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On the Twistor Bundle of De Sitter Space $\mathbb{S}_1^3$

Eduardo Hulett*

Facultad de Matemática, Astronomía y Física, FaMAF-CIEM
Ciudad Universitaria, 5000 Córdoba, Argentina
e-mail:hulett@famaf.unc.edu.ar

Abstract. We study the twistor bundle $\mathcal{Z}$ over De Sitter space $\mathbb{S}_1^3$. Viewing $\mathcal{Z}$ as an $SO(1,1)$-principal bundle over the Grassmannian $G_2(\mathbb{L}^4)$ of oriented space-like planes in Lorentz-Minkowski 4-space, the orthogonal complement of the fibers of $\pi' : \mathcal{Z} \to \mathcal{G}_\in(\mathbb{L}^\triangle)$ defines a 4-dimensional horizontal neutral (of signature $(++--)$) distribution $\mathcal{H} \subset \mathcal{T}\mathcal{Z}$. Two $SO(3,1)$-invariant almost Cauchy-Riemann structures $\mathcal{J}^\mathcal{I}$ and $\mathcal{J}^{\mathcal{I}\mathcal{I}}$ on $\mathcal{H}$ are introduced. According to which structure is considered two classes of horizontal holomorphic maps arise. These maps are projected to $\mathbb{S}_1^3$ onto space-like surfaces with different properties. We characterize both classes of horizontal maps in terms of the geometry of their projections to $\mathbb{S}_1^3$.

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1. Introduction

In this paper we consider the geometry of the twistor bundle $\mathcal{Z}$ of De Sitter 3-space $\mathbb{S}_1^3$ and study the Gauß or twistor lifts of conformal immersions of Riemann surfaces in $\mathbb{S}_1^3$. In particular the goal of the paper is to show that certain well

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known examples of space-like surfaces in $\mathbb{S}^3_1$ arise as projections of (almost) complex curves in $\mathcal{Z}$ with respect to distinct $SO(3,1)$-invariant CR-structures, where $SO(3,1)$ is the group of pseudo-isometries of $\mathbb{S}^3_1$. Since $SO(3,1)$ is a simple Lie group, some aspects of the twistor theory of $\mathbb{S}^3_1$ have no analogue in the Riemannian case. Recall that the Gauß map for surfaces in $\mathbb{R}^4$ splits up in two $\mathbb{S}^2$-valued maps according to the decomposition $SO(4) \cong SO(3) \times SO(3)$. For instance in [2] it is shown that there is no orthogonal complex structure or even a natural analogue in Lorentz-Minkowski space $\mathbb{L}^4$ playing the role of the canonical complex structure on $\mathbb{R}^4$.

The twistor space $\mathcal{Z}$ of $\mathbb{S}^3_1$ is defined here as a fiber bundle whose fiber at $p \in \mathbb{S}^3_1$ is the manifold of all oriented space-like 2-planes in $T_p\mathbb{S}^3_1$. Viewing $\mathcal{Z}$ as the total space of a principal $SO(1,1)$-bundle over the Grasmann manifold $G_2(\mathbb{L}^4)$ of oriented 2-planes of the Lorentz-Minkowski 4-space $\mathbb{L}^4$, the orthogonal complement to the fibres of the projection $\pi' : \mathcal{Z} \to \mathcal{G}_\in(\mathbb{L}^\triangle)$ defines a 4-dimensional $SO(3,1)$-invariant horizontal neutral (with signature $(++--)$) distribution $\mathcal{H} \subset \mathcal{TZ}$ which supports two different invariant CR-structures. The first one $\mathcal{J}^{\mathcal{I}}$, is π'-related to the invariant complex structure of the Grasmannian $G_2(\mathbb{L}^4) = SO(3,1)/SO(2) \times SO(1,1)$. The second one $\mathcal{J}^{\mathcal{II}}$, is obtained from $\mathcal{J}^{\mathcal{I}}$ by reversing the complex structure of the fibers of $\pi : \mathcal{Z} \to \mathbb{S}^\ni_\infty$, a characteristic feature of twistor theory.

Twistor theory was created by Roger Penrose almost thirty years ago to solve fundamental problems in Mathematical Physics. In differential geometry the methods of twistor theory contributed to the understanding and parameterization of harmonic maps of Riemann surfaces into various Riemannian symmetric spaces. For a detailed account of the theory in the Riemannian case we refer the reader to [1] and [3] and the bibliography therein.

The paper is organized as follows. In Section 2 we state the main results. Section 3 deals with the structure equations of space-like surfaces in $\mathbb{S}^3_1$. Some basic properties of the normal Gauß map are discussed and some partial independent results are obtained. In Section 4 we study the geometry of the twistor bundle $\mathcal{Z}$ over $\mathbb{S}^3_1$ and define the horizontal distribution $\mathcal{H} \subset \mathcal{TZ}$. Finally in Sections 5 and 6 we prove the main results.

With the exception of null $\mathcal{J}^{\mathcal{I}}$-holomorphic curves which are considered in subsection 5.1, the construction of explicit examples of $\mathcal{J}^{\mathcal{I}}, \mathcal{J}^{\mathcal{II}}$-holomorphic curves and their projections to $\mathbb{S}^3_1$ is not given here and will be considered elsewhere. We refer the reader to the interesting article by A. Fukioka and J. Inoguchi [4] where some of the surfaces considered here are constructed using ideas from the theory of integrable systems.

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2. Main results

Let $\mathcal{Z}$ be the twistor bundle over $\mathbb{S}^3_1$. A smooth map $\phi : M \to \mathcal{Z}$ from a Riemann surface is *horizontal* if $d\phi(TM) \subset \mathcal{H}$, where $\mathcal{H}$ is the orthogonal complement of the fibres of $\pi' : \mathcal{Z} \to \mathcal{G}_{\in}(\mathbb{L}^{\triangle})$. Let $\mathcal{J}$ be one of the almost complex structures defined before. We say that an horizontal map ϕ is $\mathcal{J}$-holomorphic if it satisfies a Cauchy-Riemann-type equation

$$\mathcal{J} \circ \lceil\phi = \lceil\phi \circ \mathcal{J}^{\mathcal{M}},$$

where J^M is the complex structure on the Riemann surface M. For technical reasons we consider only horizontal map $\phi : M \to \mathcal{Z}$ which are *substantial* i.e. ϕ is non-totally geodesic and non π-vertical. Our main result on $\mathcal{J}^{\mathcal{I}}$-holomorphic maps is the following:

Theorem A. *Let $\phi : M \to \mathcal{Z}$ be a horizontal, $\mathcal{J}^{\mathcal{I}}$-holomorphic and substantial map. Then its projection $f = \pi \circ \phi : M \to \mathbb{S}^3_1$ is a (weakly) conformal space-like totally umbilic map with constant non-zero mean curvature. Moreover, the twistor lift of f coincides with ϕ i.e. $\widehat{f} = \phi$.*
Conversely, let $f : M \to \mathbb{S}^3_1$ be a conformal totally umbilic immersion with non-zero constant mean curvature H, then

i) *its twistor lift $\widehat{f} : M \to \mathcal{Z}$ is horizontal, conformal, $\mathcal{J}^{\mathcal{I}}$-holomorphic and substantial. Moreover $\widehat{f}$ is space-like iif $H^2 < 1$, time-like iif $H^2 > 1$ and null iif $H^2 = 1$.*
ii) *the Gauß map $\gamma_f : M \to G_2(\mathbb{L}^4)$ is conformal and holomorphic (hence harmonic), and is space-like iif $H^2 < 1$, time-like iif $H^2 > 1$ and null iif $H^2 = 1$.*

Theorem A essentially says that every space-like totally umbilic map surface with constant non-zero mean curvature $f : M \to \mathbb{S}^3_1$ can be obtained as the projection of a $\mathcal{J}^{\mathcal{I}}$-holomorphic substantial curve in $\mathcal{Z}$. In particular flat totally umbilic space-like surfaces in $\mathbb{S}^3_1$ are obtained by projection of substantial null $\mathcal{J}^{\mathcal{I}}$-holomorphic curves in $\mathcal{Z}$, a fact with no analogue in the Riemannian case.

The nature of $\mathcal{J}^{\mathcal{II}}$-holomorphic curves is quite different. In fact the next result shows that $\pi : (\mathcal{Z}, \mathcal{J}^{\mathcal{II}}) \to \mathbb{S}^{\ni}_{\infty}$ is a twistor fibration in the sense of [1] and [3]).

Theorem B. *Let $\phi : M \to \mathcal{Z}$ be a horizontal, $\mathcal{J}^{\mathcal{II}}$-holomorphic substantial map. Then its projection $f = \pi \circ \phi : M \to \mathbb{S}^3_1$ is a (weakly) conformal space-like harmonic map with isolated umbilic points and satisfies $\widehat{f} = \phi$.*

Conversely, let $f : M \to \mathbb{S}^3_1$ be a conformal space-like harmonic immersion with isolated umbilic points, then its twistor lift $\widehat{f} : M \to \mathcal{Z}$ is horizontal, $\mathcal{J}^{\mathcal{II}}$-holomorphic and substantial. Moreover, the Gauß map $\gamma_f : M \to G_2(\mathbb{L}^4)$ is harmonic and non-holomorphic.

3. Conformally immersed surfaces in $\mathbb{S}^3_1$

Let $\mathbb{L}^4$ denote the 4-dimensional Lorentz-Minkowski space that is, the real vector space $\mathbb{R}^4$ equipped with the Lorentz metric

$$\langle\, , \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$$

where (x_1, x_2, x_3, x_4) are the canonical coordinates of $\mathbb{R}^4$. With respect to the canonical basis $\{e_j\}$ of $\mathbb{L}^4$ the matrix of $\langle\, ,\rangle$ is $I_{3,1} = diag(1,1,1,-1)$ and the group of all Lorenz isometries of $\mathbb{L}^4$ is $O(3,1)$. With respect to the standard basis it follows that $SO(3,1) = \{A \in GL(4,\mathbb{R}) : A^t I_{3,1} A = I_{3,1},\ a_{44} > 0\}$.

The 3-dimensional unitary De Sitter space is defined as the hyperquadric

$$\mathbb{S}^3_1 = \{x \in \mathbb{L}^4 : \langle x, x\rangle = 1\}$$

Then $\mathbb{S}^3_1$ inherits from $\mathbb{L}^4$ a Lorentzian metric denoted also by $\langle\, ,\rangle$ which has constant sectional curvature one. The Lie group $SO(3,1)$ acts transitively on $\mathbb{S}^3_1$ and fixing for convenience the base point $e_1 \in \mathbb{S}^3_1$ we see that $\mathbb{S}^3_1$ is diffeomorphic to the symmetric quotient $SO(3,1)/SO(2,1)$, where the isotropy subgroup $SO(2,1)$ of e_1 is imbedded in $SO(3,1)$ according to

$$SO(2,1) \ni A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

We fix a time orientation on $\mathbb{L}^4$ by declaring a time-like vector $X \in \mathbb{L}^4$ to be positively oriented or future-pointing if $\langle X, e_4\rangle < 0$. This time orientation induces a time orientation on $\mathbb{S}^3_1$ as follows: a time-like tangent vector $X_p \in T_p\mathbb{S}^3_1$ is future-pointing if $\langle X'_p, e_4\rangle < 0$ where X'_p is the parallel translated of X_p to the origin of $\mathbb{L}^4$. It is easily seen that the Lie group $SO(3,1)$ preserves the time orientation of $\mathbb{L}^4$ and $\mathbb{S}^3_1$.

A smooth immersion $f : M \to \mathbb{S}^3_1$ of a Riemann surface is conformal if ∂f is isotropic: $\langle \partial f, \partial f\rangle^{\mathbb{C}} = 0$, for every local complex coordinate $z = x + iy$ on M, where $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\langle\, ,\rangle^{\mathbb{C}}$ denotes the complex bilinear extension of $\langle\, ,\rangle$ to $T^{\mathbb{C}}\mathbb{S}^3_1$. Equivalently f is conformal if and only if for every local coordinate $z = x + iy$

$$\langle f_x, f_y\rangle = 0, \quad \|f_x\|^2 = \|f_y\|^2 \tag{1}$$

An immersion $f : M \to \mathbb{S}^3_1$ is said to be space-like if $f^*\langle\, ,\rangle$ is positive definite. A consequence of (1) is that every conformal immersion $f : M \to \mathbb{S}^3_1$ is a space-like map.

The time orientation of $\mathbb{S}^3_1$ allows to choose a time-like unit field $\mathbf{n}$ along f globally defined on M such that $\mathbf{n}$ is tangent to $\mathbb{S}^3_1$ along f, and $\mathbf{n}(p) \perp df_p(T_pM)$ for every $p \in M$. Thus we may assume that the immersed space-like surface M is oriented by $\mathbf{n}$. Fix on M the induced Riemannian metric $g = f^*\langle\, ,\rangle$ which is conformal and let u be a conformal parameter defined by $g = 2e^{2u}(dx^2 + dy^2)$ or equivalently $\langle \partial f, \bar\partial f\rangle^{\mathbb{C}} = e^{2u}$. Being f conformal it satisfies the following equations

$$\begin{aligned} 2\langle \bar\partial\partial f, \partial f\rangle^{\mathbb{C}} &= \bar\partial\langle \partial f, \partial f\rangle^{\mathbb{C}} = 0 \\ 2\langle \bar\partial\partial f, \bar\partial f\rangle^{\mathbb{C}} &= \partial\langle \bar\partial f, \bar\partial f\rangle^{\mathbb{C}} = 0. \end{aligned}$$

Thus $\bar\partial\partial f$ is normal to the immersed surface since it has no tangential component. On the other hand the second fundamental form of the conformal immersion is given by $II = -\langle df, d\mathbf{n}\rangle$ so that the mean curvature of f is given by $H := -e^{-2u}\langle\bar{\partial}\partial f, \mathbf{n}\rangle$. The first structural equation reads

$$\bar{\partial}\partial f = -e^{2u}f + e^{2u}H.\mathbf{n}. \tag{2}$$

If $H \equiv 0$ then the conformal immersion f is harmonic (hence minimal) if and only if

$$\bar{\partial}\partial f = -\langle\partial f, \bar{\partial} f\rangle^{\mathbb{C}} f \tag{3}$$

Introducing the complex function $\xi := -\langle\partial^2 f, \mathbf{n}\rangle^{\mathbb{C}}$, we obtain the second structural equation

$$\partial^2 f = 2\partial u.\partial f + \xi.\,\mathbf{n}, \tag{4}$$

from which we see that the complex Hopf quadratic differential $Q = \langle\partial^2 f, \partial^2 f\rangle^c dz^2$ of f is given by $Q = -\xi^2 dz^2$. Note that away from the umbilic points the unit normal vector is recovered from (4)

$$\mathbf{n} = \frac{1}{\xi}.(2\partial u.\partial f - \partial^2 f) \tag{5}$$

From (2) and (4) above we obtain the third structural equation

$$\partial\mathbf{n} = H.\partial f + e^{-2u}\xi.\bar{\partial} f. \tag{6}$$

The compatibility conditions of the above structure equations are given by the Gauß-Codazzi equations:

$$\begin{array}{ll} \text{Gauß:} & 2\bar{\partial}\partial u = (H^2-1)e^{2u} - |\xi|^2 e^{-2u}, \\ \text{Codazzi:} & \bar{\partial}\xi = e^{2u}\partial H,\ \partial\bar{\xi} = e^{2u}\bar{\partial}H \end{array} \tag{7}$$

In particular H is constant if and only if ξ is a holomorphic function.

A Lax-pair for the equations of Gauß-Codazzi arise from a local adapted frame of a conformal immersion f. So let $F : U \to SO(3,1)$ be a local frame defined on an open subset $U \subset M$ and set $F_i = F.e_i$. Then F is adapted if for every $p \in U$

$$f(p) = F(p).e_1, \quad \text{span}\{F(p).e_2, F(p).e_3\} = df_p(TM), \quad F(p).e_4 = \mathbf{n}(p),$$

A straightforward calculation using equations (2), (4) and (6) shows that the evolution of the frame F is given by

$$\begin{aligned} \partial f &= \tfrac{e^u}{\sqrt{2}}(F_2 - iF_3), \\ \partial F_2 &= -\tfrac{e^u}{\sqrt{2}}.f - i\partial u F_3 + (\tfrac{e^{-u}\xi.+e^u H}{\sqrt{2}}).\mathbf{n} \\ \partial F_3 &= i\tfrac{e^u}{\sqrt{2}}.f + i\partial u F_2 + i(\tfrac{e^{-u}\xi.-e^u H}{\sqrt{2}}).\mathbf{n} \\ \partial\mathbf{n} &= (\tfrac{e^u H+e^{-u}\xi}{\sqrt{2}}).F_2 + i(\tfrac{-e^u H+e^{-u}\xi}{\sqrt{2}}).F_3 \end{aligned} \tag{8}$$

Compactly written these equations yield the system

$$\partial F = F.A, \quad \bar{\partial} F = F.B \tag{9}$$

in which the complex matrices A and B are given by

$$A = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & i\frac{e^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & i\partial u & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} \\ -i\frac{e^u}{\sqrt{2}} & -i\partial u & 0 & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) \\ 0 & \frac{e^{-u}\xi + e^u H}{\sqrt{2}} & i(\frac{e^{-u}\xi - e^u H}{\sqrt{2}}) & 0 \end{pmatrix} \tag{10}$$

$$B = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & -i\frac{e^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & -i\bar{\partial} u & \frac{e^{-u}\bar{\xi} + e^u H}{\sqrt{2}} \\ i\frac{e^u}{\sqrt{2}} & i\bar{\partial} u & 0 & -i(\frac{e^{-u}\bar{\xi} - e^u H}{\sqrt{2}}) \\ 0 & \frac{e^{-u}\bar{\xi} + e^u H}{\sqrt{2}} & -i(\frac{e^{-u}\bar{\xi} - e^u H}{\sqrt{2}}) & 0 \end{pmatrix}. \tag{11}$$

The integrability condition of the system (9) is given by

$$\bar{\partial} A - \partial B = [A, B],$$

and is equivalent to Gauß-Codazzi's equations (7).

Let $\mathbb{H}^3_+$ be the upper hyperboloid formed by all $x \in \mathbb{L}^4$ such that $\|x\|^2 = -1$ and $x_4 > 0$. The translated to the origin of $\mathbb{L}^4$ of the time-like normal unit vector field $\mathbf{n}$ along f defines the normal Gauß map $\mathbf{n} : M \to \mathbb{H}^3_+$. Using (6) we see that

$$\|\partial \mathbf{n}\|^2 = H^2 e^{2u} + |\xi|^2 e^{-2u}, \quad \langle \partial \mathbf{n}, \partial \mathbf{n} \rangle^{\mathbb{C}} = H\xi.$$

Hence $\mathbf{n}$ is conformal when $H = 0$ or $\xi \equiv 0$. Assuming that H is constant then ξ is holomorphic by Codazzi's equation. Using the structure equations of f we obtain the following partial differential equation for $\mathbf{n}$ which was considered in [6]:

$$\bar{\partial}\partial \mathbf{n} = (e^{2u}H^2 + e^{-2u}|\xi|^2).\mathbf{n} - e^{2u}H.f \tag{12}$$

If $H = 0$ and the umbilic points of f are isolated, we see that $\mathbf{n}$ is conformal and satisfies the PDE

$$\bar{\partial}\partial \mathbf{n} = |\xi|^2 e^{-2u}.\mathbf{n} \tag{13}$$

Now from formula (6) we obtain $\langle \partial \mathbf{n}, \partial \mathbf{n} \rangle = e^{-4u}|\xi|^2 \langle \bar{\partial} f, \bar{\partial} f \rangle = e^{-2u}|\xi|^2$. Thus (13) becomes

$$\bar{\partial}\partial \mathbf{n} = \langle \partial \mathbf{n}, \partial \mathbf{n} \rangle.\mathbf{n}$$

which is just the harmonic map equation for $\mathbf{n}$. Hence we obtain

Proposition 3.1. *Let $f : M \to \mathbb{S}^3_1$ be a minimal space-like (conformal harmonic) immersion. Then its normal Gauß map $\mathbf{n} : M \to \mathbb{H}^3$ is a minimal branched immersion whose branched points are the umbilic points of f.*

Remark 3.1. Note that the metric on M induced by $\mathbf{n}$ is given by $\tilde{g} = 2e^{-2u}|\xi|^2 (dx^2 + dy^2)$. Hence $\tilde{g}$ is a branched metric whose singularities coincide with the zeros of the Hopf differential of f.

Since there are no non-constant harmonic maps from a compact manifold M into real hyperbolic spaces we obtain

Corollary 3.2. *Let $f : M \to \mathbb{S}^3_1$ be a minimal space-like immersion of a compact connected Riemann surface M. Then $f(M)$ coincides with the space-like totally geodesic 2-sphere $\mathbb{S}^3_1 \cap \mathbf{n}^\perp$.*

Proof. By hypothesis f is a conformal harmonic immersion, hence the normal Gauß map $\mathbf{n} : M \to \mathbb{H}^3_+$ is harmonic. Since M is compact, $\mathbf{n}$ must be constant and so $f(M) = \mathbb{S}^3_1 \cap \mathbf{n}^\perp$ (in particular f is totaly umbilic). $\square$

On the other hand if $H \neq 0$ and f is totally umbilical then the normal Gauß map $\mathbf{n}$ is conformal and the immersed surface $\mathbf{n} : M \to \mathbb{H}^3$ is totally umbilic since by (2), (4) and (6), $\langle \partial^2 \mathbf{n}, f\rangle = 0$. Also it is easy to see that the mean curvature of $\mathbf{n}$ is given by $H' = -\frac{\langle \bar\partial\partial \mathbf{n}, f\rangle}{\|\partial \mathbf{n}\|^2}$. Then using $\|\partial \mathbf{n}\|^2 = H^2 e^{2u}$ and (12), we see that

$$H' = \frac{\langle \bar\partial\partial \mathbf{n}, f\rangle}{H^2 e^{2u}} = -\frac{He^{2u}}{H^2 e^{2u}} = -\frac{1}{H},$$

hence the immersed surface $\mathbf{n}$ has constant mean curvature $-\frac{1}{H}$. We have then proved

Corollary 3.3. *Let $f : M \to \mathbb{S}^3_1$ be a conformal totally umbilic immersion with constant curvature $H \neq 0$. Then the normal Gauß map $\mathbf{n} : M \to \mathbb{H}^3_+$ is a conformal totally umbilic immersion with constant mean curvature $-\frac{1}{H}$.*

4. The twistor bundle of $\mathbb{S}^3_1$

Let $T\mathbb{S}^3_1 = \{(v, w) \in \mathbb{S}^3_1 \times \mathbb{L}^4 : \langle v, w\rangle = 0\}$ be the tangent bundle of $\mathbb{S}^3_1$ and consider the submanifold $\mathcal{Z} \subset \mathcal{T}\mathbb{S}^{\ni}_\infty$ formed by those (v, w) such that $\|w\|^2 = -1$, and w is future pointing. Hence $\mathcal{Z}$ is nothing but the -1-unitary sphere bundle of $\mathbb{S}^3_1$, i.e. the fiber of $\mathcal{Z}$ at a point $v \in \mathbb{S}^3_1$ is the set of vectors in $T_v\mathbb{S}^3_1$ having square norm -1. Thus $\mathcal{Z}$ is isomorphic to the following Stiefel manifold

$$\{(v, w) \in \mathbb{S}^3_1 \times \mathbb{H}^3_+ : \langle v, w\rangle = 0\} \subset \mathbb{L}^4 \times \mathbb{L}^4, \tag{14}$$

where $\mathbb{H}^3_+$ is the real 3-hyperbolic space with curvature -1 consisting of vectors $x \in \mathbb{L}^4$ satisfying $\langle x, x\rangle = -1$ and $x_4 > 0$. Define the projection $\pi : \mathcal{Z} \to \mathbb{S}^{\ni}_\infty$ by $\pi(v, w) = v$, then the fibre $\pi^{-1}(v)$ over the point $v \in \mathbb{S}^3_1$ equals $\{v\} \times (v^\perp \cap \mathbb{H}^3_+)$ which is a copy of the real 2-dimensional hyperbolic space $\mathbb{H}^2$ immersed into $\mathbb{H}^3_+$. Thus each fiber of π is naturally a complex manifold.

Given $(v, w) \in \mathcal{Z}$ the orthogonal complement $V = [v \wedge w]^\perp$ of w in $T_v\mathbb{S}^3_1$ is an oriented space-like 2-plane thus a point in the Grassmann manifold of oriented space-like 2-planes of $T_v\mathbb{S}^3_1$. Therefore $\mathcal{Z}$ may be viewed as the Gauß bundle over $\mathbb{S}^3_1$ and so it is isomorphic to the associated bundle

$$SO(\mathbb{S}^3_1) \times_{SO(2,1)} \mathbb{H}^2_+, \tag{15}$$

where $SO(\mathbb{S}^3_1) \equiv SO(3,1)$ is the principal $SO(3,1)$-bundle of oriented orthonormal frames of $\mathbb{S}^3_1$.

Let $G^+_2(\mathbb{L}^4)$ be the Grassmann manifold of oriented space-like 2-planes in $\mathbb{L}^4$. A projection map $\pi' : \mathcal{Z} \to \mathcal{G}^+_{\in}(\mathbb{L}^{\triangle})$ is defined by assigning to a point $(v,w) \in \mathcal{Z}$ the translated to the origin of $\mathbb{L}^4$ of the space-like 2-plane $V := w^\perp \cap T_v\mathbb{S}^3_1 = [v \wedge w]^\perp \subset \mathbb{L}^4$:

$$\pi'(v,w) = [v \wedge w]^\perp. \tag{16}$$

To determine the fibre of π' over $V \in G^+_2(\mathbb{L}^4)$, note that $V^\perp$ is an oriented 2-plane of signature $(+,-)$. Fix an oriented basis $\{v,w\}$ of $V^\perp$ such that $\|v\|^2 = -\|w\|^2 = 1$, $\langle v,w\rangle = 0$. Then

$$\pi'^{-1}(V) = \{(\beta(t), \beta'(t)) : t \in \mathbb{R}\},$$

where $\beta(t) = \cosh(t)v + \sinh(t)w$. Hence the fibres of π' are diffeomorphic to $\mathbb{R}$.

Another natural projection is $\pi'' : \mathcal{Z} \to \mathbb{H}^{\ni}_+$ defined by $\pi''(v,w) = w$. It is easily seen that if $f : M \to \mathbb{S}^3_1$ is a space-like surface oriented by $\mathbf{n}$, then $\pi'' \circ \widehat{f} = \mathbf{n}$ is the so-called normal Gauß map of f.

On the other hand the usual Gauß map γ_f of a conformal immersion $f : M \to \mathbb{S}^3_1$ is defined by

$$\gamma_f := \pi' \circ \widehat{f} : M \to G_2(\mathbb{L}^4). \tag{17}$$

In a local coordinate $z = x + iy$ it is given by $\gamma_f = [f \wedge \mathbf{n}]^\perp = [f_x \wedge f_y]$. If $f : M \to \mathbb{S}^3_1$ is harmonic, then f viewed as a conformal immersion into $\mathbb{L}^4$, has parallel mean curvature vector $-f$ and consequently γ_f is a harmonic map as a consequence of Theorem 2.1 in [2]. We shall see in Section 6 that there is another class of space-like surfaces in $\mathbb{S}^3_1$ with harmonic Gauß maps.

There is a right action of $SO(1,1)$ on the total space $\mathcal{Z}$ given by

$$(v,w).\begin{pmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{pmatrix} = (\cosh(t)v + \sinh(t)w, \sinh(t)v + \cosh(t)w), \tag{18}$$

This allows one to think of $\mathcal{Z}$ as an $SO(1,1)$-principal bundle over $G^+_2(\mathbb{L}^4)$. Note that the fibre of π' through $(v,w) \in \mathcal{Z}$ coincides with the orbit $\{(v,w).\exp(tX_0) : t \in \mathbb{R}\}$, where

$$X_0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \in \mathfrak{so}(1,1).$$

Let us now consider a left action of $SO(3,1)$ on $\mathcal{Z}$ given by

$$g.(v,w) = (g.v, g.w).$$

It not difficult to see that this action is transitive so that fixing for convenience the point $o := (e_1, e_4) \in \mathcal{Z}$ we obtain the homogeneous quotient representation

$$\mathcal{Z} = \frac{\mathcal{SO}(\ni, \infty)}{\mathcal{K}},$$

where

$$K = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}, A \in SO(2)\}. \tag{19}$$

is the isotropy subgroup of the element o. The Lie algebra of $SO(3,1)$ decomposes into $\mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, a \in \mathbb{R}\},$$

is the Lie algebra of $K \simeq SO(2)$, and an $Ad(K)$-invariant complement $\mathfrak{p}$ is the set of matrices of the form

$$A = \begin{pmatrix} 0 & x & y & c \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ c & z & w & 0 \end{pmatrix}, x, y, c, z, w \in \mathbb{R}.$$

The expression

$$(A, B) = -\frac{1}{2} Trace(A.B),\ A, B \in \mathfrak{p}, \tag{20}$$

defines an $Ad(K)$-invariant inner product on $\mathfrak{p}$ of signature $(++---)$. Note that the square norm of an element $A \in \mathfrak{p}$ is given by $\|A\|^2 = x^2 + y^2 - c^2 - z^2 - w^2$. This determines an $SO(3,1)$-invariant pseudo-metric on $\mathcal{Z}$, the so-called normal metric.

We define the horizontal distribution $\mathcal{H} \subset \mathcal{T}\mathcal{Z}$ as the complementary subbundle of the fibres of π':

$$\mathcal{H}_{(\sqsubseteq,\sqsupseteq)} = [\mathcal{K}]\nabla\ \lceil\pi'_{(\sqsubseteq,\sqsupseteq)}]^{\perp} \tag{21}$$

It is not difficult to see that the distribution $\mathcal{H}$ defines a connection on the principal bundle $SO(1,1) \to \mathcal{Z} \to \mathcal{G}^+_{\in}(\mathbb{L}^{\triangle})$. At the point $o = (e_1, e_4)$, $\mathcal{H}_\wr$ consists of matrices of the form

$$\begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix}, \quad x, y, z, w \in \mathbb{R}. \tag{22}$$

Thus $\mathcal{H}_,$ is $Ad(K)$-invariant and so $\mathcal{H}$ is an $SO(3,1)$-invariant distribution. Therefore

$$\mathcal{H}_{\}\mathcal{K}} := \mathcal{A}\lceil(\})\mathcal{H}_\wr,\ \forall\} \in \mathcal{SO}(\ni, \infty).$$

In particular the induced metric on $\mathcal{H}$ is neutral i.e. has signature $(++--)$.

For any $(v,w) \in \mathcal{Z}$ it is not difficult to see that $Ker\, d\pi_{(v,w)}$ and $Ker\, d\pi'_{(v,w)}$ are orthogonal subspaces thus $Ker\, d\pi_{(v,w)} \subset \mathcal{H}_{(\sqsubseteq,\sqsupseteq)}$. So we can decompose

$$\mathcal{H}_{(\sqsubseteq,\sqsupseteq)} = \mathcal{K}\rceil\nabla\,\lceil\pi_{(\sqsubseteq,\sqsupseteq)} \overset{\perp}{\oplus} \mathcal{L}_{(\sqsubseteq,\sqsupseteq)},$$

where $L_{(v,w)}$ coincides with the horizontal lift via $d\pi$ of the orthogonal complement of w in $T_v\mathbb{S}^3_1$. Note that the metric (20) restricted to $\mathcal{H}$ has signature $(++)$ on L and $(--)$ on $Ker\, d\pi$.

The complex structure $\mathcal{J}^{\mathcal{L}}_{(\sqsubseteq,\sqsupseteq)}$ on $L_{(v,w)}$ is obtained by lifting via $d\pi$ the complex structure on the orthogonal complement of w in $T_v\mathbb{S}^3_1$, which is just the positively oriented rotation of angle $\frac{\pi}{2}$ on the space-like plane $[v \wedge w]^{\perp}$. On the other hand since the fibers of π are hyperbolic 2-spaces there is another complex structure $J^{\mathcal{V}}_{(v,w)}$ on $Ker\, d\pi_{(v,w)}$. Both complex structures together yield two different almost complex structures $\mathcal{J}^{\mathcal{I}}, \mathcal{J}^{\mathcal{II}}$ on $\mathcal{H}$ given by

$$\mathcal{J}^{\mathcal{I}} = \begin{cases} J^L, & \text{on } L \\ J^{\mathcal{V}}, & \text{on } Ker\, d\pi \end{cases} \qquad \mathcal{J}^{\mathcal{II}} = \begin{cases} J^L, & \text{on } L \\ -J^{\mathcal{V}}, & \text{on } Ker\, d\pi \end{cases}$$

At the point $o = (e_1, e_4)$ it is possible to write down explicitly both almost complex structures. In fact since

$$L_o = \{\begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\}, \quad Ker\, d\pi_o = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix}\},$$

the complex structure on L_o is obtained lifting via $d\pi|_o$ the oriented rotation on $[e_1 \wedge e_4]^{\perp} \subset T_{e_1}\mathbb{S}^3_1$ given by $e_2 \mapsto e_3$, $e_3 \mapsto -e_2$. On the other hand the complex structure on $T_o(\pi^{-1}(e_1)) \cong T_{e_4}SO(2,1)/SO(2)$ is given by

$$\begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ z & w & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & -w \\ 0 & 0 & z \\ -w & z & 0 \end{pmatrix}.$$

By putting both structures together we get

$$\mathcal{J}^{\mathcal{I}}_{\wr} \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y & x & 0 \\ y & 0 & 0 & -w \\ -x & 0 & 0 & z \\ 0 & -w & z & 0 \end{pmatrix}, \tag{23}$$

which shows that $\mathcal{J}^{\mathcal{I}}$ is $Ad(K)$-invariant and orthogonal with respect to the normal metric defined by (20). Moreover it is not difficult to see that $\mathcal{J}^{\mathcal{I}}_{\wr} = \mathcal{A}\lceil(\mathcal{P}_\infty)|_{\mathcal{H}_\wr}$, where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K. \tag{24}$$

Consequently $\mathcal{J}^{\mathcal{I}}$ is an $SO(3,1)$-invariant almost complex structure on $\mathcal{H}$. It is clearly seen that $d\pi'$ projects $\mathcal{J}^{\mathcal{I}}$ onto the $SO(3,1)$-invariant complex structure $\mathcal{J}$ on $G_2^+(\mathbb{L}^4)$ determined by the Ad-action of P_1 on $T_O G_2(\mathbb{L}^4)$ through formula

$$\mathcal{J}_{\mathcal{O}} = \mathcal{A}\lceil(\mathcal{P}_\infty)|_{\mathcal{H}'_{\mathcal{O}}}, \tag{25}$$

where the base point $O \in G_2(\mathbb{L}^4)$ is the subspace $\mathbb{R}e_2 \oplus \mathbb{R}e_3$ and $\mathcal{H}'_{\mathcal{O}}$ is the subspace of matrices of $\mathfrak{so}(3,1)$ of the form (22) which identifies with $T_O G_2(\mathbb{L}^4)$. Note that the isotropy subsgroup of the point O is isomorphic to $SO(2) \times SO(1,1)$.

It is possible to show that $(\mathcal{J}^{\mathcal{I}}, \mathcal{H})$ is an integrable CR-structure on $\mathcal{Z}$ by adapting ideas from [5]. However we shall not give here the details.

The second almost complex structure is obtained by reversing the complex structure on the fibers. At $o \in \mathcal{Z}$ we have

$$\mathcal{J}^{\mathcal{II}}_{\wr} \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & z \\ -y & 0 & 0 & w \\ 0 & z & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y & x & 0 \\ y & 0 & 0 & w \\ -x & 0 & 0 & -z \\ 0 & w & -z & 0 \end{pmatrix}. \tag{26}$$

Thus like in the case of $\mathcal{J}^{\mathcal{I}}_{\wr}$ we observe that $\mathcal{J}^{\mathcal{II}}_{\wr}$ is $SO(3,1)$-invariant too. By inspection we see that $\mathcal{J}^{\mathcal{II}}_{\wr} = \mathcal{A}\lceil(\mathcal{P}_\in)|_{\mathcal{H}_\wr}$, where

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \notin K. \tag{27}$$

Hence the almost complex structure $\mathcal{J}^{\mathcal{II}}$ on $\mathcal{H}$ is not related to the complex structure of the Grasmannian $G_2(\mathbb{L}^4)$. The question of the integrability of the CR structure $(\mathcal{H}, \mathcal{J}^{\mathcal{II}})$ will be considered in a future paper. However we shall see in the proof of Theorem B that $\pi : (\mathcal{Z}, \mathcal{J}^{\mathcal{II}}) \to \mathbb{S}^{\ni}_{\infty}$ is a twistor fibration in the sense of [1] and [3].

5. Proof of Theorem A

Let $\phi : M \to \mathcal{Z}$ be a smooth horizontal map of a Riemann surface M i.e. $d\phi(TM) \subset \mathcal{H}$ and let $\mathcal{J} \in \{\mathcal{J}^{\mathcal{I}}, \mathcal{J}^{\mathcal{II}}\}$. We say that ϕ is $\mathcal{J}$-holomorphic if it satisfies the Cauchy-Riemann-type equation

$$\mathcal{J} \circ \lceil\phi = \lceil\phi \circ \mathcal{J}^{\mathcal{M}},$$

where J^M is the complex structure on the Riemann surface M.

In this section we study the class of $\mathcal{J}^{\mathcal{I}}$-holomorphic maps. Define the holomorphic horizontal distribution determined by $\mathcal{J}^{\mathcal{I}}$ by

$$\mathcal{H}^{(\infty,\prime)} = \{\mathcal{X} \in \mathcal{H}^{\mathbb{C}} : \mathcal{J}^{\mathcal{I}}\mathcal{X} = \rangle\mathcal{X}\}.$$

It is easily seen that $\phi : M \to \mathcal{Z}$ is $\mathcal{J}^{\mathcal{I}}$-holomorphic if and only if $d\phi(TM^{(1,0)}) \subseteq \mathcal{H}^{(\infty,\prime)}$. Also $\mathcal{H}^{(\infty,\prime)}$ is $SO(3,1)$-invariant since $\mathcal{J}^{\mathcal{I}}$ is $SO(3,1)$-invariant. From (23) it follows that $\mathcal{H}_{\wr}^{(\infty,\prime)}$ consists of matrices of the form

$$\begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & -ib \\ 0 & b & -ib & 0 \end{pmatrix}, \ a, b \in \mathbb{C},$$

hence

$$\mathcal{H}_{\mathcal{F}.\wr}^{(\infty,\prime)} = \mathcal{A}\lceil(\mathcal{F})\mathcal{H}_{\wr}^{(\infty,\prime)},$$

for every $F \in SO(3,1)$. Moreover $\mathcal{H}^{(\infty,\prime)}$ is an isotropic subbundle of $T^{\mathbb{C}}\mathcal{Z}$ as is clear from the invariance of $\mathcal{H}^{(\infty,\prime)}$ and the fact that $(X, X) = 0$ for every $X \in \mathcal{H}_{\wr}^{(\infty,\prime)}$.

Now recall the reductive decomposition $\mathfrak{so}(3,1) = \mathfrak{k} \oplus \mathfrak{p}$ considered before. Then since $\mathfrak{p}$ is $Ad(K)$-invariant, we have an $SO(3,1)$-invariant vector bundle

$$[\mathfrak{p}] = SO(3,1) \times_K \mathfrak{p} \subset \mathcal{Z} \times \mathfrak{so}(\ni, \infty),$$

whose fiber at $F.o \in \mathcal{Z}$ is $[\mathfrak{p}]_{F.o} = \{F.o\} \times Ad(F)(\mathfrak{p})$, for any $F \in SO(3,1)$.

Since $\mathcal{Z}$ is naturally reductive there is a bundle isomorphism $T\mathcal{Z} \xrightarrow{\beta} [\mathfrak{p}]$ which is the inverse of the application $\mathfrak{so}(3,1) \to T_x\mathcal{Z}$ defined by $X \mapsto \frac{d}{dt}|_{t=0} \exp(tX).x$. Thus β may be viewed as a one form on $\mathcal{Z}$ with values in $\mathfrak{so}(3,1)$. Given a map $\phi : M \to \mathcal{Z}$, there is a useful formula to compute the differential of ϕ in terms of β and the Maurer-Cartan form of $SO(3,1)$, namely

$$\phi^*\beta = Ad(F)\alpha_{\mathfrak{p}}, \tag{28}$$

where $\phi = F.o$, i.e. F is a frame of ϕ, and $\alpha_{\mathfrak{p}} = [F^{-1}dF]_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of the pull-back of the Maurer-Cartan one form of $SO(3,1)$ by the frame F. More details on this construction are provided in [1].

Let $\phi : M \to \mathcal{Z}$ be an horizontal map. By the preceding discussion ϕ is $\mathcal{J}^{\mathcal{I}}$-holomorphic if and only if

$$(\phi^*\beta)(\frac{\partial}{\partial z}) \in \mathcal{H}_{\mathcal{F}.\wr}^{(\infty,\prime)},$$

for every frame F of ϕ and every local complex coordinate z. Hence ϕ is $\mathcal{J}^{\mathcal{I}}$-holomorphic if and only if for every frame F of ϕ and every complex coordinate z there there are smooth complex functions $a, b, c : U \to \mathbb{C}$ such that

$$\alpha(\frac{\partial}{\partial z}) = F^{-1}\partial F = \begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & -c & b \\ ia & c & 0 & -ib \\ 0 & b & -ib & 0 \end{pmatrix} = \alpha_{\mathfrak{k}}(\frac{\partial}{\partial z}) + \alpha_{\mathfrak{p}}(\frac{\partial}{\partial z}), \tag{29}$$

whence

$$(\phi^*\beta)(\frac{\partial}{\partial z}) = Ad(F)[\alpha_{\mathfrak{p}}(\frac{\partial}{\partial z})] = Ad(F)\begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & -ib \\ 0 & b & -ib & 0 \end{pmatrix}. \tag{30}$$

It is useful to assume that ϕ is *substantial* which means that the complex functions a, b have isolated zeros. Since $\mathcal{H}^{(\infty,\prime)}$ is isotropic we see from the above equation that ϕ is conformal. Using β we have $(\phi^*\beta)(\frac{\partial}{\partial z}) = d\phi(\frac{\partial}{\partial z}) = \partial\phi$, and so by (30) we have

$$\|d\phi(\frac{\partial}{\partial z})\|^2 = 2(|a|^2 - |b|^2). \tag{31}$$

On the other hand from (29) we obtain the ∂-derivative of the columns of the frame F:

$$\begin{array}{ll} \text{i)} & \partial F_1 = -a(F_2 - iF_3) \\ \text{ii)} & \partial F_2 = aF_1 + cF_3 + bF_4 \\ \text{iii)} & \partial F_3 = -iaF_1 - cF_2 - ibF_4 \\ \text{iv)} & \partial F_4 = bF_2 - ibF_3 \end{array}$$

Since F a frame of ϕ we have $\phi = (F_1, F_4)$ and that F is also a frame of the projection $f = F_1 = \pi \circ \phi : U \to \mathbb{S}_1^3$, in particular $\mathbf{n} = F_4$ is $\mathbb{H}_+^3$-valued and $\widehat{f} = \phi$. We can read off the structure equations of $f = F_1$ from equations i),ii), iii) and iv) above. From the first one we see that ∂f is isotropic: $\langle \partial f, \partial f\rangle^{\mathbb{C}} = 0$, and $\|\partial f\|^2 = 2|a|^2 > 0$, thus f is a conformal space-like immersion. There is no loss of generality in assuming that a is real and negative (otherwise one can perform a gauge transformation on F consisting on a rotation on the plane generated by F_2, F_3). Also we compute

$$\partial^2 f = (-a)(F_2 - iF_3) - a(\partial F_2 - i\partial F_3) = -\partial a(F_2 - iF_3) - a(cF_3 + icF_2),$$

hence $\langle \partial^2 f, F_4\rangle^{\mathbb{C}} \equiv 0$, which shows that the Hopf differential vanishes, thus f is totally umbilic. Since f conformal $\bar\partial\partial f$ has no tangential component and is normal to $f(M)$. Hence from i), ii), iii) it follows that

$$\begin{aligned} &\partial a = iac, \\ &\bar\partial\partial f = -2|a|^2 f - 2a\bar{b}F_4. \end{aligned}$$

From the second equation above we obtain that the mean curvature of f is given by $H = -\frac{1}{2a^2}\langle \bar\partial\partial f, F_4\rangle^{\mathbb{C}} = \frac{\bar{b}}{a}$. Hence H must be a constant as a consequence of Codazzi equation for f, and so $b = Ha$. In particular a and b have the same zeros. Thus $f = F_1 : M \to \mathbb{S}_1^3$ is a (weakly) conformal totally umbilic map with constant non-zero mean curvature $H = \frac{b}{a}$.

Conversely, let $f : M \to \mathbb{S}_1^3$ be a given conformal totally umbilic immersion then by Codazzi's equation the mean curvature H of f is constant. Introduce a conformal parameter u by $\langle \partial f, \partial f\rangle = e^{2u}$ and let $\mathbf{n}$ be a tangent vector field on $\mathbb{S}_1^3$ such that $\|\mathbf{n}\|^2 = -1$ and which is normal to $df(TM)$ along f. The structure equations of f are given by (2), (4) and (6) in which $\xi = 0$. Taking the frame

$F = (f, F_2, F_3, \mathbf{n})$ of f we may assume that $F \in SO(3,1)$ (changing $\mathbf{n} \to -\mathbf{n}$ if necessary) and $\partial f = \frac{e^u}{\sqrt{2}}(F_2 - iF_3)$. Let $A = F^{-1}\partial F$ and $B = F^{-1}\bar{\partial} F$ then from (10) we see that the matrix A decomposes into $A = A_0 + A_1 \in \mathfrak{k}^{\mathbb{C}} \oplus \mathcal{H}_{\wr}^{(\infty,\prime)}$. Then $(\widehat{f}^*\beta)(\frac{\partial}{\partial z}) = Ad(F)(A_1)$ and so $\widehat{f}$ is $\mathcal{J}^{\mathcal{I}}$-holomorphic. Note that respect to the normal metric $\|\partial \widehat{f}\|^2 = e^{2u}(1 - H^2)$. Thus $\widehat{f}$ is space-like iif $H^2 < 1$, time-like iif $H^2 > 1$ and null iif $H = \pm 1$.

On the other hand the differential of the projection map $\pi' : \mathcal{Z} \to \mathcal{G}_{\in}(\mathbb{L}^{\triangle})$ restricted to the distribution $\mathcal{H}$ satisfies

$$d\pi' \circ \mathcal{J}^{\mathcal{I}} = \mathcal{J} \circ \lceil \pi',$$

where $\mathcal{J}$ is the $SO(3,1)$-invariant complex structure of $G_2(\mathbb{L}^4)$ defined by (25). In particular the Gauß map $\gamma_f = \pi' \circ \widehat{f} : M \to G_2(\mathbb{L}^4)$ is conformal and holomorphic (hence harmonic) since

$$\mathcal{J} \circ \lceil\gamma_{\{} = \mathcal{J} \circ \lceil\pi' \circ \lceil\widehat{\{} = \lceil\pi' \circ \mathcal{J}^{\mathcal{I}} \circ \lceil\widehat{\{} = \lceil\pi' \circ \lceil\widehat{\{} \circ \mathcal{J}^{\mathcal{M}} = \lceil\gamma_{\{} \circ \mathcal{J}^{\mathcal{M}}.$$

Also using that $\widehat{f}$ is horizontal and satisfies $\|\partial \widehat{f}\|^2 = e^{2u}(1 - H^2)$, it is easy to conclude that $\|\partial \gamma_f\|^2 = e^{2u}(1 - H^2)$. Hence γ_f is space-like iif $H^2 < 1$, time-like iif $H^2 > 1$ and null iif $H = \pm 1$.

5.1. Flat surfaces

The Gauß equation (7) of a conformal totally umbilic immersion $f : M \to \mathbb{S}^3_1$ reduces to the Liouville-type equation

$$2\bar{\partial}\partial u = (H^2 - 1)e^{2u}. \tag{32}$$

Since the Gaußian curvature of the induced metric $g = f^*\langle\,,\,\rangle$ is given by $K = -2e^{-2u}\bar{\partial}\partial u$, the Gauß equation becomes $K = 1 - H^2$. Thus K is constant if and only if H is constant. Non-trivial examples of totally umbilic flat space-like surfaces in $\mathbb{S}^3_1$ with constant mean curvature ± 1 may be constructed from a non-constant harmonic function $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ by projecting the substantial null $\mathcal{J}^{\mathcal{I}}$-holomorphic curve

$$\phi(z, \bar{z}) = \exp(\int^z A + \int^{\bar{z}} B).o,$$

to $\mathbb{S}^3_1$ where A, B are given by

$$A = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & \frac{ie^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & i\partial u & \frac{e^u}{\sqrt{2}}H \\ -\frac{ie^u}{\sqrt{2}} & -i\partial u & 0 & -\frac{ie^u}{\sqrt{2}}H \\ 0 & \frac{e^u}{\sqrt{2}}H & -\frac{ie^u}{\sqrt{2}}H & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{e^u}{\sqrt{2}} & -\frac{ie^u}{\sqrt{2}} & 0 \\ \frac{e^u}{\sqrt{2}} & 0 & -i\bar{\partial} u & \frac{e^u}{\sqrt{2}}H \\ \frac{ie^u}{\sqrt{2}} & i\bar{\partial} u & 0 & \frac{ie^u}{\sqrt{2}}H \\ 0 & \frac{e^u}{\sqrt{2}}H & \frac{ie^u}{\sqrt{2}}H & 0 \end{pmatrix}, \tag{33}$$

with $H^2 = 1$. Then the simplest examples of flat totally umbilic space-like surfaces in $\mathbb{S}^3_1$ occur by taking $u = const.$ The immersion in this case is obtained by projecting the null curve

$$\phi(z, \bar{z}) = \exp(zA + \bar{z}B).o,$$

where the (constant) matrices A, B are given by (33) in which $\partial u = \bar{\partial} u = 0$.

6. Proof of Theorem B

The setup for $\mathcal{J}^{\mathcal{II}}$-holomorphic maps $\phi : M \to \mathcal{Z}$ is analogous to that of $\mathcal{J}^{\mathcal{I}}$-holomorphic maps. The holomorphic subbundle $\mathcal{H}^{(\infty,\prime)} \subset \mathcal{H}^{\mathbb{C}}$ determined by $\mathcal{J}^{\mathcal{II}}$ consists of those $X \in \mathcal{H}^{\mathbb{C}}$ such that $\mathcal{J}^{\mathcal{II}}\mathcal{X} = \rangle\mathcal{X}$, which is fixed throughout this section. From (26) we see that $\mathcal{H}^{(\infty,\prime)}_{\wr}$ consists of complex matrices of the form

$$\begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & ib \\ 0 & b & ib & 0 \end{pmatrix}, \; a, b \in \mathbb{C}$$

Thus an horizontal map $\phi : M \to \mathcal{Z}$ is $\mathcal{J}^{\mathcal{II}}$-holomorphic if and only if for every frame F of ϕ and every local complex coordinate z there exist complex functions $a, b, c : M \to \mathbb{C}$ with such that

$$\alpha(\frac{\partial}{\partial z}) = F^{-1}\partial F = \begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & -c & b \\ ia & c & 0 & ib \\ 0 & b & ib & 0 \end{pmatrix} = \alpha_{\mathfrak{k}}(\frac{\partial}{\partial z}) + \alpha_{\mathfrak{p}}(\frac{\partial}{\partial z}), \tag{34}$$

and consequently

$$(\phi^*\beta)(\frac{\partial}{\partial z}) = Ad(F)[\alpha_{\mathfrak{p}}(\frac{\partial}{\partial z})] = Ad(F) \begin{pmatrix} 0 & a & -ia & 0 \\ -a & 0 & 0 & b \\ ia & 0 & 0 & ib \\ 0 & b & ib & 0 \end{pmatrix}.$$

We assume that the complex functions a and b have isolated zeros i.e. that ϕ is *substantial*. Since $\mathcal{H}^{(\infty,\prime)}$ is isotropic ϕ is conformal and satisfies $\|\partial\phi\|^2 = 2(|a|^2 - |b|^2)$. Using (34) we extract the ∂-derivative of the columns of the frame F:

i′) $\partial F_1 = -a(F_2 - iF_3)$
ii′) $\partial F_2 = aF_1 + cF_3 + bF_4$
iii′) $\partial F_3 = -iaF_1 - cF_2 + ibF_4$
iv′) $\partial F_4 = bF_2 + ibF_3$,

from which we read off the structure equations of $f = F_1$. The first equation implies that f is a (weakly)conformal space-like map since ∂f is isotropic and

$\|\partial f\|^2 = 2|a|^2 > 0$. Like in the proof of Theorem A we can arrange that a be real and negative. Also from i'), ii') and iii') we compute

$$\partial^2 f = (-\partial a)(F_2 - iF_3) - a(\partial F_2 - i\partial F_3) = -\partial a(F_2 - iF_3) - a(cF_3 + 2bF_4 + icF_2),$$

thus $\xi = \langle \partial^2 f, F_4\rangle^{\mathbb{C}} = 2ab$ defines the Hopf differential of f: $Q = \xi^2 dz^2$. On the other hand since f is conformal $\bar\partial\partial f$ must be normal to the immersed surface. Thus

$$\begin{aligned} \partial a &= iac \\ \bar\partial\partial f &= -2a^2 f. \end{aligned}$$

The second equation implies that f has zero mean curvature so that $f : M \to \mathbb{S}^3_1$ results harmonic. Thus f is a (branched) minimal space-like surface in $\mathbb{S}^3_1$ with isolated umbilic points.

On the other hand let $f : M \to \mathbb{S}^3_1$ be a conformal minimal immersion with isolated umbilic points. Define a conformal parameter u by $\langle \partial f, \partial f\rangle = e^{2u}$ and let $\mathbf{n}$ be a tangent vector field on $\mathbb{S}^3_1$ which is normal to $df(TM)$ along f and satisfies $\|\mathbf{n}\|^2 = -1$. Let $F = (f, F_2, F_3, \mathbf{n})$ be an adapted frame of f such that $\partial f = \frac{e^u}{\sqrt{2}}(F_2 - iF_3)$. One may choose $\mathbf{n}$ so that $F \in SO(3,1)$. Note that F is also a frame of $\widehat{f}$. Consider the evolution matrices $A = F^{-1}\partial F$ and $B = F^{-1}\bar\partial F$. Then A and B have the form (10) (11) with $H = 0$. Hence we see in particular that A decomposes into $A = A_0 + A_1$, with $A_0 \in \mathfrak{k}^{\mathbb{C}}$ and $A_1 \in \mathcal{H}_{\wr}^{(\infty,\prime)}$ so that $(\widehat{f}^*\beta)(\frac{\partial}{\partial z}) = Ad(F)(A_1)$, which shows that $\widehat{f}$ is $\mathcal{J}^{\mathcal{II}}$-holomorphic. Note here that even when $\widehat{f}$ is conformal its ∂-derivative is not necessarily definite since $\|\partial\widehat{f}\|^2 = e^{2u} - e^{-2u}|\xi|^2$.

The Gauß map $\gamma_f : M \to G_2(\mathbb{L}^4)$ results harmonic as consequence of Theorem 2.1 in [2], since f viewed as an immersion in $\mathbb{L}^4$ has parallel second fundamental form. However in this case it is clear that γ_f is not holomorphic.

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