^2(\lambda)\Gamma^{(3)}(\lambda) - 3\Gamma(\lambda)\Gamma'(\lambda)\Gamma''(\lambda) + 2(\Gamma'(\lambda))^3)}{2\Gamma(\lambda)(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}(\frac{d\mu}{dt})^2 = 0.
\end{aligned} \tag{3.16}$$

Proof. By Definition 2.9 and using Γ_{ij}^k which we have calculated above, we can get the geodesic equations immediately.

In particular, for fixed λ, from (3.16) we can get the solution with respect to μ that $\mu = constant$ or $c_1 t + c_2 = 1$.

Similarly, for fixed μ, from (3.15) we can get the solution with respect to λ that λ =constant or

$$\int \exp\{-\int \varphi(\lambda,\mu)d\lambda\}d\lambda = t,$$

where

$$\varphi(\lambda,\mu) = \frac{2\lambda\Gamma(\lambda)\Gamma''(\lambda) - 2\lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda)}{2\mu(\lambda\Gamma(\lambda)\Gamma''(\lambda) - \lambda(\Gamma'(\lambda))^2 - \Gamma^2(\lambda))}.$$

4. The affine immersion

Let M be an m-dimensional manifold, f be an immersion from M to $\mathbb{R}^{m+1}$, and ξ be a transversal vector field along f. We can identify $T_x R^{m+1} \equiv \mathbb{R}^{m+1}$ for $\forall x \in R^{m+1}$. Then the pair $\{f,\xi\}$ is said to be an affine immersion from M to $\mathbb{R}^{m+1}$, if for each point $P \in M$, the following formula holds

$$T_{f(P)}R^{m+1} = f_*(T_P M) \oplus \operatorname{span}\xi_P.$$

We denote the standard flat affine connection of $\mathbb{R}^{m+1}$ with D. Identifying the covariant derivative along f with D, we have the following decompositions

$$\begin{aligned}
D_X f_* Y &= f_*(\nabla_X Y) + h(X,Y)\xi, \\
D_X \xi &= -f_*(Sh(X)) + \tau(X)\xi.
\end{aligned}$$

The induced objects ∇, h, Sh and τ are the induced connection, the affine fundamental form, the affine shape operator and the transversal connection form, respectively.

Since the inverse Gamma distribution can be written as

$$\ln p(x, \lambda, \mu) = -\frac{\mu}{x} - (\lambda + 1) \ln x - (\ln \Gamma(\lambda) - \lambda \ln \mu), \ x > 0, \lambda > 0, \mu > 0,$$

where $\psi(\theta) = \ln \Gamma(\lambda) - \lambda \ln \mu$ is called the potential function, then we get the following

Proposition 4.1. *Denote by $\theta = (\theta^1, \theta^2) = (\mu, \lambda + 1)$ a natural coordinate system. Then the inverse Gamma manifold can be realized in $\mathbb{R}^3$ by the graph of a θ-potential function, namely, the inverse Gamma manifold can be realized by the affine immersion $\{f, \xi\}$:*

$$f : \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} \mu \\ \lambda + 1 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ \psi(\theta) \end{pmatrix} = \begin{pmatrix} \mu \\ \lambda + 1 \\ \psi(\theta) \end{pmatrix}, \ \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\psi(\theta)$ is the potential function.

5. Applications

The inverse Gamma distribution is very famous and it can be applied to various fields. In [5], the authors used the inverse Gamma distribution to model the overall distribution of total chip leakage. The Gamma and inverse Gamma texture distribution ([6]) were derived after the general CRB expressions under an arbitrary texture model were simplified. Then the generalized Gauss-Laguerre quadrature was used to compute the CRBs for gamma texture whereas the CRBs for the inverse gamma texture. In non-Rayleigh distributed radar images, the number of scatterers was viewed as a Poisson distributed random variable with the mean itself random. Then in [7], the authors added three new possible distributions for this mean, inverse Gamma, Beta of the first kind and Beta of the second kind, and showed that new intensity distributions so obtained could be estimated, with the interest of the extension validated on a real image.

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