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Homogeneous Spaces and Isoparametric Hypersurfaces

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Abstract. In this note we establish a relation between isoparametric hypersurfaces with four distinct principal curvatures in spheres and homogeneous spaces. Let m_1, m_2 denote the multiplicities of the principal curvatures. Then the orbit $N(m_1, m_2) = \{A \in \mathfrak{h}(2m_1 + m_2) \mid \mathrm{tr}(A) = 0,\ \mathrm{rank}(A) = 2m_1, \text{ and } A^3 = A\}$ of the action of the orthogonal group $\mathrm{O}(2m_1 + m_2)$ on the real symmetric matrices $\mathfrak{h}(2m_1 + m_2)$ contains a totally geodesic, m_2-dimensional round sphere. Here $N(m_1, m_2)$ is endowed with the metric induced by a scalar product on $\mathfrak{h}(2m_1 + m_2)$ defined by the trace.

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1. Introduction

In this note we establish a relation between isoparametric hypersurfaces and homogeneous spaces. Let M denote an isoparametric hypersurface with four distinct principal curvatures in a sphere. Then the principal curvatures of M have at most two distinct multiplicities m_1, m_2, see [6], Satz 1. By [7], [1] the only possible pairs (m_1, m_2) with $m_1 = m_2$ are $(1, 1)$ and $(2, 2)$. For the possible pairs (m_1, m_2) with $m_1 < m_2$ we have $(m_1, m_2) = (4, 5)$ or $2^{\phi(m_1-1)}$ divides $m_1 + m_2 + 1$ (see [8]), where $\phi : \mathbb{N} \to \mathbb{N}$ is given by

$$\phi(m) = |\{i \mid 1 \le i \le m \text{ and } i \equiv 0, 1, 2, 4 \pmod 8\}|.$$

Conversely, each pair (m_1, m_2) that satisfies these conditions occurs as a pair of multiplicities. In the sequel we will not assume that these pairs are ordered such that $m_1 \leq m_2$.

By using this characterization of possible pairs (m_1, m_2) the result on homogeneous spaces presented in this note may be stated without referring to isoparametric hypersurfaces. Our proof, however, is based on the algebraic structure of isoparametric triple systems associated with isoparametric hypersurfaces with four distinct principal curvatures in spheres. We are now going to explain this theorem.

The orthogonal group $\mathrm{O}(2m_1 + m_2)$ acts on the vector space $\mathfrak{h}(2m_1 + m_2)$ of symmetric matrices, the self-adjoint endomorphisms of the Euclidean vector space $\mathbb{R}^{2m_1+m_2}$, by conjugation. The orbit

$$N(m_1, m_2) = \{A \in \mathfrak{h}(2m_1 + m_2) \mid \mathrm{tr}(A) = 0, \mathrm{rank}(A) = 2m_1, \text{ and } A^3 = A\}$$

consists of all symmetric matrices that have the eigenvalues ± 1 with multiplicity m_1 and the eigenvalue 0 with multiplicity m_2. We endow $\mathfrak{h}(2m_1 + m_2)$ with the scalar product defined by $(A, B) \mapsto (1/2m_1)\mathrm{tr}(AB)$. Then $\mathrm{O}(2m_1 + m_2)$ acts on $\mathfrak{h}(2m_1 + m_2)$ by isometries and, with respect to the induced metric, $N(m_1, m_2) \approx \mathrm{O}(2m_1 + m_2) / \mathrm{O}(m_1) \times \mathrm{O}(m_1) \times \mathrm{O}(m_2)$ is a homogeneous manifold contained in the unit sphere $\mathbb{S}_{\mathfrak{h}(2m_1+m_2)}$. In this paper, we prove the following

Theorem 1.1. *Let m_1, m_2 denote the multiplicities of the four distinct principal curvatures of an isoparametric hypersurface in a sphere. Then the homogeneous manifold $N(m_1, m_2) \subseteq \mathfrak{h}(2m_1 + m_2)$ contains an m_2-dimensional totally geodesic round sphere $\mathbb{S}_{N(m_1,m_2)}$ that is a great sphere of $\mathbb{S}_{\mathfrak{h}(2m_1+m_2)}$.*

It would be interesting to find a proof of this result within the theory of homogeneous manifolds and to analyze how this result can be improved and extended to other values of m_1, m_2. Conversely it might then be possible to derive restrictions on the multiplicities of the four distinct principal curvatures independently of [7], [1] and [8].

2. Isoparametric triple systems

The proof of Theorem 1.1 is based on the theory of isoparametric triple systems developed by Dorfmeister and Neher in [2] and subsequent papers. In this section we explain how these algebraic structures are related to isoparametric hypersurfaces with four distinct principal curvatures in spheres and we describe their basic properties. The proof of Theorem 1.1 then follows easily from Theorem 2.1 below.

An *isoparametric hypersurface M* in the unit sphere $\mathbb{S}$ of a Euclidean vector space V is a (compact, connected) smooth hypersurface in $\mathbb{S}$ with constant principal curvatures. The sphere $\mathbb{S}$ is foliated by the isoparametric hypersurfaces parallel to M and the two focal manifolds, see [6], Theorem 4. A great circle S that intersects one of these hypersurfaces or focal manifolds orthogonally at one point intersects these submanifolds of $\mathbb{S}$ orthogonally at each intersection point. Moreover, the intersection points of S with the two focal manifolds follow alternatingly at spherical distance π/g, where g denotes the number of distinct principal curvatures of M, see [6], Section 6, cf. also [5], Proposition 3.2. We call such a great circle S a *normal circle*.

For $g = 4$, there exists a triple product structure associated with the isoparametric hypersurface M, i.e. a symmetric, trilinear map $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$, satisfying $\langle\{x, y, z\}, w\rangle = \langle x, \{y, z, w\}\rangle$ for all $x, y, z, w \in V$ and some other conditions. In this way, we get an *isoparametric triple system* $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$, see [2] or [3], [4]. The focal manifolds are given by

$$M_+ = \{x \in \mathbb{S} \mid \{x, x, x\} = 3x\} \text{ and } M_- = \{y \in \mathbb{S} \mid \{y, y, y\} = 6y\}.$$

For $x, y \in V$ and $\lambda \in \mathbb{R}$ we set $T(x, y) : V \to V : z \mapsto \{x, y, z\}$, $T(x) = T(x, x)$, and $V_\lambda(x) = \{z \in V \mid T(x)(z) = \lambda z, \langle x, z\rangle = 0\}$. For $p \in M_+$ and $r \in M_-$ we have *Peirce decompositions*

$$V = \operatorname{span}\{p\} \oplus V_3(p) \oplus V_1(p) = \operatorname{span}\{r\} \oplus V_0(r) \oplus V_2(r).$$

The dimensions of the *Peirce spaces* $V_3(p)$, $V_1(p)$, $V_0(r)$, and $V_2(r)$ are given by $m_1 + 1$, $m_1 + 2m_2$, $m_2 + 1$, and $2m_1 + m_2$, respectively, where m_1 and m_2 denote the multiplicities of the principal curvatures of M, see [2], Theorem 2.2. Let us now explain the geometric meaning of these Peirce spaces. By differentiating the identity $\{x, x, x\} = 3x$ for $x \in M_+$ we see that for every $p \in M_+$ the tangent space T_pM_+ is contained in $V_1(p)$. For reasons of dimension we get $\mathrm{T}_pM_+ = V_1(p)$, cf. [6], proof of Satz 4. As a consequence, $V_3(p)$ is the normal space N_pM_+ of T_pM_+ in $\mathrm{T}_p\mathbb{S}$. Analogously, for every $r \in M_-$ we get $\mathrm{T}_rM_- = V_2(r)$ and $\mathrm{N}_rM_- = V_0(r)$. By [4], Theorem 2.1, we have the following

Theorem 2.1. *Let $(V, \langle\cdot,\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be an isoparametric triple system. Let S be a normal circle of $\mathbb{S}$ that intersects M_+ at the four points $\pm p$, $\pm q$ and M_- at the four points $\pm r$, $\pm s$. Then V decomposes as an orthogonal sum*

$$V = \operatorname{span}(S) \oplus V_3'(p) \oplus V_3'(q) \oplus V_0'(r) \oplus V_0'(s),$$

where $V_3'(p)$, $V_3'(q)$, $V_0'(r)$, $V_0'(s)$ are defined by $V_3(p) = V_3'(p) \oplus \operatorname{span}\{q\}$, $V_3(q) = V_3'(q) \oplus \operatorname{span}\{p\}$, $V_0(r) = V_0'(r) \oplus \operatorname{span}\{s\}$, and $V_0(s) = V_0'(s) \oplus \operatorname{span}\{r\}$.

In Theorem 2.1 we may assume without loss of generality that $r = (1/\sqrt{2})(p + q)$ and $q = (1/\sqrt{2})(r + s)$. Then the linear map $T(r, s)$ leaves the subspaces $V_3'(p)$, $V_3'(q)$, $V_0'(r)$, $V_0'(s)$ invariant with $T(r, s)|_{V_0'(r)} = T(r, s)|_{V_0'(s)} = 0$, $T(r, s)|_{V_3'(q)} = \mathrm{id}$, and $T(r, s)|_{V_3'(p)} = -\mathrm{id}$. Analogously we have $T(p, q)|_{V_3'(p)} = T(p, q)|_{V_3'(q)} = 0$, $T(p, q)|_{V_0'(r)} = -\mathrm{id}$, and $T(p, q)|_{V_0'(s)} = \mathrm{id}$, see [4], proof of Theorem 2.1.

Proof of Theorem 1.1. Choose $r \in M_-$ and $s \in V_0(r) \cap \mathbb{S}$ arbitrarily. Then the great circle S through r and s is a normal circle because of $V_0(r) = \mathrm{N}_rM_-$. As a consequence, we have $s \in M_-$ and $T(r,s)$ leaves the orthogonal complement $V_2(r)$ of $\operatorname{span}\{r\} \oplus V_0(r) = \operatorname{span}(S) \oplus V_0'(r)$ invariant, see Theorem 2.1. Furthermore, $T(r,s)|_{V_2(r)}$ is self-adjoint and has the eigenvalues ± 1 with multiplicity m_1 and the eigenvalue 0 with multiplicity m_2. In the sequel we identify $(V_2(r), \langle\cdot,\cdot\rangle)$ with the Euclidean vector space $\mathbb{R}^{2m_1+m_2}$. Then the linear map

$$\gamma : V_0(r) \to \mathfrak{h}(2m_1 + m_2) : s \mapsto T(r,s)|_{V_2(r)}$$

is injective and the image of $V_0(r) \cap \mathbb{S}$ is contained in $N(m_1, m_2)$. We set $\mathbb{S}_{N(m_1,m_2)} = \gamma(V_0(r) \cap \mathbb{S}) = \gamma(V_0(r)) \cap \mathbb{S}_{\mathfrak{h}(2m_1+m_2)}$. Then $\mathbb{S}_{N(m_1,m_2)}$ is a great sphere of $\mathbb{S}_{\mathfrak{h}(2m_1+m_2)}$ and, in particular, an m_2-dimensional totally geodesic round sphere in $N(m_1, m_2)$. $\square$

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