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Stanley Filtrations and Strongly Stable Ideals

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Abstract. We give a new short proof of the fact that the Castelnuovo-Mumford regularity of a strongly stable ideal is the highest degree of a minimal monomial generator. Our proof depends on results due to D. Maclagan and G. Smith on multigraded regularity. More precisely, we construct a Stanley filtration for strongly stable ideals which provides a bound for the Castelnuovo-Mumford regularity.

1. Introduction

Throughout the paper, let k be a field, let

$$S = k[x_0, \dots, x_n]$$

be the polynomial ring, and let $I \subset S$ be a monomial ideal. Recall that I is strongly stable if for every monomial $a \in I$ one has $x_i a/x_j \in I$ if a is divisible by x_j and $i < j$. For a graded S-module M, the Castelnuovo-Mumford regularity is defined by

$$\operatorname{reg}(M) := \max\{u - i : Tor_i(M, k)_u \neq 0\}.$$

We give a new proof of the fact that the Castelnuovo-Mumford regularity of a strongly stable ideal is the highest degree of a minimal monomial generator. This result was proved in [6] where a minimal free resolution of a strongly stable ideal is given, see also [3]. Our proof depends on results due to D. Maclagan and G. Smith on multigraded regularity ([8], [9]). More precisely, we construct a Stanley filtration as defined in [9] for strongly stable ideals which provides a bound for the Castelnuovo-Mumford regularity.

Strongly stable ideals are important since generic initial ideals are strongly stable if k has characteristic zero, provided that the variables are ordered in the standard way (see e.g. [7] for an introduction to the theory of Hilbert functions and generic initial ideals).

Some arguments in this article are closely related to observations in [4] and [5], see also [2]. However, for the reader's convenience we give complete proofs.

It should be remarked that there are different definitions of Stanley decomposition in the literature, although the concepts are related. For an algorithmic approach to Stanley decompositions, see e.g. [1] and [10].

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2. Preliminaries

In addition to the notation introduced in the introduction, we use the following notation and definitions.

Let $\prec$ be the pure lexicographic order on S with

$$x_0 \succ \cdots \succ x_n.$$

Let M denote the monomials in S and let M_u be the monomials in S of degree u. For $a \in M$,

$$m_x(a) = \max\{i : x_i|a\}, \ \min_x(a) = \min\{i : x_i|a\}.$$

For a monomial $a \in M_u$, the set

$$L(a) = \{b \in M_u : b \succeq a\}$$

is called a *lexsegment*.

Recall the following notation and definitions from [9]. For a monomial $a \in S$ and $\sigma \subseteq \{0, \ldots, n\}$, the pair (a, σ) denotes the set of all monomials of the form ax^v where $\text{supp}(v) := \{i : v_i \neq 0\} \subseteq \sigma$.

A *Stanley decomposition* for S/I is a set $\mathfrak{G}$ of pairs (a, σ) such that $\mathfrak{G}$ constitutes a partition of the monomials of S not in I.

A *Stanley filtration* is a Stanley decomposition with an ordering of the pairs $\{(a_i, \sigma_i) : 1 \leq i \leq m\}$ such that for all $1 \leq j \leq m$ the set $\{(a_i, \sigma_i) : 1 \leq i \leq j\}$ is a Stanley decomposition for $S/(I + \langle a_{j+1}, \ldots, a_m \rangle)$.

For the sake of brevity, we use the following notation

$$(a, \{\geq l\}) := (a, \{l, l+1, \ldots, n\}).$$

Example 2.1. Let $I = \langle x_0^2, x_0x_1, x_1^3 \rangle \subset k[x_0, x_1, x_2, x_3]$. The set of pairs below, ordered as listed, constitutes a Stanley filtration for S/I.

$(1, \emptyset), (x_3, \emptyset), (x_2, \emptyset), (x_1, \emptyset), (x_0, \emptyset),$
$(x_3^2, \{3\}), (x_2x_3, \{3\}), (x_2^2, \{2, 3\}), (x_1x_3, \{3\}), (x_1x_2, \{2, 3\}),$
$(x_1^2, \{2, 3\}), (x_0x_3, \{3\}), (x_0x_2, \{2, 3\}).$

The Stanley filtration is constructed as follows. Let I^c denote the complement of the ideal in S. For each monomial $a \in I^c$ such that $\deg(a) < 2$, the pair $(a, \emptyset) \in \mathfrak{G}$. For each monomial $a \in I^c$ such that $\deg(a) = 2$, the pair $(a, \{i, i+1, \ldots, 3\}) \in \mathfrak{G}$, where i is the least integer such that $i \geq m_x(a)$ and $ax_i \in I^c$. In particular, $(x_1^2, \{2, 3\}) \in \mathfrak{G}$ since $x_1^3 \in I, x_1^2x_2 \notin I$.

Consider the pairs in $\mathfrak{G}$ consisting of infinitely many monomials. The last pair $(x_0x_2, \{2, 3\})$ consists of the greatest monomials with respect to $\prec$. The pair $(x_0x_3, \{3\})$ consists of the greatest monomials with respect to $\prec$ not in $(x_0x_2, \{2, 3\})$, and so forth.

3. Proof of the main result

Throughout the section, let I be a strongly stable ideal and let u be the highest degree of a minimal monomial generator for I. In order to prove $\text{reg}(I) = u$ we need some preliminary results. The case $u = 1$ being trivial, we assume $u \geq 2$.

Lemma 3.1. *Let $a \in M_v$, where $v \geq u$. Consider the unique decomposition*

$$a = \underline{a}\overline{a}, \quad m_x(\underline{a}) \leq min_x(\overline{a}), \quad \underline{a} \in M_u.$$

Then $a \in I$ if and only if $\underline{a} \in I$.

Proof. Let $a \in I$. It is sufficient to show that $\underline{a} \in I$. Let $a = cd$ where

$$c \in I \cap M_u, \quad d \in M_{v-u}.$$

If $m_x(c) > \min_x(d)$, put

$$c' = cx_{\min_x(d)}/x_{m_x(c)}, \quad d' = dx_{m_x(c)}/x_{\min_x(d)}.$$

Then

$$cd = c'd', \quad c' \in I \cap M_u$$

since I is strongly stable. Moreover,

$$m_x(c') \leq m_x(c), \quad \min_x(d') \geq \min_x(d).$$

If $m_x(c') = m_x(c)$, then the $x_{m_x(c)}$-power is less for c' than for c.

By repeating the procedure, one eventually finds the monomials $\underline{a}$ and $\overline{a}$ as in the lemma, and it is clear that $\underline{a} \in I$. ☐

Lemma 3.2. *Let a, a' be two monomials of the same degree where $a \succ a'$. Then the monomials in the set $(a, \{\geq m_x(a)\})$ are all greater with respect to $\prec$ than the monomials in $(a', \{\geq m_x(a')\})$.*

Proof. Let j be the least index such that the x_j-powers for a and a' differ. Then $m_x(a') > j$, since a and a' have the same degree. Consider the sets

$$(a, \{\geq m_x(a)\}), \quad (a', \{\geq m_x(a')\}).$$

The x_i-powers are the same for the monomials in these sets whenever $i < j$, but the x_j-power for a monomial in the former set is greater than the x_j-power for any monomial in the latter set. The result follows by definition of $\prec$. □

Definition 3.3. *For a monomial $a \in I^c$ of degree at most $u-1$, the set $g(a)$ is defined as follows.*

If $\deg(a) = u-1$ then $g(a) = (a, \{\geq i\})$ where i is the least integer such that $i \geq m_x(a)$ and $ax_i \in I^c$, provided that such an integer exists. If $ax_n \in I$ (or equivalently, $ax_i \in I$ for all i) then $g(a) = (a, \emptyset)$.

If $\deg(a) < u-1$ then $g(a) = (a, \emptyset)$.

Let

$$\mathfrak{G}^{u-1} = \{g(a) : a \in M \cap I^c, \deg(a) \leq u-1\}.$$

Lemma 3.4. *The set $\mathfrak{G}^{u-1}$ is a Stanley decomposition for S/I.*

Proof. We first show that the intersection of any set in $\mathfrak{G}^{u-1}$ and I is empty. Using the same notation as in the definition, if $g(a) = (a, \emptyset)$ clearly $g(a) \cap I = \emptyset$. Otherwise, let $b \neq a \in g(a)$. It is sufficient to show $b \in I^c$. By definition, $b \in (a, \{\geq i\})$ where i is the least integer such that $i \geq m_x(a)$ and $ax_i \in I^c$. The fact that $ax_i \in I^c$ implies that $ax_j \in I^c$ for all $j \geq i$, since I is strongly stable.

Obviously, $b \in (ax_l, \{\geq l\})$ for some $l \geq i \geq m_x(a)$. It follows that

$$b \in (ax_l, \{\geq l\}) = (ax_l, \{\geq m_x(ax_l)\}).$$

Consider the decomposition of b as in Lemma 3.1

$$b = \underline{b}\overline{b}, \quad \underline{b} \in M_u, \quad m_x(\underline{b}) \leq \min_x(\overline{b}).$$

Obviously $\underline{b} = ax_l \in I^c$. Hence $b \in I^c$ by Lemma 3.1.

Next we show that any monomial in the complement of I is contained in some set in $\mathfrak{G}^{u-1}$. Let b be a monomial such that $b \in I^c$ and $\deg(b) \geq u$. Let $b = \underline{b}\overline{b}$ where $\underline{b} \in M_u$ be the decomposition of b as in Lemma 3.1. It follows that

$$b \in g(\underline{b}/x_{m_x(\underline{b})}) \in \mathfrak{G}^{u-1}.$$

If $b \in I^c$ where $\deg(b) < u$, then $g(b) \in \mathfrak{G}^{u-1}$ by construction.

It remains to show that the pairs in $\mathfrak{G}^{u-1}$ are pairwise disjoint. Let $b, c \in M_{u-1} \cap I^c$ where $b \succ c$. It is sufficient to show that $g(b) \cap g(c) = \emptyset$. By definition

$$g(b) \subseteq (b, \{\geq m_x(b)\}), \quad g(c) \subseteq (c, \{\geq m_x(c)\}).$$

Consequently, if $b' \in g(b)$, $c' \in g(c)$ then $b' \succ c'$ by Lemma 3.2, so that $g(b) \cap g(c) = \emptyset$. □

We will use the following easy observations.

Remark 3.5. For $w > u$ the result obtained by replacing u by w in Lemma 3.1 holds as well. For $v \geq u-1$ let $\mathfrak{G}^v$ denote the set obtained by replacing $u-1$ by v in Definition 3.3. For $v \geq u-1$, the set $\mathfrak{G}^v$ is a Stanley decomposition for S/I.

Remark 3.6. A monomial ideal J is strongly stable if for every minimal monomial generator $a \in J$ one has $x_i a/x_j \in J$ if a is divisible by x_j and $i < j$.

Theorem 3.7. *Consider the order where $g(a)$ precedes $g(b)$ if $\deg(a) < \deg(b)$ or if $\deg(a) = \deg(b)$ and $a \prec b$. The Stanley decomposition $\mathfrak{G}^{u-1}$ with this order is a Stanley filtration for S/I.*

Proof. Let $a_1, \dots, a_m$ be the monomials of degree at most $u-1$ in I^c. Put $\mathfrak{G}^{u-1} = \{g(a_i) \mid i = 1, \dots, m\}$, where $g(a_i)$ precedes $g(a_{i+1})$. We need to show that $\{g(a_i) \mid i = 1, \dots, j\}$, is a Stanley decomposition for $S/(I + \langle a_{j+1}, \dots, a_m \rangle)$ for all $1 \leq j \leq m$.

First we show that $I + \langle a_m \rangle$ is strongly stable. By Remark 3.6, it is sufficient to check the condition of strong stability on minimal generators. In this case it is sufficient to consider a_m since I is strongly stable. By construction, $L(a_m) \subseteq I$ and thus $x_r a_m / x_s \in I$ if a_m is divisible by x_s and $r < s$. Hence $I + \langle a_m \rangle$ is strongly stable. By induction, $I + \langle a_{j+1}, \dots, a_m \rangle$ is strongly stable for all $1 \leq j \leq m$.

Obviously the monomials of degree at most $u-1$ in the complement of $I + \langle a_{j+1}, \dots, a_m \rangle$ are $a_1, \dots, a_j$. By Lemma 3.4 and Remark 3.5, there exists a Stanley decomposition

$$\tilde{\mathfrak{G}}^{u-1} = \{\tilde{g}(a_i) \mid i = 1, \dots, j\}$$

of $S/(I + \langle a_{j+1}, \dots, a_m \rangle)$ as described in Lemma 3.4. It is thus sufficient to show that $\tilde{g}(a_1) = g(a_1), \dots, \tilde{g}(a_j) = g(a_j)$.

If $\deg(a_i) < u-1$ for some $1 \leq i \leq j$, then $\tilde{g}(a_i) = g(a_i) = (a_i, \emptyset)$.

If $\deg(a_i) = u-1$ for some $1 \leq i \leq j$, then $a_{i+1}, \dots, a_m \in M_{u-1}$. It is easily seen that $\tilde{g}(a_i) = g(a_i)$ provided that the following condition is satisfied: If $a_i x_l \in I + \langle a_{j+1}, \dots, a_m \rangle$ for some $l \geq m_x(a_i)$, then $a_i x_l \in I$. Let $a_i x_l \in I + \langle a_{j+1}, \dots, a_m \rangle$ for some $l \geq m_x(a_i)$. Since $a_i x_l \in (a_i, \{\geq m_x(a_i)\})$ and $a_i \prec a_{j+1}$ it follows from Lemma 3.2 that $a_i x_l \prec a_{j+1} \prec \cdots \prec a_m$. Hence $a_i x_l \notin \langle a_{j+1}, \dots, a_m \rangle$, so that $a_i x_l \in I$. Consequently $\tilde{g}(a_i) = g(a_i)$ for all $1 \leq i \leq j$, which completes the proof. □

We can now give a new proof of the fact that $\operatorname{reg}(I) = u$.

Corollary 3.8. (Eliahou-Kervaire) *The Castelnuovo-Mumford regularity of a strongly stable ideal I is the highest degree of a minimal monomial generator.*

Proof. The Stanley decomposition $\mathfrak{G}^{u-1}$ in Theorem 3.7 consists of pairs (a, σ_a) such that $\deg(a) \leq u-1$. It follows from [9, Theorem 4.1] that $\operatorname{reg}(I) \leq u$. Since I has a minimal monomial generator of degree u, one concludes that $\operatorname{reg}(I) = u$. □

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