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# Banach-Mazur Distance of Central Sections of a Centrally Symmetric Convex Body

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**Abstract.** We prove that the Banach-Mazur distance between arbitrary two central sections of co-dimension $c$ of any centrally symmetric convex body in $E^n$ is at most $(2c+1)^2$.

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As usual, by a convex body of Euclidean $n$-space $E^n$ we mean a compact convex set with non-empty interior. Denote by $\mathcal{B}^n$ the family of all centrally symmetric convex bodies of $E^n$ which are centered at the center $o$ of $E^n$. Let $E_1^k$ and $E_2^k$ be $k$-dimensional subspaces of $E^n$, let $C_1$ be a convex body of $E_1^k$ centered at $o$ and let $C_2$ be a convex body of $E_2^k$ centered at $o$. The *Banach-Mazur distance between* $C_1$ and $C_2$ is the number

$$\delta(C_1, C_2) = \inf\{\lambda;\ a(C_2) \subset C_1 \subset \lambda\, a(C_2)\},$$

where $a$ stands for an affine transformation, and $\lambda A$ stands for the image of a set $A$ under the homothety with center $o$ and a positive ratio $\lambda$.

Extensive surveys of results on Banach-Mazur distance are given by Thompson [5] and by Tomczak-Jaegermann [6]. See also the last section of the recent book by Brass, Moser, and Pach [1]. The classic paper of Dvoretzky [2] stipulated intensive research on Banach-Mazur distance between central sections of centrally symmetric convex bodies. In particular, Rudelson [4] considers asymptotic behavior of Banach-Mazur distance between $k$-dimensional sections of bodies of $\mathcal{B}^n$.

Our aim is to prove the upper bound $(2c+1)^2$ of the Banach-Mazur distance between every two $(n-c)$-dimensional central sections of an arbitrary body of $\mathcal{B}^n$. Let us point out that this estimate depends only on the co-dimension $c$ of the sections. So our estimate does not grow when the dimension $n$ tends to infinity.

The proof of our Theorem is based on Lemma whose formulation requires some notation. Let $C \in \mathcal{B}^n$. Let $S$ be a central section of $C$ of co-dimension $c$, this is of dimension $n-c$. By compactness arguments we see that there exist $c$ segments $I_1, \dots, I_c$ centered at $o$ whose end-points are in the boundary of $C$ such that the convex hull

$$P = \operatorname{conv}(I_1 \cup \dots \cup I_c \cup S) \tag{1}$$

has the maximum volume from amongst all convex hulls of this form, where $S$ is fixed. Clearly, $P \subset C$.

Since the Banach-Mazur distance is invariant with respect to affine transformations, without loss of generality further we assume that $I_i$ is the segment of length 2 contained in the $i$-th coordinate axis of $E^n$ and centered at $o$ for $i \in \{1, \dots, c\}$, and that $S$ is in the $(n-c)$-dimensional subspace containing the remaining coordinate axes of $E^n$.

**Lemma.** *Let $C \in \mathcal{B}^n$ and let $S$ be an $(n-c)$-dimensional central section of $C$. For the cylinder $K = I_1 \times \dots \times I_c \times (c+1)S$, where $I_1, \dots, I_c$ are defined above, we have*

$$\delta(C, K) \le 2c + 1.$$

*Proof.* For every $i \in \{1, \dots, c\}$ we denote by $g_i$ and $h_i$ the end-points of $I_i$. We provide through every $g_i$ the hyperplane $G_i$ parallel to the hyperplane containing $S$ and all the segments from amongst $I_1, \dots, I_c$ which are different from $I_i$. Analogously, through every $h_i$ we provide the hyperplane $H_i$ parallel to $G_i$. In order to see that

$$G_1, \dots, G_c, H_1, \dots, H_c \text{ are supporting hyperplanes of } C, \tag{2}$$

assume the opposite. Then the central symmetry of $C$ and of our construction implies that a $j \in \{1, \dots, c\}$ exists such that $G_j, H_j$ are not supporting hyperplanes of $C$. As a consequence, we can find a segment $J_j \subset C$ centered at $o$ such that its end-points are out of the strip between $G_j$ and $H_j$. Thus $\operatorname{conv}(J_1 \cup \dots \cup J_c \cup S)$, where $J_m = I_m$ for all $m \in \{1, \dots, c\}$ different from $j$, has volume greater than $P$, see (1). So our opposite assumption contradicts the choice of $I_1, \dots, I_c$, see (1). Thus (2) is true.

We intend to show that

$$C \subset K. \tag{3}$$

Assume that this is not true, i.e. assume that there exists a point $u \in C$ such that $u \notin K$. Since $u$ is out of $K$, from (2) we conclude that $u = (a_1, \dots, a_c, qa_{c+1}, \dots, qa_n)$, where $q > c+1$, such that $|a_1| \le 1, \dots, |a_c| \le 1$ and such that $w = (0, \dots, 0, a_{c+1}, \dots, a_n)$ is a point of the relative boundary of $S$. We provide the straight line through $u$ and $w$. Its parametric equation is $x_1 = ta_1, \dots, x_c =$

$ta_c$, $x_{c+1} = ((q-1)t+1)a_{c+1}, \dots, x_n = ((q-1)t+1)a_n$, where $-\infty < t < \infty$. For $t = -\frac{1}{q-1}$ we get the point $z = (-\frac{1}{q-1}a_1, \dots, -\frac{1}{q-1}a_c, 0, \dots, 0)$. Since $|-\frac{1}{q-1}a_1| + \dots + |-\frac{1}{q-1}a_c| = \frac{1}{q-1}(|a_1| + \dots + |a_c|) \le \frac{c}{q-1} < 1$, we conclude that $z$ is an interior point of $P$. From $P \subset C$ we see that $z$ is an interior point of $C$. Hence the assumption that $u \in C$ and the fact that $w$ is a point of the segment $uz$ different from $u$ imply that $w$ is an interior point of $C$. This contradicts the fact that $w$ is a point of the relative boundary of $S$. As a consequence, (3) holds true.

Now we will show that

$$\frac{1}{2c+1}K \subset P. \tag{4}$$

Since every convex body is the convex hull of its extreme points, it is sufficient to show that all extreme points of $\frac{1}{2c+1}K$ are in $P$. Every extreme point of $\frac{1}{2c+1}K$ has the form $e' = (\frac{1}{2c+1}e_1, \dots, \frac{1}{2c+1}e_n)$, where $e = (e_1, \dots, e_n)$ is an extreme point of $K$. Then $|e_1| = \dots = |e_c| = 1$ and $(0, \dots, 0, e_{c+1}, \dots e_n)$ is in the relative boundary of $(c+1)S$.

The segment $oe$ has the equation $x_1 = te_1, \dots, x_n = te_n$, where $0 \le t \le 1$. The equation of the boundary bd(P) of $P$ is $|x_1| + \dots + |x_c| + ||(0, \dots, 0, x_{c+1}, \dots, x_n)|| = 1$, where $||\ \ ||$ denotes the norm of the normed space whose unit ball is $C$. In order to find the point of the intersection of the segment $oe$ with bd(P) we substitute the above equation of $oe$ into the above equation of bd(P). We obtain $ct + ||(0, \dots, 0, te_{c+1}, \dots, te_n)|| = 1$. Since $(0, \dots, 0, e_{c+1}, \dots, e_n)$ belongs to the relative boundary of $(c+1)S$ which is a subset of the boundary of $(c+1)C$, we get $ct + (c+1)t = 1$. Hence for $t' = \frac{1}{2c+1}$ we obtain a common point of $oe$ and bd(P). Substituting $t'$ into the parametric equation of the segment $oe$, we see that this point is just $e'$. We conclude that every extreme point $e'$ of $\frac{1}{2c+1}K$ belongs to $P$. So (4) has been shown.

From (3), (4) and from $P \subset C$ we obtain that

$$\frac{1}{2c+1}K \subset C \subset K.$$

This implies the thesis of Lemma. □

**Theorem.** *Let $S_1$ and $S_2$ be central sections of co-dimension $c$ of a centrally symmetric convex body in $E^n$. Then*

$$\delta(S_1, S_2) \le (2c+1)^2.$$

*Proof.* Assume that $S_1 \ne S_2$ and that $S_0 = S_1 \cap S_2$ is $(n-d)$-dimensional. Of course, $d \le 2c$. Clearly $S_0$ is an $(n-d)$-dimensional central section of $S_i$, where $i \in \{1, 2\}$. We apply Lemma taking $S_i$, where $i \in \{1, 2\}$, in the part of $C$. Since $S_i$ is of co-dimension $c$, the present $n-c$ plays the part of $n$ from Lemma. Moreover, we take $S_0$ in the part of $S$. For the section $S_0$ of $S_i$, where $i \in \{1, 2\}$, we define a cylinder $K_i$ analogically like the cylinder $K$ is defined for $S$ in Lemma. Since $S_i$ is $(n-c)$-dimensional, $K_i$ is $(n-c)$-dimensional for $i \in \{1, 2\}$. From $(n-c) - (n-d) = d - c$ and by Lemma we get $\delta(S_1, K_1) \le 2(d-c) + 1$ and

$\delta(S_2, K_2) \le 2(d-c)+1$. These inequalities, the obvious equality $\delta(K_1, K_2) = 1$ and $0 \le d \le 2c$ imply $\delta(S_1, S_2) \le (2(d-c)+1)^2 \cdot 1 = (2d-2c+1)^2 \le (2c+1)^2$. $\square$

By John's [3] theorem, $\delta(S_1, S_2) \le n-c$ under the assumptions of Theorem. Thus the estimate from Theorem is better only when $(2c+1)^2 < n-c$. So for $n > (2c+1)^2 + c$. In particular, for $n > 10$ when $c = 1$, and for $n > 27$ when $c = 2$.

From the proof of Theorem we conclude the following more precise corollary. Theorem is its special case for $d = 2c$.

**Corollary.** *Let $S_1$ and $S_2$ be central sections of co-dimension $c$ of a centrally symmetric convex body in $E^n$ such that $S_1 \cap S_2$ is of co-dimension $d$. Then*

$$\delta(S_1, S_2) \le (2d-2c+1)^2.$$

The author expects that the estimates from Theorem and Corollary are not the best possible and would not be surprised if the bound $2c+1$ or better holds true. *The problem is to improve the estimate obtained in Theorem.* Especially for $c = 1$. Just for $c = 1$ our Theorem gives the estimate 9, while the author is not able to find an $n$ and a $C \in \mathcal{B}^n$ with two central $(n-1)$-dimensional sections whose Banach-Mazur distance is over 2.

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