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Distance Preserving Mappings of Grassmann Graphs

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Abstract. Distance preserving (non-surjective) mappings between Grassmann graphs will be determined.

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1. Introduction

Let V be an n-dimensional vector space over a division ring. We write $\mathcal{G}_k(V)$ for the Grassmannian consisting of all k-dimensional subspaces of V. Two elements of $\mathcal{G}_k(V)$ are called *adjacent* if their intersection is $(k-1)$-dimensional (this is equivalent to the fact that the sum of the subspaces is $(k+1)$-dimensional). It is trivial that any two distinct elements of $\mathcal{G}_k(V)$ are adjacent if $k = 1, n-1$. The *Grassmann graph* $\Gamma_k(V)$ is the graph whose vertex set is $\mathcal{G}_k(V)$ and whose edges are pairs of adjacent k-dimensional subspaces. By duality, the graphs $\Gamma_k(V)$ and $\Gamma_{n-k}(V^*)$ are canonically isomorphic (V^* is the dual vector space). The Grassmann graph is connected and the distance $\mathrm{d}(S,U)$ between $S, U \in \mathcal{G}_k(V)$ is defined as the minimal number i such that there is an i-path (path of length i) connecting S and U. We have the following distance formula:

$$\mathrm{d}(S,U) = \dim(S+U) - k = k - \dim(S \cap U);$$

in particular, the diameter of the graph $\Gamma_k(V)$ is equal to k if $2k \leq n$, and $n-k$ otherwise.

Now let V' be an n'-dimensional vector space over a division ring and

$$1 < k < \min\{n, n'\} - 1.$$

Let also $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ be adjacency preserving mapping (two elements of $\mathcal{G}_k(V)$ are adjacent if and only if their f-images are adjacent). If f is surjective then it is an isomorphism of $\Gamma_k(V)$ to $\Gamma_k(V')$ and Chow's theorem [2] (see also [3, 13]) guarantees that f is induced by a semilinear isomorphism of V to V' or V'^* (the second possibility can be realized only in the case when $n = 2k$); some results closely related with Chow's theorem can be found in [1, 9, 7, 8, 11, 12]. In the general case, f induces a mapping

$$f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_i(V'), \quad i = k \pm 1$$

(Section 3); thus there are precisely two types of adjacency preserving mappings (associated with mappings of $\mathcal{G}_{k-1}(V)$ to $\mathcal{G}_{k-1}(V')$ or $\mathcal{G}_{k+1}(V')$, respectively). If f is not surjective then f_{k-1} need not to be adjacency preserving. However, if f is distance preserving (a non-surjective adjacency preserving mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$ need not to be distance preserving, in general) then the same holds for f_{k-1}; this is related with the fact that every minimal path of $\Gamma_k(V)$ induces minimal paths in $\Gamma_{k-1}(V)$ and $\Gamma_{k+1}(V)$ (Section 4). Using this observation we determine distance preserving mappings of each of two types given above in the case when $2k \leq \min\{n, n'\}$ (Sections 5 and 6).

2. Semilinear embeddings

Throughout the paper we suppose that V and V' are left vector spaces over division rings R and R', respectively. An additive mapping $l : V \to V'$ is called *semilinear* if there exists a homomorphism $\sigma : R \to R'$ such that

$$l(ax) = \sigma(a)l(x)$$

for all $x \in V$ and all $a \in R$. If l is non-zero then there is only one homomorphism satisfying this condition. Note also that non-zero homomorphisms of division rings are injective.

Let $k \leq \min\{n, n'\}$. A semilinear injection $l : V \to V'$ is said to be a k-*embedding* if it sends any k linearly independent vectors to linearly independent vectors. This condition implies that for any subspace $S \subset V$ whose dimension is not greater than k we have

$$\dim\langle l(S)\rangle = \dim S$$

(we write $\langle X\rangle$ for the subspace spanned by a set X). We get a mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$ which transfers $S \in \mathcal{G}_k(V)$ to the subspace spanned by $l(S)$, in what follows this mapping will be denoted by $(l)_k$. In general, the mapping $(l)_k$ need not to be injective.

Proposition 2.1. *If $l : V \to V'$ is a semilinear $(k+1)$-embedding then $(l)_k$ is an injection sending adjacent subspaces to adjacent subspaces.*

Proof. If there exist two distinct subspaces $S, U \in \mathcal{G}_k(V)$ such that $\langle l(S)\rangle$ and $\langle l(U)\rangle$ are coincident then the l-image of the $(k+1)$-dimensional subspace spanned by S and a vector $x \in U \setminus S$ is contained in the k-dimensional subspace

$$\langle l(S+U)\rangle = \langle l(S)\rangle = \langle l(U)\rangle$$

which contradicts the fact that l is a $(k+1)$-embedding. Thus $(l)_k$ is injective. Similarly, we show that it maps adjacent subspaces to adjacent subspaces. □

It must be pointed out that the mapping from Proposition 2.1 need not to be distance preserving.

Proposition 2.2. *If $2k \le \min\{n, n'\}$ and $l : V \to V'$ is a semilinear $(2k)$-embedding then $(l)_k$ is distance preserving.*

Proof. Let S and U be k-dimensional subspaces of V. The dimension of $S+U$ is not greater than $2k$. Since l is a $(2k)$-embedding,

$$\dim(S+U) = \dim\langle l(S) + l(U)\rangle$$

and the distance formula (cf. Introduction) shows that the distance between S and U is equal to the distance between $\langle l(S)\rangle$ and $\langle l(U)\rangle$. □

Examples of semilinear k-embeddings which are not $(k+1)$-embeddings can be found in [10].

3. Two types of adjacency preserving mappings

First we give some trivial facts concerning cliques of Grassmann graphs. Let S be a subspace of V. We write $[S]_k$ for the set of all k-dimensional subspaces of V incident with S. This set is called a *star* or a *top* if $1 < k < n-1$ and the dimension of S is equal to $k-1$ or $k+1$, respectively.

Proposition 3.1. [2] *In the case when $1 < k < n-1$, stars and tops are maximal cliques of the graph $\Gamma_k(V)$, and every maximal clique of $\Gamma_k(V)$ is a star or a top. Moreover, each clique is contained in a maximal clique.*

The canonical isomorphism of $\Gamma_k(V)$ to $\Gamma_{n-k}(V^*)$ maps stars to tops and tops to stars.

Proposition 3.2. *Let*

$$1 < k < \min\{n, n'\} - 1.$$

If a mapping $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ is adjacency preserving then one of the following possibilities is realized:

(A) *stars go to subsets of stars and tops go to subsets of tops,*

(B) *stars go to subsets of tops and tops go to subsets of stars.*

Moreover, every maximal clique of $\Gamma_k(V')$ *contains at most one of the* f*-images of maximal cliques of* $\Gamma_k(V)$ *and the* f*-image of every maximal clique of* $\Gamma_k(V)$ *is contained in precisely one maximal clique of* $\Gamma_k(V')$.

To prove Proposition 3.2 we use the following trivial observation.

Lemma 3.1. *If* $1 < k < n-1$ *then the following assertions are satisfied:*

(1) *The intersection of two maximal cliques of* $\Gamma_k(V)$ *contains more than one element if and only if these cliques are of different types* (*one of them is a star and the other is a top*) *and the associated* $(k+1)$*-dimensional and* $(k-1)$*-dimensional subspaces are incident.*

(2) *The intersection of two distinct maximal cliques of the same type is empty or one-element, and the second possibility is realized only in the case when the associated* $(k \pm 1)$*-dimensional subspaces are adjacent.*

Proof of Proposition 3.2. It is clear that f transfers cliques to cliques. First we show that every maximal clique of $\Gamma_k(V')$ contains at most one of the f-images of maximal cliques of $\Gamma_k(V)$.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are distinct maximal cliques of $\Gamma_k(V)$ whose f-images are contained in a certain maximal clique of $\Gamma_k(V')$. Since f is adjacency preserving and its restriction to every clique is injective, an element $S \in \mathcal{X}$ is adjacent with all $U \in \mathcal{Y}$ which satisfy $f(S) \neq f(U)$ and there is at most one $U \in \mathcal{Y}$ satisfying $f(S) = f(U)$. In other words, every element of $\mathcal{X}$ is adjacent with all elements of $\mathcal{Y}$ or with all except one. An easy verification shows that this is impossible for subspaces belonging to $\mathcal{X} \setminus \mathcal{Y}$.

Now we establish that the f-image of every maximal clique of $\Gamma_k(V)$ is contained only in one maximal clique of $\Gamma_k(V')$.

Suppose that $\mathcal{X} \subset \mathcal{G}_k(V)$ is a maximal clique whose f-image is contained in two distinct maximal cliques $\mathcal{X}'$ and $\mathcal{Y}'$ of $\Gamma_k(V')$. By Lemma 3.1, one of these cliques is a star and the other is a top. We choose a maximal clique $\mathcal{Y} \subset \mathcal{G}_k(V)$ such that $\mathcal{Y} \neq \mathcal{X}$ and the intersection $\mathcal{X} \cap \mathcal{Y}$ has more than one element. Since the restriction of f to every clique of $\Gamma_k(V)$ is injective, $f(\mathcal{Y})$ intersects $\mathcal{X}'$ and $\mathcal{Y}'$ by subsets containing more than one element. Lemma 3.1 shows that a maximal clique containing $f(\mathcal{Y})$ coincides with $\mathcal{X}'$ or $\mathcal{Y}'$. Thus one of these cliques contains the f-images of two distinct maximal cliques which is impossible.

Finally, we prove that one of the given above cases is realized.

Let S be a $(k-1)$-dimensional subspace of V such that the f-image of the star $[S]_k$ is contained in a star. Consider a $(k-1)$-dimensional subspace $U \subset V$ adjacent with S and choose a $(k+1)$-dimensional subspace N containing S and U. The stars $[S]_k$ and $[U]_k$ intersect the top $[N]_k$ by sets containing more than one element. The f-image of $[S]_k$ is contained in a star (by assumption), and Lemma 3.1 implies that f transfers $[N]_k$ to a subset of a top; hence $[U]_k$ goes to a subset of a star. The same holds for every $U \in \mathcal{G}_{k-1}(V)$, since it can be connected with

S by a path in the graph $\Gamma_{k-1}(V)$. The same arguments show that the images of tops are contained in tops.

Similarly, we establish that the case (B) is realized if the f-image of the star $[S]_k$ is a subset of a top. $\square$

Let f be as in Proposition 3.2. We say that f is of *type* (A) or (B) if the corresponding case is realized. By Proposition 3.2, in the first case there exists an injection

$$f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_{k-1}(V')$$

such that

$$f([S]_k) \subset [f_{k-1}(S)]_k \tag{3.1}$$

for every $S \in \mathcal{G}_{k-1}(V)$. Then

$$f_{k-1}([U]_{k-1}) \subset [f(U)]_{k-1}$$

for every $U \in \mathcal{G}_k(V)$; in other words, f_{k-1} sends tops to subsets of tops. The latter means that f_{k-1} transfers adjacent subspaces to adjacent subspaces (since two distinct elements of a Grassmannian are adjacent if and only if there is a top containing them). However, if M and N are adjacent elements of $f_{k-1}(\mathcal{G}_{k-1}(V))$ then $M+N$ need not to be an element of $f(\mathcal{G}_k(V))$ and we cannot assert that f_{k-1} is adjacency preserving. It will be established in Section 5 that f_{k-1} is distance preserving if f is distance preserving; under this assumption we show that f is induced by a semilinear $(2k)$-embedding of V to V'.

If f is of type (B) then we get an injection of $\mathcal{G}_{k-1}(V)$ to $\mathcal{G}_{k+1}(V')$. This more complicated case will be considered in Section 6.

4. Path lemmas

In this section we prove some simple lemmas concerning minimal paths in Grassmann graphs. A path connecting $S, U \in \mathcal{G}_k(V)$ in $\Gamma_k(V)$ is said to be *minimal* if it consists of precisely $\mathrm{d}(S,U)$ edges.

Lemma 4.1. *If $S_0, S_1, \dots, S_i$ is a minimal path in $\Gamma_k(V)$ then*

$$S_0 + S_i = S_0 + S_1 + \cdots + S_i \ \ \textit{and} \ \ S_0 \cap S_i = S_0 \cap S_1 \cap \cdots \cap S_i.$$

Proof. Since

$$\dim(S_0 + S_1) = k+1 \ \text{ and } \ \dim(S_0 + \cdots + S_{j+1}) \le \dim(S_0 + \cdots + S_j) + 1,$$

we have

$$\dim(S_0 + \cdots + S_i) \le k + i = k + \mathrm{d}(S_0, S_i) = \dim(S_0 + S_i)$$

which implies the first equality. Similarly,

$$\dim(S_0 \cap S_1) = k-1 \ \text{ and } \ \dim(S_0 \cap \cdots \cap S_{j+1}) \ge \dim(S_0 \cap \cdots \cap S_j) - 1$$

show that

$$\dim(S_0 \cap \cdots \cap S_i) \geq k - i = k - \mathrm{d}(S_0, S_i) = \dim(S_0 \cap S_i)$$

and we get the second equality. □

Lemma 4.2. *Let $1 < k < n-1$. If $S_0, S_1, \ldots, S_i$ is a minimal path in $\Gamma_k(V)$ then*

$$(S_0 \cap S_1), \ldots, (S_{i-1} \cap S_i) \;\; \textit{and} \;\; (S_0 + S_1), \ldots, (S_{i-1} + S_i) \tag{4.1}$$

are minimal paths in $\Gamma_{k-1}(V)$ and $\Gamma_{k+1}(V)$, respectively.

Proof. By Lemma 4.1,

$$S_0 \cap S_i = (S_0 \cap S_1) \cap (S_{i-1} \cap S_i).$$

The dimension of this subspace is equal to $k - i$ and the distance formula shows that

$$\mathrm{d}((S_0 \cap S_1), (S_{i-1} \cap S_i)) = k - 1 - \dim((S_0 \cap S_1) \cap (S_{i-1} \cap S_i)) = i - 1.$$

Similarly,

$$S_0 + S_i = (S_0 + S_1) + (S_{i-1} + S_i)$$

is a $(k+i)$-dimensional subspace and

$$\mathrm{d}((S_0 + S_1), (S_{i-1} + S_i)) = \dim((S_0 + S_1) + (S_{i-1} + S_i)) - k - 1 = i - 1$$

(by the distance formula). Since (4.1) are paths of length $i - 1$, we get the claim. □

Lemma 4.3. *If $2k \leq n$ then for any minimal path $S_0, S_1, \ldots, S_i$ in $\Gamma_{k-1}(V)$ there is a minimal path $U_0, U_1, \ldots, U_{i+1}$ in $\Gamma_k(V)$ such that*

$$S_j = U_j \cap U_{j+1}$$

for each $j \in \{0, \ldots, i\}$.

Proof. First we define

$$U_{j+1} := S_j + S_{j+1}$$

for $j \in \{0, \ldots, i-1\}$. Since $S_0, S_1, \ldots, S_i$ is a minimal path, the dimension of

$$U_0 + U_1 + \cdots + U_i = S_0 + S_1 + \cdots + S_i = S_0 + S_i$$

is not greater than $2k - 2 < n$ and we choose two vectors

$$x \in V \setminus (S_0 + S_i) \;\; \text{and} \; y \in V \setminus \langle S_0 + S_i, x \rangle.$$

The path

$$\langle S_0, x \rangle, U_1, \ldots, U_i, \langle S_i, y \rangle$$

is as required. □

5. Distance preserving mappings of type (A)

Theorem 5.1. *Let*

$$2 < 2k \leq \min\{n, n'\}$$

and $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ *be a distance preserving mapping of type* (A). *Then* f *is induced by a* $(2k)$*-embedding of* V *to* V'.

Proof. Let

$$f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_{k-1}(V')$$

be as in the end of Section 3. The equation (3.1) shows that

$$f_{k-1}(M \cap N) = f(M) \cap f(N)$$

for every adjacent $M, N \in \mathcal{G}_k(V)$.

Let $S_0, S_1, \dots, S_i$ be a minimal path in $\Gamma_{k-1}(V)$. By Lemma 4.3, there exists a minimal path $U_0, U_1, \dots, U_{i+1}$ in $\Gamma_k(V)$ such that

$$S_j = U_j \cap U_{j+1}$$

for each $j \in \{0, \dots, i\}$. Then

$$f(U_0), f(U_1), \dots, f(U_{i+1})$$

is a minimal path of $\Gamma_k(V')$ (since f is distance preserving) and Lemma 4.2 guarantees that

$$f(U_0) \cap f(U_1) = f_{k-1}(S_0), \dots, f(U_i) \cap f(U_{i+1}) = f_{k-1}(S_i)$$

is a minimal path in $\Gamma_{k-1}(V')$. Thus f_{k-1} is distance preserving.

The mapping f_{k-1} maps tops to subsets of tops, hence it is of type (A). Step by step we get a sequence of distance preserving mappings

$$f_i : \mathcal{G}_i(V) \to \mathcal{G}_i(V') \quad i = k, \dots, 1,$$

where $f_k = f$ and for any $(i-1)$-dimensional subspace $S \subset V$

$$f_i([S]_i) \subset [f_{i-1}(S)]_i$$

if $1 < i \leq k$. The latter implies that

$$f_{i-1}([U]_{i-1}) \subset [f_i(U)]_{i-1}$$

for any i-dimensional subspace $U \subset V$. It follows from Faure-Frölicher-Havlicek's version of the Fundamental Theorem of Projective Geometry [4, 5, 6] that f_1 is induced by a certain semilinear mapping $l : V \to V'$. It is clear that

$$f(S) = \langle l(S) \rangle$$

for every $S \in \mathcal{G}_k(V)$. Since f is distance preserving, the l-image of every $(2k)$-dimensional subspace of V spans a $(2k)$-dimensional subspace of V'. This means that l is a semilinear $(2k)$-embedding. $\square$

Remark 5.1. Suppose that $n = n' \leq 2k$ (as above we require that $1 < k < n-1$) and $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ is a distance preserving mapping of type (A). By duality, f can be considered as a distance preserving mapping of $\mathcal{G}_{n-k}(V^*)$ to $\mathcal{G}_{n-k}(V'^*)$. The latter mapping is also of type (A) and $2(n-k) \leq n$, Theorem 5.1 guarantees that it is induced by a semilinear $(2n-2k)$-embedding of V^* to V'^*. In particular, if $n = 2k$ then we get the mapping induced by a semilinear n-embedding of V^* to V'^*

Theorem 5.2. *Let $n \geq n' = 2k$ and $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ be a distance preserving mapping. Then f is induced by a semilinear $(2k)$-embedding of V to V' or V'^*.*

Proof. If f is of type (A) then it is induced by a semilinear $(2k)$-embedding of V to V' (Theorem 5.1). Suppose that f is of type (B). Since $n' = 2k$, we can consider f as a distance preserving mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V'^*)$ (by duality). This mapping is of type (A), hence it is induced by a semilinear $(2k)$-embedding of V to V'^*. □

6. Distance preserving mappings of type (B)

As in the previous section we suppose that

$$2k \leq \min\{n, n'\}.$$

Let W be a $(2k)$-dimensional subspace of V'. Consider W as a left vector space over R' and denote by W^* the dual vector space. Now let $l : V \to W^*$ be a semilinear $(2k)$-embedding. By duality,

$$(l)_k : \mathcal{G}_k(V) \to \mathcal{G}_k(W^*)$$

can be considered as a distance preserving mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(W)$ (recall that $\dim W = 2k$). The latter mapping is of type (B). Since $\mathcal{G}_k(W)$ is contained in $\mathcal{G}_k(V')$, we get a distance preserving mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$, it is of type (B).

Theorem 6.1. *Let*

$$2 < 2k \leq \min\{n, n'\}$$

and $f : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$ be a distance preserving mapping of type (B). *Then there exists a $(2k)$-dimensional subspace $W \subset V'$ such that f is induced by a $(2k)$-embedding of V to W^*.*

Proof. In this case, we have a mapping

$$f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_{k+1}(V')$$

such that

$$f([S]_k) \subset [f_{k-1}(S)]_k$$

for every $S \in \mathcal{G}_{k-1}(V)$. Then

$$f_{k-1}([U]_{k-1}) \subset [f(U)]_{k+1}$$

for every $U \in \mathcal{G}_k(V)$. This implies that

$$f_{k-1}(M \cap N) = f(M) + f(N)$$

for every adjacent $M, N \in \mathcal{G}_k(V)$.

First, we show that f_{k-1} preserves the distance between subspaces. Consider a minimal path $S_0, S_1, \dots, S_i$ in $\Gamma_{k-1}(V)$. As in the proof of Theorem 5.1, we choose a minimal path $U_0, U_1, \dots, U_{i+1}$ in $\Gamma_k(V)$ such that

$$S_j = U_j \cap U_{j+1}$$

for each $j \in \{0, \dots, i\}$. Since

$$f(U_0), \dots, f(U_{i+1})$$

is a minimal path in $\Gamma_k(V')$, Lemma 4.2 implies that

$$f(U_0) + f(U_1) = f_{k-1}(S_0), \dots, f(U_i) + f(U_{i+1}) = f_{k-1}(S_i)$$

is a minimal path.

The mapping f_{k-1} transfers tops to subsets of stars; as in the proof of Proposition 3.2, we establish that f_{k-1} sends stars to subsets of tops. Step by step we get a sequence of distance preserving mappings

$$f_i : \mathcal{G}_i(V) \to \mathcal{G}_{2k-i}(V') \quad i = k, \dots, 1,$$

where $f_k = f$ and for any $(i-1)$-dimensional subspace $S \subset V$

$$f_i([S]_i) \subset [f_{i-1}(S)]_{2k-i}$$

if $1 < i \leq k$. Then

$$f_{i-1}([U]_{i-1}) \subset [f_i(U)]_{2k-i-1}$$

for any i-dimensional subspace $U \subset V$.

The mapping f_1 is distance preserving and the image of $\mathcal{G}_1(V)$ is a clique of $\Gamma_{2k-1}(V')$. This clique is not contained in a star (indeed, if it is a subset of the star associated with a $(2k-2)$-dimensional subspace $M \subset V'$ then $f_1(N) = M$ for any 2-dimensional subspace $N \subset V$ which is is impossible, since f_1 is distance preserving). Thus there exists a $(2k)$-dimensional subspace $W \subset V'$ such that the associated top contains the f_1-image of $\mathcal{G}_1(V)$. By Faure-Frölicher-Havlicek's result [4, 5, 6], f_1 is induces by a semilinear mapping $l : V \to W^*$. An easy verification shows that l is a $2k$-embedding inducing f. □

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