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A Geometric Embedding for Standard Analytic Modules

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Abstract. In this manuscript we make a general study of the representations realized, for a reductive Lie group of Harish-Chandra class, on the compactly supported sheaf cohomology groups of an irreducible finite-rank polarized homogeneous vector bundle defined in a generalized complex flag space. In particular, we show that the representations obtained are minimal globalizations of Harish-Chandra modules and that there exists a whole number q, depending only on the orbit, such that all cohomologies vanish in degree less than q. The representation realized on the q-th cohomology group is called a standard analytic module. Our main result is a geometric proof that a standard analytic module embeds naturally in an associated standard module defined on the full flag space of Borel subalgebras. As an application, we give geometric realizations for irreducible submodules of some principal series representations in case the group is complex.

1. Introduction

Even though the irreducible admissible representations of a real reductive Lie group have been classified for about 30 years, one does not yet have an explicit realization of these representations. Historically speaking, two of the first constructions of admissible representations were the parabolically induced modules and Schmid's realization of the discrete series [12], generalizing the Borel-Weil-Bott Theorem [3]. Developed in the 1980s, the localization technique of Beilinson and

Bernstein [1] gave a general prescription for realizing irreducible Harish-Chandra modules. However, when the irreducible representation is not obtained as a standard module, it can be quite difficult to understand the corresponding geometric realization. Since a specific criteria for the irreducibility of some standard modules is known [10], the Beilinson-Bernstein result does obtain a concrete realization for many irreducible representations, although the object produced is a Harish-Chandra module and not a group representation. Using the methods of analytic localization, Hecht and Taylor gave an analytic version of the Beilinson-Bernstein theory [6], including a concrete construction of standard modules that yields full-fledged group representations globalizing the standard Beilinson-Bernstein modules. The standard analytic modules defined by Hecht and Taylor are realized on the compactly supported sheaf cohomology groups of polarized homogeneous vector bundles. One knows that their construction is sufficiently general, for example, to provide geometric realizations for all the tempered representations.

The Hecht-Taylor construction is directly related to the geometry of a full flag space. In order to realize more irreducible representations, in this manuscript we consider the natural generalization of their construction to the setting of an arbitrary flag space. Examples show that, because of certain geometric conditions on the orbits, the underlying Harish-Chandra module of this natural generalization is not universally described by the analog of a standard Beilinson-Bernstein module. In fact, generally speaking, one knows little about the underlying Harish-Chandra module. Alternatively, what we propose to establish is an embedding theorem, showing in general that the standard analytic modules appear naturally as submodules of certain standard modules defined in a full flag space. In particular, the standard analytic modules defined in an arbitrary flag space appear as refinements to certain standard modules occurring in the Beilinson-Bernstein classification scheme. Our main results along these lines are Theorems 4.1 and 4.3, appearing in Section 4. In particular, the embedding in Theorem 4.1 applies to any finite-rank, irreducible polarized homogeneous vector bundle defined in a complex flag space. Theorem 4.3 adds a nonvanishing result when an additional hypothesis is assumed. In the last section, as an example, we will apply the embedding to realize irreducible submodules of certain principal series in the case of a complex reductive group.

The paper is organized as follows. The first section is the introduction. In the second section we introduce the complex flag spaces and the polarized homogeneous vector bundles. A main result is the construction of the associated bundle. In the third section we deal with some topological considerations, introduce the minimal globalization [11] and show that representations obtained on the compactly supported sheaf cohomology groups of a polarized homogeneous vector bundle are naturally isomorphic to minimal globalizations of Harish-Chandra modules. We also establish a vanishing result. In Section 4 we prove Theorems 4.1 and 4.3 as mentioned previously. In Section 5 we apply our result to realize irreducible subrepresentations of some principal series in the case of a complex reductive group.

2. Polarized homogeneous vector bundle

We let G_0 denote a reductive Lie group of Harish-Chandra class with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. G denotes the complex adjoint group of $\mathfrak{g}$. This notation is a bit misleading, since the adjoint representation defines an isomorphism between the Lie algebra of G and $[\mathfrak{g}, \mathfrak{g}]$.

Flag spaces for G_0. We define a *flag space for* G_0 to be a complex projective, homogeneous G-space. The flag spaces can be constructed as follows. By definition, a *Borel subalgebra of* $\mathfrak{g}$ is a maximal solvable subalgebra. One knows that G acts transitively on the set of Borel subalgebras and that the resulting homogeneous space is a complex projective variety called the *full flag space for* G_0. A complex subalgebra that contains a Borel subalgebra is called a *parabolic subalgebra of* $\mathfrak{g}$. The space of G-conjugates to a given parabolic subalgebra is a complex projective variety, and each flag space is realized in this way.

From the construction it follows that points in the flag spaces are identified with parabolic subalgebras of $\mathfrak{g}$. Let X denote the full flag space and suppose Y is a flag space. Given $x \in X$ and $y \in Y$ we let $\mathfrak{b}_x$ and $\mathfrak{p}_y$ denote, respectively, the corresponding Borel and parabolic subalgebras. For each $x \in X$ it is known there is a unique $y \in Y$ such that $\mathfrak{b}_x \subseteq \mathfrak{p}_y$. Thus there is a unique G-equivariant projection

$$\pi : X \to Y$$

called the *natural projection.*

We note that the quantity of G_0-orbits in Y is finite [13]. In particular, G_0-orbits are locally closed in Y and define regular analytic submanifolds.

The Cartan dual and infinitesimal characters. Suppose x is a point in the full flag space X and let $\mathfrak{n}_x$ denote the nilradical of the corresponding Borel subalgebra $\mathfrak{b}_x$. Put

$$\mathfrak{h}_x = \mathfrak{b}_x / \mathfrak{n}_x$$

and let $\mathfrak{h}_x^*$ denote the complex dual of $\mathfrak{h}_x$. Since the stabilizer of x in G acts trivially on $\mathfrak{h}_x^*$ the corresponding G-homogeneous holomorphic vector bundle on X is trivial. Thus the space $\mathfrak{h}^*$ of global sections is isomorphic $\mathfrak{h}_x^*$ via the the evaluation at x. We refer to $\mathfrak{h}^*$ as the *Cartan dual for* $\mathfrak{g}$. If $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_x$ then projection of $\mathfrak{c}$ onto $\mathfrak{h}_x$ coupled with evaluation at x defines an isomorphism of $\mathfrak{h}^*$ onto $\mathfrak{c}^*$ called the *specialization of* $\mathfrak{h}^*$ *onto* $\mathfrak{c}^*$ *at* x. Using the specializations we can identify a *root subset*

$$\Sigma \subseteq \mathfrak{h}^*$$

for $\mathfrak{g}$ in $\mathfrak{h}^*$ and a *set of positive roots* $\Sigma^+ \subseteq \Sigma$ where a root is called *positive* if it is identified with a root of $\mathfrak{c}$ in $\mathfrak{b}_x$. The given subsets are independent of the specialization used to define them. For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{h}^*$ we can also define the complex values $\overset{\vee}{\alpha}(\lambda)$ and a corresponding *Weyl group* W *of* $\mathfrak{g}$ *in* $\mathfrak{h}^*$. $\lambda \in \mathfrak{h}^*$ is called *regular* if the stabilizer of λ in W is trivial. We say that λ is *antidominant* provided

$$\overset{\vee}{\alpha}(\lambda) \notin \{1, 2, 3, \ldots\} \quad \text{for each } \alpha \in \Sigma^+.$$

Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. Then a $\mathfrak{g}$-*infinitesimal character* is a homomorphism of algebras

$$\Theta : Z(\mathfrak{g}) \to \mathbb{C}.$$

A well known result of Harish-Chandra identifies the set of $\mathfrak{g}$-infinitesimal characters with the quotient

$$\mathfrak{h}^*/W\ .$$

Suppose Θ is an infinitesimal character and $\lambda \in \mathfrak{h}^*$. Then we write $\Theta = W \cdot \lambda$ when Harish-Chandra's correspondence identifies Θ with the Weyl group orbit of λ in $\mathfrak{h}^*$.

Let Y be a flag space and suppose $y \in Y$. We let $\mathfrak{p}_y$ denote the corresponding parabolic subalgebra and let $\mathfrak{u}_y$ denote the nilradical of $\mathfrak{p}_y$. The corresponding *Levi quotient* is defined by

$$\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y\ .$$

Thus $\mathfrak{l}_y$ is a complex reductive Lie algebra. In fact, the natural projection identifies the Borel subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}_y$ with the Borel subalgebras of $\mathfrak{l}_y$. In this way, we can identify $\mathfrak{h}^*$ with the Cartan dual for $\mathfrak{l}_y$. Therefore we have a corresponding *set of roots*

$$\Sigma_Y \subseteq \Sigma$$

for $\mathfrak{h}^*$ in $\mathfrak{l}_y$ and a *Weyl group for the Levi quotient* $W_Y \subseteq W$ generated by reflections from the roots in Σ_Y. As the notation suggests, these subsets are independent of the point y. The *positive roots* Σ_Y^+ of $\mathfrak{h}^*$ in $\mathfrak{l}_y$ are defined by

$$\Sigma_Y^+ = \Sigma_Y \cap \Sigma^+.$$

An element $\lambda \in \mathfrak{h}^*$ is called *antidominant for* Y provided there exists an element $w \in W_Y$ such that $w \cdot \lambda$ is antidominant. An equivalent condition is that $\overset{\vee}{\alpha}(\lambda)$ not be a positive integer for each $\alpha \in \Sigma^+ - \Sigma_Y^+$. Finally we note that the Harish-Chandra parametrization identifies set of $\mathfrak{l}_y$-infinitesimal characters with the set of Weyl group orbits

$$\mathfrak{h}^*/W_Y\ .$$

Polarized modules for the stabilizer. Suppose Y is a flag space. Fix $y \in Y$. We have the parabolic subalgebra $\mathfrak{p}_y$ with nilradical $\mathfrak{u}_y$ and Levi quotient $\mathfrak{l}_y$. Let $G_0[y]$ denote the stabilizer of y in G_0. Suppose

$$\omega : G_0[y] \to GL(V)$$

is a representation in a finite-dimensional complex vector space V. A compatible representation of $\mathfrak{p}_y$ in V is called *a polarization* if the nilradical $\mathfrak{u}_y$ acts trivially. We note that a polarization need not exist, even if the $G_0[y]$-module V

is irreducible. However a polarization is unique when it does exist, because the compatibility condition assures that two possibly distinct $\mathfrak{l}_y$-actions agree on a parabolic subalgebra of $\mathfrak{l}_y$ and therefore are identical.

Since a $G_0[y]$-invariant subspace of a polarized module need not be invariant under the corresponding $\mathfrak{l}_y$-action, we define *a morphism of polarized modules* to be a linear map that intertwines both the $G_0[y]$ and $\mathfrak{l}_y$-action. Thus the category of polarized $G_0[y]$-modules is nothing but the category of finite-dimensional $(\mathfrak{l}_y, G_0[y])$-modules.

We briefly consider the structure of an irreducible polarized module V. Since G_0 is Harish-Chandra class it follows that V has an $\mathfrak{l}_y$-infinitesimal character. Since there is only one finite-dimensional irreducible $\mathfrak{l}_y$-module with a given infinitesimal character, it follows that V is a direct sum of several copies of an irreducible highest weight module. In fact, if one makes a linear assumption on the group G_0, then an irreducible polarized module is nothing but an irreducible finite-dimensional $\mathfrak{l}_y$-module with compatible $G_0[y]$-action.

Polarized homogeneous vector bundle. By definition, a polarized homogeneous vector bundle is a homogeneous analytic vector bundle that comes equipped with a canonical way to define the notion of a restricted holomorphic section. By definition these restricted holomorphic sections form the corresponding sheaf of polarized sections. For example, in the case of a trivial line bundle, the polarized sections are exactly the restricted holomorphic functions.

Let S be a G_0-orbit and fix $y \in S$. Suppose

$$\omega : G_0[y] \to GL(V)$$

is a polarized module for the stabilizer $G_0[y]$. For $\xi \in \mathfrak{p}_y$ and $v \in V$, we use the notation

$$\omega(\xi)v$$

to indicate the compatible $\mathfrak{p}_y$-action. Let

$$\begin{array}{c}\mathbb{V}\\ \downarrow\\ S\end{array}$$

be the G_0-homogeneous vector bundle with fiber V and let

$$\phi : G_0 \to S \quad \text{be the projection } \phi(g) = g \cdot y.$$

Then we can identify a section of $\mathbb{V}$ over an open set $U \subseteq S$ with a real analytic function

$$f : \phi^{-1}(U) \to V \ \text{ such that } \ f(gp) = \omega(p^{-1})f(g) \quad \forall p \in G_0[y].$$

The section is said to be *polarized* if

$$\frac{d}{dt}\,|_{t=0}\ f(g\exp(t\xi_1)) + i\frac{d}{dt}\,|_{t=0}\ f(g\exp(t\xi_2)) = -\omega(\xi_1 + i\xi_2)f(g)$$

for all ξ_1, $\xi_2 \in \mathfrak{g}_0$ such that $\xi_1 + i\xi_2 \in \mathfrak{p}_y$.

Let $\mathcal{P}(y, V)$ denote the sheaf of polarized sections and let $\mathcal{O}_Y \mid_S$ be the sheaf of restricted holomorphic functions on S. As a sheaf of $\mathcal{O}_Y \mid_S$-modules, $\mathcal{P}(y, V)$ is locally isomorphic to $\mathcal{O}_Y \mid_S \otimes V$ [6]. The left translation defines a G_0, and thus a $\mathfrak{g}$-action on $\mathcal{P}(y, V)$. We will be interested in the representations obtained on the compactly supported sheaf cohomologies

$$H^p_{\mathrm{c}}(S, \mathcal{P}(y, V)) \quad p = 0, 1, 2, \ldots.$$

One knows that the sheaf cohomologies are dual nuclear Fréchet (DNF) $\mathfrak{g}$-modules with a compatible G_0-action, provided certain naturally defined topologies are Hausdorff [6].

The associated bundle. Let S be a G_0-orbit and suppose

$$\begin{array}{c} \mathbb{V} \\ \downarrow \\ S \end{array}$$

is the homogeneous polarized vector bundle corresponding to an irreducible polarized module for the stabilizer. We let $\mathcal{P}(\mathbb{V})$ denote the corresponding sheaf of polarized sections. Let X denote the full flag space and let

$$\pi : X \to Y$$

be the natural projection. Put

$$N = \pi^{-1}(S)$$

the *associated manifold*. In order to study the compactly supported cohomologies of $\mathcal{P}(\mathbb{V})$, we now define a certain G_0-equivariant bundle over N that comes naturally equipped with a corresponding polarization.

For $y \in S$ we let $\mathbb{V}_y$ denote the fiber of $\mathbb{V}$ over y. Thus $\mathbb{V}_y$ is an irreducible polarized module for the stabilizer $G_0[y]$. Put

$$X_y = \pi^{-1}(\{y\})$$

and let $x \in X_y$. Define

$$\mathbb{W}_x = \frac{\mathbb{V}_y}{\mathfrak{n}_x \mathbb{V}_y}$$

where $\mathfrak{n}_x$ is the nilradical of the Borel subalgebra $\mathfrak{b}_x$. Thus $\mathbb{W}_x$ is an irreducible polarized module for $G_0[x]$. Let $\mathbb{W}$ be the bundle over N given by

$$\mathbb{W} = \bigcup_{x \in N} \mathbb{W}_x.$$

To provide the corresponding trivializations, we define a holomorphic extension of $\mathbb{W}$ over a neighborhood of x in X. Since our result is local, we may assume that G_0 is a real form of G. Let B_x denote the stabilizer of x in G. Then the representation of $\mathfrak{b}_x$ in $\mathbb{W}_x$ determines a local holomorphic representation of B_x in $\mathbb{W}_x$. The standard construction of a G-homogeneous holomorphic vector bundle limited to an appropriate neighborhood of the identity in G then provides the desired holomorphic extension of $\mathbb{W}$ over a neighborhood of x in X.

We now consider the polarized sections. For $\xi \in \mathfrak{g}_0$ and σ an analytic section of $\mathbb{W}$ defined on an open set $U \subseteq N$, put

$$(\xi * \sigma)(x) = \frac{d}{dt} \mid_{t=0} \exp(t\xi)\sigma(\exp(-t\xi)x), \quad x \in U$$

where we use juxtaposition to indicate the action of G_0 on $\mathbb{W}$. This definition determines a unique complex linear action of $\mathfrak{g}$ on the sheaf of analytic sections of $\mathbb{W}$. The section σ is said to be *polarized* if

$$(\xi * \sigma)(x) = \omega_x(\xi)\sigma(x) \;\text{ for each } x \in U \text{ and each } \xi \in \mathfrak{b}_x$$

where ω_x indicates the $\mathfrak{b}_x$-action on the fiber. In case

$$f : U \to \mathbb{C}$$

is a real analytic function, observe that

$$(\xi * f)(x) = \frac{d}{dt} \mid_{t=0} f(\exp(-t\xi)x), \quad x \in U \quad \text{and} \quad \xi \in \mathfrak{g}_0.$$

Let $\mathcal{P}(\mathbb{W})$ denote the sheaf of polarized sections and let $\mathcal{O}_X \mid_N$ be the sheaf of restricted holomorphic functions on N. The following lemma will be used to show that $\mathcal{P}(\mathbb{W})$ is a locally free sheaf of $\mathcal{O}_X \mid_N$-modules.

Lemma 2.1. *Suppose U is open in N and let*

$$f : U \to \mathbb{C}$$

be an analytic function. Then f is a restricted holomorphic function if and only if

$$(\xi * f)(x) = 0 \quad \textit{for each } x \in U \quad \textit{and} \quad \xi \in \mathfrak{b}_x.$$

Proof. If f is a restricted holomorphic function it is clear that f satisfies the equation in Lemma 2.1. On the other hand, suppose that f satisfies the given equation. We may assume that G_0 is a real form of G. To simplify notation, we assume that f is defined on all of N. Let Q be a G_0-orbit in N. We first show that there is a holomorphic function defined on a neighborhood of Q in X that restricts to f on Q. Fix $x \in Q$ and consider the function $g \mapsto f(gx)$ defined on G_0. Observe that this function extends to a holomorphic function φ defined on an open set of G. For each $g \in G_0$ and each $\xi \in \mathfrak{b}_{gx}$ we have

$$(\xi * \varphi)(gx) = \frac{d}{dt} \mid_{t=0} \varphi(g \exp(-t\mathrm{Ad}(g^{-1})\xi)x) = 0.$$

Thus, for g in an open set of G containing G_0, we have

$$\varphi(gb) = \varphi(g) \text{ for each } b \in B_x$$

so that φ defines a holomorphic extension of f to a neighborhood of Q in X.

It follows that there exists a holomorphic function on an open set in X which restricts to f on a dense open set of N (the union of all the open G_0-orbits). Choose $y \in \pi(N)$ and put

$$X_y = \pi^{-1}(\{y\}).$$

Define

$$\psi : G_0 \times X_y \to \mathbb{C} \text{ by } \psi(g, z) = f(gz).$$

We claim that, for each fixed g, the function $z \mapsto \varphi(g, z)$ is holomorphic. By the above considerations, the given function is holomorphic on a dense open subset of X_y. But therefore the claim follows, since the given function is annihilated by the $\overline{\partial}$-operator the complex manifold X_y.

Thus ψ extends to a holomorphic function defined on a neighborhood of $G_0 \times X_y$ in $G \times X_y$. Using the same letter ψ to denote the holomorphic extension, observe that for each $\xi \in \mathfrak{b}_x$ and each $(g, z) \in G_0 \times X_y$ we have

$$\frac{d}{dt} |_{t=0} \psi(g \exp(t\xi), z) = \frac{d}{dt} |_{t=0} \psi(g, \exp(t\xi)z).$$

Thus, for each $b \in B_x$, we have

$$\psi(gb, z) = \psi(g, bz).$$

It follows that ψ defines a holomorphic extension of f to a neighborhood of N in X. □

From the previous lemma and the chain rule, it follows that if σ is a local polarized section and f is a restricted polarized section then $f\sigma$ is a polarized section. We finish this section with the following proposition.

Proposition 2.2. *Fix $x \in X$. As a sheaf of $\mathcal{O}_X |_N$-modules $\mathcal{P}(\mathbb{W})$ is locally isomorphic to $\mathcal{O}_X |_N \otimes \mathbb{W}_x$.*

Proof. Since the result is local, and using the fact $\mathbb{W}$ locally extends to a holomorphic vector bundle, we may assume that $\mathbb{W}$ extends to a G-homogeneous vector bundle on all of X. Let

$$\Phi : G \to X \text{ be the projection } \Phi(g) = gx$$

and suppose

$$\eta : U \to G$$

is a corresponding holomorphic section defined on a neighborhood U of x in X. We need to check that an analytic section σ of $\mathbb{W}$ is polarized if and only if the function

$$f(z) = \eta(z)^{-1}\sigma(z)$$

is holomorphic. Using the differential of the action map

$$G \times \mathbb{W}_z \to \mathbb{W}$$

we define a multiplication by complex scalars in the tangent space to $\mathbb{W}$ at point $v \in \mathbb{V}_z$. Let $\xi_1, \xi_2 \in \mathfrak{g}_0$ such that $\xi = \xi_1 + i\xi_2 \in \mathfrak{b}_z$ and put $v = \sigma(z)$. Since

$$\frac{d}{dt} \mid_{t=0} \exp(t\xi_1)v + i\frac{d}{dt} \mid_{t=0} \exp(t\xi_2)v = \omega(\xi)v$$

it follows from the product rule that σ satisfies the polarized condition at z if and only if

$$\frac{d}{dt} \mid_{t=0} \sigma(\exp(-t\xi_1)z) + i\frac{d}{dt} \mid_{t=0} \sigma(\exp(-t\xi_2)z) = 0.$$

Since

$$\frac{d}{dt} \mid_{t=0} \eta(\exp(-t\xi_1)z)^{-1}v + i\frac{d}{dt} \mid_{t=0} \eta(\exp(-t\xi_2)z)^{-1}v = 0$$

the desired result follows from the product rule as applied to the function f. $\square$

$\mathbb{W}$ is called the *associated bundle on* N.

3. Topological considerations and vanishing

In order to show that the compactly supported sheaf cohomologies of a polarized vector bundle are minimal globalizations, we will apply some of the results by Hecht and Taylor for certain topological sheaves of $\mathfrak{g}$-modules, with compatible G_0-action [6]. We begin this section by considering some of those results. The following notations will be adopted. If $S \subseteq Y$ is a locally closed subspace and $\mathcal{F}$ is a sheaf on Y then $\mathcal{F} \mid_S$ denotes the restriction of $\mathcal{F}$ to S. If $\mathcal{F}$ is a sheaf on S then $\mathcal{F}^Y$ denotes the extension by zero of $\mathcal{F}$ to Y.

DNF sheaves and analytic modules. By definition, a *DNF space* is a complete, locally convex topological vector space whose continuous dual, when equipped with the strong topology is a nuclear Fréchet space. In the obvious fashion, one can define the notion of a DNF algebra and a corresponding category of DNF modules.

We will be interested in sheaves with DNF structure. A *DNF sheaf of algebras* is a sheaf of algebras such that the space of germs of sections over each compact subset is a DNF algebra and such that the restriction map for any nested pair of compact subsets is continuous. When $\mathcal{A}$ is a DNF sheaf of algebras, then one can define, in the obvious fashion a corresponding concept of *DNF sheaf of* $\mathcal{A}$*-modules*. A corresponding *morphism* is a morphism of sheaves of $\mathcal{A}$-modules that is continuous for all the given topologies. One can show that the continuity condition is completely determined by continuity on the stalks. By the open mapping theorem, a continuous sheaf isomorphism has a continuous inverse. We note that when $\mathcal{F}$ is a DNF sheaf of $\mathcal{A}$-modules on Y and $S \subseteq Y$ is a locally closed subset then $(\mathcal{F} \mid_S)^Y$ is a DNF sheaf of $\mathcal{A}$-modules supported in S.

The sheaf $\mathcal{O}$ of holomorphic functions is an important example of DNF sheaf of algebras. Hecht and Taylor show that when $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules whose geometric fibers have countable dimension and whose stalks are free modules over the stalks of $\mathcal{O}$ then $\mathcal{F}$ carries a unique DNF structure consistent with the natural DNF structure that exists on stalks. With this structure $\mathcal{F}$ becomes a DNF sheaf of $\mathcal{O}$-modules.

Suppose $\mathcal{A}$ is a DNF sheaf of algebras on a flag space Y. Since Y is compact the space of global sections

$$A = \Gamma(Y, \mathcal{A})$$

is a DNF algebra. Let $\mathbb{M}_{\mathrm{DNF}}(\mathcal{A})$ denote the category of DNF sheaves of $\mathcal{A}$-modules and let $\mathbb{M}_{\mathrm{DNF}}(A)$ denote the category of DNF A-modules. Thus the global sections define a functor

$$\Gamma : \mathbb{M}_{\mathrm{DNF}}(\mathcal{A}) \to \mathbb{M}_{\mathrm{DNF}}(A).$$

Hecht and Taylor, using a modified Čech resolution, show that the category $\mathbb{M}_{\mathrm{DNF}}(\mathcal{A})$ has enough Γ-acyclic objects. For $\mathcal{M} \in \mathbb{M}_{\mathrm{DNF}}(\mathcal{A})$ they prove that the sheaf cohomology groups

$$H^p(Y, \mathcal{M}) \quad p = 1, 2, \ldots$$

will be also DNF A-modules, provided the associated topologies are Hausdorff. Indeed, their construction is functorial and their work shows that the topological properties for the sheaf cohomologies are independent of the choice of Γ-acyclic resolution, as long as these resolutions originate from the given category.

We will be interested in the following DNF sheaves of algebras on Y. To each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein have shown how to define a twisted sheaf of differential operators (TDO) on the full flag space X [1]. We let $\mathcal{D}_\lambda$ denote the corresponding TDO with holomorphic coefficients. With our parametrization, when ρ indicates one-half the sum of the positive roots in $\mathfrak{h}^*$ then $\mathcal{D}_{-\rho}$ is the sheaf of holomorphic differential operators on X. One knows that $\mathcal{D}_\lambda$ is a DNF sheaf of algebras on X. Let

$$\pi : X \to Y$$

denote the natural projection and let $\pi_*(\mathcal{D}_\lambda)$ denote the corresponding direct image in the category of sheaves. Since X and Y are compact, $\pi_*(\mathcal{D}_\lambda)$ is a DNF sheaf of algebras on Y. If W_Y denotes the Weyl group for the Levi quotient then one knows that $\pi_*(\mathcal{D}_{w\lambda}) \cong \pi_*(\mathcal{D}_\lambda)$ for each $w \in W_Y$.

To characterize the sheaf cohomologies of $\pi_*(\mathcal{D}_\lambda)$ we introduce the $\mathfrak{g}$-infinitesimal character $\Theta = W \cdot \lambda$. Let U_Θ be the quotient of $U(\mathfrak{g})$ by the ideal generated from the kernel of Θ. Using the result of Beilinson and Bernstein, Hecht and Taylor show that

$$\Gamma(Y, \pi_*(\mathcal{D}_\lambda)) \cong U_\Theta \quad \text{and} \quad H^p(Y, \pi_*(\mathcal{D}_\lambda)) = 0 \quad \text{for } p > 0.$$

Let $y \in Y$ and suppose V is an irreducible polarized module for the stabilizer $G_0[y]$. Let $\mathcal{P}(y, V)$ denote the corresponding sheaf of polarized sections defined on the G_0-orbit S of y. Let $\mathfrak{l}_y$ denote the Levi quotient and let $\lambda \in \mathfrak{h}^*$ be a parameter for the $\mathfrak{l}_y$-infinitesimal character on V. Then $\mathcal{P}(y,V)^Y$ is a DNF sheaf of $\pi_*(\mathcal{D}_\lambda)$-modules. Thus the compactly supported sheaf cohomologies

$$H^p_c(S,\mathcal{P}(y,V)) \cong H^p(Y,\mathcal{P}(y,V)^Y)$$

will be DNF U_Θ-modules, provided the corresponding topologies are Hausdorff.

In order to characterize the G_0-action on $H^p_c(S,\mathcal{P}(y,V))$, Hecht and Taylor develop a formalism for DNF sheaves of $\pi_*(\mathcal{D}_\lambda)$-modules with compatible analytic G_0-action. The corresponding objects obtained are referred to as *analytic sheaves of* $(\pi_*(\mathcal{D}_\lambda),G_0)$*-modules.* The sheaf $\mathcal{P}(y,V)^Y$ provides the fundamental example of these sorts of objects. By definition a *morphism of analytic sheaves of* $(\pi_*(\mathcal{D}_\lambda),G_0)$*-modules* is a morphism of DNF $\pi_*(\mathcal{D}_\lambda)$-modules which is equivariant for the G_0-action.

Suppose M is a DNF U_Θ-module equipped with a continuous, linear G_0-action ω. M is called an *analytic* (G_0,U_Θ)*-module* if:

(1) for each $v\in M$ the function

$$g\mapsto\omega(g)v$$

is an analytic function from G_0 to M,

(2) the derivative of the G_0-action agrees with the $\mathfrak{g}$-action.

A morphism of analytic (G_0,U_Θ)*-modules* is a continuous linear map which is equivariant with respect to the G_0-actions.

It is clear from definitions that the global sections functor determines a functor from the category of analytic sheaves of $(\pi_*(\mathcal{D}_\lambda),G_0)$-modules to the category of analytic (G_0,U_Θ)-modules. What Hecht and Taylor show with their formalism is that, with respect to the construction of sheaf cohomology, the analytic group action is functorial, provided that the sheaf cohomology has a Hausdorff topology.

Minimal globalization. Let $K_0\subseteq G_0$ be a maximal compact subgroup and suppose M is a Harish-Chandra module for $(K_0,\mathfrak{g})$. By definition a globalization M_{glob} of M is an admissible representation of G_0 in a complete locally convex space whose underlying subspace of K_0-finite vectors is isomorphic to M as a $(K_0,\mathfrak{g})$-module. By now one knows that there exist several canonical and functorial globalizations. In this article we shall be interested in realizing the remarkable minimal globalization, $M_{\min}$ whose existence was first proved by W. Schmid in [11]. When M_{glob} is a globalization of M then the inclusion $M\to M_{\text{glob}}$ lifts to a continuous G_0-equivariant linear inclusion

$$M_{\min}\to M_{\text{glob}}.$$

This lifting defines an isomorphism of $M_{\min}$ onto the analytic vectors in a Banach space globalization [8]. In particular, $M_{\min}$ is an analytic module. One also knows that the functor of minimal globalization is exact [8].

Vanishing number. It turns out there is a specific non-negative integer q associated to a G_0-orbit S, such that the compactly supported sheaf cohomologies of any polarized homogeneous vector bundle will vanish in degrees less than q, but such that the q-th cohomology will not vanish for some polarized homogeneous vector bundles on S. This value q will be called *the vanishing number* of S.

In order to define this number, we introduce some definitions and facts related to the Matsuki duality [9]. Fix a maximal compact subgroup $K_0 \subseteq G_0$ and let K be the complexification of K_0. The group K acts algebraically on Y and the choice of K_0 determines an involutive automorphism

$$\theta : \mathfrak{g} \to \mathfrak{g}$$

called the *Cartan involution* of $\mathfrak{g}$. A point $y \in Y$ is called *special* if $\mathfrak{p}_y$ contains a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such $\theta(\mathfrak{c}) = \mathfrak{c}$ and such that $\mathfrak{g}_0 \cap \mathfrak{c}$ is a real form for $\mathfrak{c}$. A K-orbit is said to be *Matsuki dual* to a G_0-orbit if their intersection contains a special point. Matsuki has shown that each G_0-orbit (each K-orbit) contains a special point and that K_0 acts transitively on the special points in a given G_0-orbit (in a given K-orbit). Thus Matsuki duality establishes a one to one correspondence between the G_0-orbits and the K-orbits in Y. In particular each G_0-orbit has a unique Matsuki dual.

Let S be a G_0-orbit and let Q be its Matsuki dual. The *vanishing number of* S is defined to be the codimension q of Q in Y.

The compactly supported cohomologies of a polarized homogeneous vector bundle. We begin this subsection with the following lemma.

Lemma 3.1. *Suppose*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of analytic sheaves of $(\pi_*(\mathcal{D}_\lambda), G_0)$-modules on Y. Assume that for each p, the sheaf cohomology groups $H^p(Y, \mathcal{F})$ and $H^p(Y, \mathcal{H})$ are admissible analytic G_0-modules, isomorphic to the minimal globalizations of their underlying Harish-Chandra modules. Then the same is true of $H^p(Y, \mathcal{G})$.

Proof. Consider the sequence

$$H^{p-1}(Y, \mathcal{H}) \to H^p(Y, \mathcal{F}) \to H^p(Y, \mathcal{G}).$$

Since a morphism of minimal globalizations has closed range, it follows that the kernel of the second morphism is closed in $H^p(Y, \mathcal{F})$. Therefore, by applying [6, Corollary A.11] to the sequence

$$H^p(Y, \mathcal{F}) \to H^p(Y, \mathcal{G}) \to H^p(Y, \mathcal{H})$$

one deduces that $H^p(Y, \mathcal{G})$ is Hausdorff. Thus $H^p(Y, \mathcal{G})$ is an analytic G_0-module. Hence the desired result follows from [6, Lemma 10.11]. $\square$

With respect to the polarized homogeneous vector bundles in the full flag space, one has the following result.

Proposition 3.2. *Suppose X is the full flag space for G_0 and $S \subseteq X$ is a G_0-orbit. Choose a point $x \in S$ and suppose V is a polarized module for $G_0[x]$. Let $\mathcal{P}(x, V)$ denote the corresponding sheaf of polarized sections. Then we have the following:*

(a) *For each $p = 0, 1, 2, \ldots$, the compactly supported sheaf cohomology group*

$$H_c^p(S, \mathcal{P}(x, V))$$

is an analytic G_0-module, naturally isomorphic to the minimal globalization of its underlying Harish-Chandra module.

(b) *Let q denote the vanishing number of S. Then*

$$H_c^p(S, \mathcal{P}(x, V)) = 0 \quad \text{for } p < q.$$

Proof. When V is a polarized module with infinitesimal character, then this result is proved in [4]. Since the correspondence

$$W \longmapsto \mathcal{P}(x, W)^X$$

defines an exact functor from the category of polarized $G_0[x]$-modules to the category of analytic sheaves of $(\pi_*(\mathcal{D}_\lambda), G_0)$-modules, using the previous lemma, the general case follows. $\square$

Fix $y \in Y$, let V be an irreducible polarized module for $G_0[y]$, put $S = G_0 \cdot y$ and let $N = \pi^{-1}(S) \subseteq X$ be the associated manifold. In order to study the compactly supported sheaf cohomologies of the polarized sections $\mathcal{P}(y, V)$ we introduce the associated vector bundle $\mathbb{W}$, defined on N.

Suppose $y \in S$ is special and let $K[y]$ denote the stabilizer of y in K. Then the duality between G_0 and K-orbits on X descends to a 1-1 correspondence between $G_0[y]$ and $K[y]$-orbits on X_y [9]. In particular, there exists a unique closed $G_0[y]$-orbit on X_y whose Matsuki dual is the open $K[y]$-orbit. It follows there is a unique G_0-orbit $O \subseteq N$ which is closed in N. Observe that the vanishing number of O is equal to the vanishing number of S. We call O *the associated orbit.*

Applying the previous two results we now establish the following.

Lemma 3.3. *Fix $y \in Y$, put $S = G_0 \cdot y$ and let V be an irreducible polarized module for $G_0[y]$. Let $\mathbb{W}$ be the associated vector bundle defined on $N = \pi^{-1}(S) \subseteq X$ and let $\mathcal{P}_{\mathbb{W}}$ denote the corresponding sheaf of polarized sections. Then we have the following.*

(a) *For each $p = 0, 1, 2, \ldots$, the compactly supported sheaf cohomology group*

$$H_c^p(N, \mathcal{P}_{\mathbb{W}})$$

is an analytic G_0-module, naturally isomorphic to the minimal globalization of its underlying Harish-Chandra module.

(b) *Let q denote the vanishing number of S and let O be the associated orbit in N. Then*

$$H_c^p(N, \mathcal{P}_{\mathbb{W}})) = 0 \quad \textit{for } p < q$$

and there exists a natural, G_0-equivariant, continuous linear inclusion

$$H_c^q(N, \mathcal{P}_{\mathbb{W}}) \to H_c^q(O, \mathcal{P}_{\mathbb{W}}\mid_O).$$

Proof. Put $U = N - O$ and consider the short exact sequence

$$0 \to (\mathcal{P}_{\mathbb{W}}\mid_U)^X \to \mathcal{P}_{\mathbb{W}}^X \to (\mathcal{P}_{\mathbb{W}}\mid_O)^X \to 0.$$

Thus (a) follows from the previous two results and the long exact sequence in cohomology provided we know the result for the sheaf cohomologies of $(\mathcal{P}_{\mathbb{W}}\mid_U)^X$. By choosing a second G_0-orbit $S_2 \subseteq U$, that is closed in U and defining $U_2 = U - S_2$ we obtain the exact sequence

$$0 \to (\mathcal{P}_{\mathbb{W}}\mid_{U_2})^X \to (\mathcal{P}_{\mathbb{W}}\mid_U)^X \to (\mathcal{P}_{\mathbb{W}}\mid_{S_2})^X \to 0.$$

Thus we reduce to the case where U is a G_0-orbit open in N and the result follows by Proposition 3.2.

Similar considerations can be used to establish (b). In particular, we only need to add the observation that if S' is a G_0-orbit contained in U then

$$\dim(O) < \dim(S')$$

and thus Matsuki duality implies that the vanishing number q' of S' satisfies $q' > q$. Thus, where $U = N - O$, we obtain

$$H^p(X, (\mathcal{P}_{\mathbb{W}}\mid_U)^X) = 0 \quad \text{for } p < q$$

and the desired emdedding is obtained from the long exact sequence in cohomology applied to the first of the previous two short exact sequences. □

Theorem 3.4. *Let Y be a flag space for G_0 and suppose $y \in Y$. Let V be a polarized $G_0[y]$-module and let $\mathcal{P}(y, V)$ denote the corresponding sheaf of polarized sections. Suppose S is the G_0-orbit of y and let q denote the vanishing number of S. Then we have the following.*

(a) *For each $p = 0, 1, 2, \ldots$, the compactly supported sheaf cohomology group*

$$H_c^p(S, \mathcal{P}(y, V))$$

is an admissible analytic G_0-module, naturally isomorphic to the minimal globalization of its underlying Harish-Chandra module.

(b) *Let q denote the vanishing number of S. Then $H_c^p(S, \mathcal{P}(y, V))$ vanishes for $p < q$.*

Proof. Using Lemma 3.1 we can reduce to the case where V is irreducible. Let

$$\pi : X \to Y$$

be the canonical projection, put $N = \pi^{-1}(S)$ and let

$$\begin{array}{c} \mathbb{W} \\ \downarrow \\ N \end{array}$$

be the associated vector bundle. Let $\mathcal{P}_{\mathbb{W}}$ denote the corresponding sheaf of polarized sections.

For $x \in X_y = \pi^{-1}(\{y\})$ let $\mathfrak{b}_x \subseteq \mathfrak{p}_y$ be the corresponding Borel subalgebra. Let μ be the element of $\mathfrak{h}^*$ that specializes to the the $\mathfrak{b}_x$-action on $V/\mathfrak{n}_x V$. Thus μ is the lowest weight in V and the corresponding $\mathfrak{l}_y$-infinitesimal character is given by $\lambda = \mu - \rho$. Thus $\mathcal{P}(y,V)^Y$ is an analytic sheaf of $(\pi_*(\mathcal{D}_\lambda), G_0)$-modules and $\mathcal{P}_{\mathbb{W}}^X$ is an analytic sheaf of $(\mathcal{D}_\lambda, G_0)$-modules.

Observe that the direct image functor π_* induces a functor from the category of DNF sheaves of $\mathcal{D}_\lambda$-modules to the category of DNF sheaves of $\pi_*(\mathcal{D}_\lambda)$-modules. More specifically, $\pi_*\mathcal{P}_{\mathbb{W}}^X$ is an analytic sheaf of $(\pi_*(\mathcal{D}_\lambda), G_0)$-modules. We will deduce Theorem 3.4 from the following lemma.

Lemma 3.5. *Maintain the above notations. Then we have the following:*

(a) $\pi_*\mathcal{P}_{\mathbb{W}}^X \cong \mathcal{P}(y,V)^Y$ *as an analytic sheaf of* $(\pi_*(\mathcal{D}_\lambda), G_0)$*-modules.*

(b) *Let* $R^p\pi_*$ *denote the p-th derived functor for the direct image. Then* $R^p\pi_*\left(\mathcal{P}_{\mathbb{W}}^X\right) = 0$ *for* $p > 0$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} X_y & \xrightarrow{i} & N & \to & X \\ \downarrow & & \downarrow \pi & & \downarrow \\ \{y\} & \to & S & \to & Y \end{array}$$

Since the map π is proper, by standard sheaf cohomology considerations it suffices to show that

$$\pi_*\left(\mathcal{P}_{\mathbb{W}}\right) \cong \mathcal{P}(y,V) \text{ and that } R^p\pi_*\left(\mathcal{P}_{\mathbb{W}}\right) = 0 \text{ for } p > 0.$$

To establish the isomorphism $\pi_*\left(\mathcal{P}_{\mathbb{W}}\right) \cong \mathcal{P}(y,V)$, suppose that $z \in S$ and let $\Gamma(X_z, \mathbb{W}\mid_{X_z})$ denote the global holomorphic sections of the holomorphic vector bundle

$$\begin{array}{c} \mathbb{W}\mid_{X_z} \\ \downarrow \\ X_z \end{array}$$

Let $\mathbb{V}$ denote the polarized homogeneous vector bundle over S corresponding to V. Then there is a canonical isomorphism of polarized $G_0[y]$-modules

$$\mathbb{V}_y \cong \Gamma(X_y, \mathbb{W}\mid_{X_y}).$$

In particular, if $v \in \mathbb{V}_y$ and $x \in X_y$ then the natural projection of v into

$$\mathbb{W}_x = \frac{\mathbb{V}_y}{\mathfrak{n}_x \mathbb{V}_y}$$

determines a global section $\sigma : X_y \to \mathbb{W}$. Using this isomorphism, together with the natural G_0-action on

$$\bigcup_{z \in S} \Gamma(X_z, \mathbb{W} \mid_{X_z})$$

we identify this union with $\mathbb{V}$. Observe that the natural $\mathfrak{p}_z$-action in $\Gamma(X_z, \mathbb{W} \mid_{X_z})$ corresponds to the $\mathfrak{p}_z$-action on $\mathbb{V}_z$. Suppose U is an open subset of S. We define the isomorphism from $\pi_* (\mathcal{P}_{\mathbb{W}}) (U)$ to $\mathcal{P}(y, V)(U)$ as follows. Let $\sigma \in \pi_* (\mathcal{P}_{\mathbb{W}}) (U)$ and suppose $z \in U$. Thus the restriction $\sigma \mid_{X_z}$ is a global holomorphic section of the bundle $\mathbb{W} \mid_{X_z}$. Put

$$\gamma_\sigma(z) = \sigma \mid_{X_z} \quad \text{for } z \in U.$$

Then one checks that γ_σ is a polarized section of $\mathbb{V}$ and that the correspondence

$$\sigma \mapsto \gamma_\sigma$$

defines the desired isomorphism.

To establish the vanishing, we first note that the sheaf $\mathcal{P}_{\mathbb{W}}$ is locally free as a sheaf of $\mathcal{O}_X \mid_N$-modules and that the map π is proper. Therefore the desired result can be deduced from the Borel-Weil-Bott theorem and Grauert's results for the direct image. In particular, for $z \in S$ let $\mathcal{O}_{X_z}$ denote the sheaf of holomorphic functions on X_z and put

$$\mathcal{W}(z) = \mathcal{O}_{X_z} \bigotimes_{\mathcal{O}_X \mid_{X_z}} \mathcal{P}_{\mathbb{W}} \mid_{X_z} .$$

Then $\mathcal{W}(z)$ is naturally isomorphic to the sheaf of holomorphic sections of the bundle

$$\begin{array}{c} \mathbb{W} \mid_{X_z} \\ \downarrow \\ X_z \end{array}$$

By the Borel-Weil-Bott theorem [3]

$$H^P(X_z, \mathcal{W}(z)) = 0 \ \text{ for } p > 0.$$

Thus the desired result is deduced by standard techniques from Grauert's Theorem, [2]. $\square$

To finish the proof of Theorem 3.4, observe that part (a) follows immediately from Lemma 3.3, the previous lemma and the Leray spectral sequence. To establish the vanishing from part (b), let O be the associated orbit in N. Then Matsuki duality implies that the vanishing number of O is equal to q. Thus (b) also follows from Lemma 3.3, Lemma 3.5 and the Leray spectral sequence. $\square$

When V is an irreducible polarized $G_0[y]$-module and q is the vanishing number of the G_0-orbit $S = G_0 \cdot y$ then the representation

$$H^q_{\mathrm{c}}(S, \mathcal{P}(y, V))$$

will be called *the standard analytic G_0-module induced from V.*

4. Geometric embedding

Suppose $S \subseteq Y$ is a G_0-orbit and let q denote the vanishing number of S. Choose $y \in S$ and let V be an irreducible polarized module for $G_0[y]$. We now continue our analysis of the standard analytic module $H^q_c(S, \mathcal{P}(y, V))$.

Let $N = \pi^{-1}(S)$ be the associated manifold and let O be the associated orbit in N. Thus O has vanishing number q. Choose $x \in O \cap X_y$ and let $V/\mathfrak{n}_x V$ be the corresponding polarized module for $G_0[x]$. Then we have the *the associated standard analytic module $H^q_{\mathrm{c}}(O, \mathcal{P}(x, V/\mathfrak{n}_x V))$ determined by V.*

The geometric embedding of $H^q_{\mathrm{c}}(S, \mathcal{P}(y, V))$ in $H^q_{\mathrm{c}}(O, \mathcal{P}(x, V/\mathfrak{n}_x V))$ now follows immediately from our previous analysis.

Theorem 4.1. *There exists a natural, G_0-equivariant, continuous linear inclusion*

$$H^q_c(S, \mathcal{P}(y, V)) \to H^q_c(O, \mathcal{P}(x, V/\mathfrak{n}_x V)).$$

Proof. Let

$$\begin{array}{c}\mathbb{W}\\ \downarrow\\ N\end{array}$$

be the polarized equivariant vector bundle in X determined from V and let $\mathcal{P}_{\mathbb{W}}$ denote the sheaf of polarized sections of $\mathbb{W}$. Then

$$\mathcal{P}_{\mathbb{W}} \mid_O \cong \mathcal{P}(x, V/\mathfrak{n}_x V).$$

Thus the desired result follows from Lemma 3.3, Lemma 3.5 and the Leray spectral sequence. $\square$

The standard Beilinson-Bernstein modules. In general, not much is known about the underlying Harish-Chandra module of $H^q_{\mathrm{c}}(S, \mathcal{P}(y, V))$. However we can establish non-vanishing result for $H^q_{\mathrm{c}}(S, \mathcal{P}(y, V))$ using information about the Harish-Chandra module of $H^q_{\mathrm{c}}(O, \mathcal{P}(x, V/\mathfrak{n}_x V))$. In order to state and prove the result, we need to utilize the Beilinson-Bernstein construction of standard Harish-Chandra modules on the full flag space, which is known to describe the Harish-Chandra module of a standard analytic module in that case. We now review some relevant points surrounding the Beilinson-Bernstein construction. We remark that many sheaves in this section are defined on the topological space X^{alg}, where X^{alg} denotes the full flag space equipped with the G-invariant Zariski topology.

For $x \in X^{\mathrm{alg}}$ let $K[x]$ denote the stabilizer of x in K. By definition, *a polarized algebraic $K[x]$-module* is a finite dimensional algebraic $(K[x], \mathfrak{b}_x)$-module such that the nilradical $\mathfrak{n}_x$ acts trivially. A morphism of polarized algebraic modules is a linear map, equivariant for both the $K[x]$ and the $\mathfrak{b}_x$-actions. Suppose that x is special. One knows there exists a natural equivalence between the categories of polarized algebraic $K[x]$-modules and the category of polarized $G_0[x]$-modules.

Let $\lambda \in \mathfrak{h}^*$ and let $\mathcal{D}_\lambda^{\text{alg}}$ be the corresponding twisted sheaf of differential operators with regular coefficients, defined on the algebraic variety X^{alg}. By definition, a *Harish-Chandra sheaf* with parameter λ is a coherent sheaf of $\mathcal{D}_\lambda^{\text{alg}}$-modules with compatible algebraic K-action [7]. Let $\mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K)$ denote the corresponding category of Harish-Chandra sheaves. When $\mathcal{M} \in \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K)$ then the sheaf cohomologies

$$H^p(X^{\text{alg}}, \mathcal{M}) \quad p = 0, 1, 2, \ldots$$

are Harish-Chandra modules with infinitesimal character $\Theta = W \cdot \lambda$.

Let $\mathbb{M}_{\text{HC}}(U_\Theta, K)$ denote the category of Harish-Chandra modules for $(\mathfrak{g}, K)$ with infinitesimal character $\Theta = W \cdot \lambda$. The global sections define a functor

$$\Gamma : \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K) \to \mathbb{M}_{\text{HC}}(U_\Theta, K).$$

Beilinson and Bernstein observed that Γ has the left adjoint

$$\Delta_\lambda : \mathbb{M}_{\text{HC}}(U_\Theta, K) \to \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K)$$

given by

$$\Delta_\lambda(M) = \mathcal{D}_\lambda^{\text{alg}} \bigotimes_{U_\Theta} M$$

where M is an object in $\mathbb{M}_{\text{HC}}(U_\Theta, K)$ and $U_\Theta = \Gamma(X^{\text{alg}}, \mathcal{D}_\lambda^{\text{alg}})$. In fact, when $\lambda \in \mathfrak{h}^*$ is antidominant, Beilinson and Bernstein have shown that the functor

$$\Gamma : \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K) \to \mathbb{M}_{\text{HC}}(U_\Theta, K)$$

is exact and that the adjointness morphism defines an isomorphism between $\Gamma \circ \Delta_\lambda$ and the identity functor on $\mathbb{M}_{\text{HC}}(U_\Theta, K)$.

Let ρ be one half the sum of the positive roots and let W be a polarized algebraic $K[x]$-module. When $\mathfrak{b}_x$ acts by the specialization of $\lambda + \rho \in \mathfrak{h}^*$ then Beilinson and Bernstein define a corresponding standard Harish-Chandra sheaf

$$\mathcal{I}(x, W) \in \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K)$$

[7]. Suppose that W is irreducible. Then there is a unique $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{b}_x$ acts by $\lambda + \rho$. One knows that $\mathcal{I}(x, W)$ has a unique irreducible Harish-Chandra subsheaf $\mathcal{J}(x, W) \in \mathbb{M}_{\text{coh}}(\mathcal{D}_\lambda^{\text{alg}}, K)$. Put

$$I(x, W) = \Gamma(X^{\text{alg}}, \mathcal{I}(x, W)) \text{ and } J(x, W) = \Gamma(X^{\text{alg}}, \mathcal{J}(x, W)).$$

The pair (x, W) will be called a *classifying data* if λ is antidominant and if $J(x, W) \neq 0$. When (x, W) is a classifying data then $J(x, W)$ is the unique irreducible $(\mathfrak{g}, K)$-submodule of $I(x, W)$. An exact criteria for when (x, W) is a classifying data is known [10].

Lemma 4.2. *Suppose that (x, W) is a classifying data. Let $\mathcal{J}(x, W)$ denote the corresponding irreducible Harish-Chandra sheaf in $\mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$ and let $J(x, W)$ be the global sections of $\mathcal{J}(x, W)$. Suppose $\mathcal{K}$ is an object from $\mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$ contained in $\Delta_\lambda(J(x, W))$, whose global sections are zero. Then $\mathcal{K}$ is contained in the kernel of the adjointness morphism*

$$\Delta_\lambda(J(x, W)) \to \mathcal{J}(x, W).$$

Proof. Let $\mathcal{N}$ denote the kernel of the adjointness morphism. Since $J(x, W) \neq 0$, we have the exact sequence

$$0 \to \mathcal{N} \to \Delta_\lambda(J(x, W)) \to \mathcal{J}(x, W) \to 0.$$

Since $\mathcal{J}(x, W)$ is irreducible, $\mathcal{N}$ is a maximal, K-equivariant subsheaf of coherent $\mathcal{D}_\lambda$-submodules in $\Delta_\lambda(J(x, W))$. Since $\Gamma \circ \Delta_\lambda$ is the identity, the global sections of $\mathcal{N}$ are zero. Consider the sheaf

$$\mathcal{M} = \mathcal{K} + \mathcal{N} \subseteq \Delta_\lambda(J(x, W)).$$

Then $\mathcal{M}$ is a K-equivariant subsheaf of coherent $\mathcal{D}_\lambda$-submodules of $\Delta_\lambda(J(x, W))$ containing $\mathcal{N}$. Since $\mathcal{M}$ has global sections zero, $\mathcal{M} \neq \Delta_\lambda(J(x, W))$. Thus $\mathcal{M} = \mathcal{N}$ and $\mathcal{K} \subseteq \mathcal{N}$. □

Nonvanishing for the geometric embedding. Suppose $x \in X$ is special. Then there is a unique θ-stable Cartan subgroup H_0 of G_0 contained in $G_0[x]$. In particular, if N_0 indicates the connected subgroup of $G_0[x]$ with Lie algebra $\mathfrak{g}_0 \cap \mathfrak{n}_x$ then

$$G_0[x] = H_0 \cdot N_0 \quad \text{with } H_0 \cap N_0 = \{0\}\,.$$

Thus a finite-dimensional irreducible polarized representation W of $G_0[x]$ is nothing but a finite-dimensional irreducible representation of H_0 whose derivative we write as $\lambda + \rho \in \mathfrak{h}^*$. The representation W in turn carries a uniquely compatible structure as an irreducible polarized algebraic $K[x]$-module. Let $\mathcal{P}(x, W)$ be the corresponding induced sheaf of polarized sections on $O = G_0 \cdot x$ and put

$$I(x, W) = \Gamma(X^{\mathrm{alg}}, \mathcal{I}(x, W)).$$

Let q denote the vanishing number of O. Then Hecht and Taylor have shown that $H_c^q(O, \mathcal{P}(x, W))$ is naturally isomorphic to the minimal globalization of $I(x, W)$. Using this result, we are now ready to establish the nonvanishing clause of the geometric embedding.

Theorem 4.3. *Let S be a G_0-orbit on Y and suppose q denotes the vanishing number of S. Choose a special point $y \in S$ and let V be an irreducible polarized representation for $G_0[y]$. Let O be the associated orbit in X and choose $x \in O \cap X_y$. Assume $(x, V/\mathfrak{n}_x V)$ is a classifying data. Then $H_c^q(S, \mathcal{P}(y, V))$ is non-zero. In particular, $H_c^q(S, \mathcal{P}(y, V))$ is naturally isomorphic to a topologically closed, nonzero submodule of $H_c^q(O, \mathcal{P}(x, V/\mathfrak{n}_x V))$.*

Proof. Let N be the associated manifold in X and put $U = N - O$. Let

$$\begin{array}{c}\mathbb{W}\\ \downarrow\\ N\end{array}$$

associated vector bundle and let $\mathcal{P}_{\mathbb{W}}$ denote the corresponding sheaf of polarized sections. Consider the short exact sequence

$$0 \to (\mathcal{P}_{\mathbb{W}}\mid_U)^X \to (\mathcal{P}_{\mathbb{W}})^X \to (\mathcal{P}_{\mathbb{W}}\mid_O)^X \to 0.$$

The proof is by contradiction. Suppose $H^q_{\mathrm{c}}(S, \mathcal{P}(y, V)) = 0$. Since

$$H^q_{\mathrm{c}}(S, \mathcal{P}(y, V)) \cong H^q(X, (\mathcal{P}_{\mathbb{W}})^X)$$

we thus obtain a continuous G_0-equivariant inclusion

$$H^q_{\mathrm{c}}(O, \mathcal{P}(x, V/\mathfrak{n}_xV)) \cong H^q_{\mathrm{c}}(O, \mathcal{P}_{\mathbb{W}}\mid_O) \to H^{q+1}_{\mathrm{c}}(U, \mathcal{P}_{\mathbb{W}}\mid_U).$$

Put $\Lambda = \{G_0 - \text{orbits } E \subseteq N : E \text{ has vanishing number } q + 1\}$. Then, arguing as in the previous section, we obtain a continuous G_0-equivariant inclusion

$$H^{q+1}_{\mathrm{c}}(U, \mathcal{P}_{\mathbb{W}}\mid_U) \to \bigoplus_{E\in\Lambda} H^{q+1}_{\mathrm{c}}(E, \mathcal{P}_{\mathbb{W}}\mid_E).$$

Let $J(x, V/\mathfrak{n}_xV)$ denote the unique irreducible Harish-Chandra submodule of $I(x, V/\mathfrak{n}_xV)$ and let $\overline{J(x, V/\mathfrak{n}_xV)}$ denote the closure of $J(x, V/\mathfrak{n}_xV)$ in $H^q_{\mathrm{c}}(O, \mathcal{P}(x, V/\mathfrak{n}_xV))$. By composing the two inclusions above, one obtains an $E \in \Lambda$ and a continuous G_0-equivariant inclusion

$$\overline{J(x, V/\mathfrak{n}_xV)} \to H^{q+1}_{\mathrm{c}}(E, \mathcal{P}_{\mathbb{W}}\mid_E).$$

Choose a special point $z \in E \cap X_y$. Then

$$H^{q+1}_{\mathrm{c}}(E, \mathcal{P}_{\mathbb{W}}\mid_E) \cong H^{q+1}_{\mathrm{c}}(E, \mathcal{P}(z, V/\mathfrak{n}_zV))$$

so that the underlying Harish-Chandra module of $H^{q+1}_{\mathrm{c}}(E, \mathcal{P}_{\mathbb{W}}\mid_E)$ is the Beilinson-Bernstein module $I(z, V/\mathfrak{n}_zV)$. We therefore have an inclusion of Harish-Chandra modules

$$J(x, V/\mathfrak{n}_xV) \to I(z, V/\mathfrak{n}_zV).$$

Let $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{b}_x$ and $\mathfrak{b}_z$ act on $V/\mathfrak{n}_xV$ and $V/\mathfrak{n}_zV$, respectively by the specialization of $\lambda + \rho$. Put $\Theta = W \cdot \lambda$ and consider the localization functor

$$\Delta_\lambda : \mathbb{M}_{\mathrm{HC}}(U_\Theta, K) \to \mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K).$$

By our hypothesis, λ is antidominant. Let $\mathcal{I}(z, V/\mathfrak{n}_zV)$ be the standard Harish-Chandra sheaf from $\mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$ such that

$$I(z, V/\mathfrak{n}_zV) = \Gamma(X, \mathcal{I}(z, V/\mathfrak{n}_zV)).$$

By the adjointness property, the inclusion of $J(x, V/\mathfrak{n}_x V)$ in $I(z, V/\mathfrak{n}_z V)$ defines a non-zero morphism

$$\varphi : \Delta_\lambda(J(x, V/\mathfrak{n}_x V)) \to \mathcal{I}(z, V/\mathfrak{n}_z V).$$

Thus we have the following exact sequence in $\mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$

$$0 \to \ker(\varphi) \to \Delta_\lambda(J(x, V/\mathfrak{n}_x V)) \xrightarrow{\varphi} \mathrm{im}(\varphi) \to 0$$

where $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ denote the kernel and the image of φ, respectively. Since the K-orbit of x has open intersection with the fiber X_y, it follows that x does not belong to the closure of the K-orbit of z, thus the stalk $\mathrm{im}(\varphi)_x$ is zero. On the other hand, we claim that the stalk of the quotient

$$\left(\Delta_\lambda(J(x, V/\mathfrak{n}_x V))/\ker(\varphi)\right)_x$$

is not zero. To establish this claim, observe that since $\Gamma \circ \Delta_\lambda$ is the identity, it follows that

$$\Gamma(X, \ker(\varphi)) = 0.$$

Let $\mathcal{J}(x, V/\mathfrak{n}_x V)$ denote the unique irreducible object in $\mathbb{M}_{\mathrm{coh}}(\mathcal{D}_\lambda, K)$ such that

$$J(x, V/\mathfrak{n}_x V) = \Gamma(X, \mathcal{J}(x, V/\mathfrak{n}_x V)).$$

Therefore, by Lemma 4.2, $\ker(\varphi)$ is contained in the kernel of the canonical surjection

$$\Delta_\lambda(J(x, V/\mathfrak{n}_x V)) \to \mathcal{J}(x, V/\mathfrak{n}_x V) \to 0.$$

Therefore we have a surjection

$$\Delta_\lambda(J(x, V/\mathfrak{n}_x V))/\ker(\varphi) \to \mathcal{J}(x, V/\mathfrak{n}_x V) \to 0.$$

This last surjection implies the claim, since $\mathcal{J}(x, V/\mathfrak{n}_x V)_x \neq 0$. □

5. Realizing some irreducible subrepresentations of principal series in a complex reductive group

In this section we let G_0 be a connected complex reductive Lie group. In order to illustrate the embedding of the previous section, we will realize some irreducible subrepresentations of the principal series.

Let $C_0 \subseteq G_0$ be a Cartan subgroup with Lie algebra $\mathfrak{c}_0$ and let $\mathfrak{c} \subseteq \mathfrak{g}$ be the corresponding complexification. Fix a Borel subgroup B_0 of G_0 containing C_0. Let $\mathfrak{b}_0$ denote the Lie algebra of B_0 and let $\mathfrak{b} \subseteq \mathfrak{g}$ indicate the complexification. Thus the G_0-orbit of $\mathfrak{b}$ in X is closed. Based on this choice there is a simple 1-1 correspondence between the Weyl group W_0 of $\mathfrak{c}_0$ in $\mathfrak{g}_0$ and the set of G_0-orbits in X as follows. Let $\Sigma(\mathfrak{c})$ indicate the set of roots $\mathfrak{c}$ in $\mathfrak{g}$ and let α_0 be a root of $\mathfrak{c}_0$ in $\mathfrak{g}_0$. Then the extension of α_0 to a complex linear functional $\alpha \in \mathfrak{c}^*$ is a root of $\mathfrak{c}$ in $\mathfrak{g}$. Thus we identify the roots Σ_0 of $\mathfrak{c}_0$ in $\mathfrak{g}_0$ with a subset of $\Sigma(\mathfrak{c})$. In turn W_0 is identified with a subgroup of the Weyl group of $\mathfrak{c}$ in $\mathfrak{g}$. Thus for $w \in W_0$ we can define the Borel subalgebra $w \cdot \mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{c}$. The application assigning $w \in W_0$ to the G_0-orbit of $w \cdot \mathfrak{b}$ defines the desired 1-1 correspondence. Observe that the dimension of the G_0-orbit grows with the length of w. In particular, the open G_0-orbit on X corresponds to the longest element of W_0.

Let

$$\chi : H_0 \to \mathbb{C}$$

be a continuous character. Then χ extends uniquely to a character of the stabilizer $G_0[w \cdot \mathfrak{b}]$ of $w \cdot \mathfrak{b}$ in X and the derivative $d\chi$ of χ extends uniquely to a 1-dimensional complex representation of $w \cdot \mathfrak{b}$. Thus χ determines an irreducible polarized module for $G_0[w \cdot \mathfrak{b}]$. We use the notation

$$A(w, \chi)$$

to indicate the corresponding standard analytic module. When w is the identity we simply write $A(\chi)$. In this case $A(\chi)$ is the principal series representation consisting of the space of real analytic functions

$$f : G_0 \to \mathbb{C} \text{ such that } f(gb) = \chi(b^{-1})f(g) \text{ for } g \in G_0 \text{ and } b \in B_0.$$

Let $d\chi$ indicate the natural extension of the differential of χ to an element of $\mathfrak{c}^*$ and identify $d\chi$ with an element of $\mathfrak{h}^*$ via the specialization to $\mathfrak{c}^*$ at $w \cdot \mathfrak{b}$. Put

$$\lambda = d\chi - \rho$$

the shifted differential in $\mathfrak{h}^*$. In particular, the sheaf used to define $A(w, \chi)$ is a sheaf of $\mathcal{D}_\lambda$-modules.

We will need the following result, which follows directly by an application of the analytic version of the intertwining functor, established in [6, Proposition 9.10] (also consider [10]).

Proposition 5.1. *Suppose $\lambda = d\chi - \rho$. Let $\alpha \in \Sigma$ be a simple route and suppose*

$$\overset{\vee}{\alpha}(\lambda) \notin \mathbb{Z}.$$

Specializing to $\mathfrak{c}^*$ via the point $w \cdot \mathfrak{b}$ let $\chi_\alpha : H_0 \to \mathbb{C}$ be the corresponding character and define

$$\gamma : H_0 \to \mathbb{C} \quad \text{by} \quad \chi \cdot \chi_\alpha^{-1}.$$

Then there exists a natural isomorphism

$$A(w, \chi) \cong A(s_\alpha \cdot w, \gamma).$$

Observe that, via the identification made above, the shifted differential

$$\tau = d\gamma - \rho \in \mathfrak{h}^*$$

satisfies the equation $\tau = s_\alpha(\lambda) = \lambda - \overset{\vee}{\alpha}(\lambda)\alpha$. Hence, when λ is antidominant, then so is τ. In particular, suppose β is a root. Then

$$\overset{\vee}{\beta}(s_\alpha(\lambda)) = 2\frac{\langle \beta, s_\alpha(\lambda)\rangle}{\langle \beta, \beta\rangle} = 2\frac{\langle s_\alpha(\beta), \lambda\rangle}{\langle s_\alpha(\beta), s_\alpha(\beta)\rangle} = s_\alpha(\overset{\vee}{\beta})(\lambda).$$

Thus the claim is established, since $s_\alpha \cdot \Sigma^+ = (\Sigma^+ - \{\alpha\}) \cup \{-\alpha\}$ and since $\overset{\vee}{\alpha}(\lambda) \notin \mathbb{Z}$.

We assume that λ is antidominant and that $A(w,\chi)$ is a classifying module. Thus $A(w,\chi)$ contains a unique irreducible submodule $J(w,\chi)$. In this section we treat the case where w is the identity and consider the possibility of realizing the irreducible submodule $J(\chi)$ of $A(\chi)$.

Recall we have identified the roots Σ_0 of $\mathfrak{c}_0$ in $\mathfrak{g}_0$ with subset of $\Sigma(\mathfrak{c}) \subseteq \mathfrak{c}^*$. Put

$$\Sigma_0(\chi) = \left\{\alpha \in \Sigma_0 : \overset{\vee}{\alpha}(d\chi) \in \mathbb{Z}\right\}.$$

One knows that $\Sigma_0(\chi)$ is a root subspace of Σ_0. The roots of $\mathfrak{c}$ in $\mathfrak{b}$ define positive systems Σ_0^+ and $\Sigma_0^+(\chi) = \Sigma_0(\chi) \cap \Sigma_0^+$. $\Sigma_0(\chi)$ will be called *parabolic* (*at* $\mathfrak{b}$) if every simple root in $\Sigma_0^+(\chi)$ is simple for Σ_0^+.

Suppose $\Sigma_0(\chi)$ is parabolic. Put

$$\Sigma_0(\mathfrak{u}) = \Sigma_0^+ - \Sigma_0^+(\chi)$$

and define

$$S^+ = \Sigma_0^+(\chi) \cup -\Sigma_0(\mathfrak{u}).$$

Thus S^+ is a positive set of roots for Σ_0. In particular, there exists a unique $w \in W_0$ such that

$$w\Sigma_0^+ = S^+.$$

Define

$$\mathfrak{p} = w \cdot \mathfrak{b} + \sum_{\alpha \in \Sigma_0^+(\chi)} \mathfrak{g}^{-\alpha}$$

where $\mathfrak{g}^{-\alpha} \subseteq \mathfrak{g}$ is the corresponding root subspace. Thus $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor

$$\mathfrak{l} = \mathfrak{c} + \sum_{\alpha \in \Sigma_0^+(\chi)} \mathfrak{g}^{\alpha} + \sum_{\alpha \in \Sigma_0^+(\chi)} \mathfrak{g}^{-\alpha} \quad \text{and nilradical } \mathfrak{u} = \sum_{\alpha \in \Sigma_0(\mathfrak{u})} \mathfrak{g}^{-\alpha}.$$

Lemma 5.2. *Let k be the number of roots in $\Sigma_0(\mathfrak{u})$. Then there is a sequence of roots*

$$\alpha_1, \dots, \alpha_k \quad \textit{with } \alpha_j \in \Sigma_0(\mathfrak{u})$$

such that α_{j+1} is simple with respect to the Borel subalgebra

$$s_{\alpha_1} \cdots s_{\alpha_j} \mathfrak{b}.$$

Proof. In the case where $\Sigma_0(\mathfrak{u}) = \Sigma_0^+$ and $\Sigma_0^+(\chi) = \varnothing$ a reference for this result can be found in [6, Lemma 10.6]. The more general case is proved in the same way. ☐

Observe that

$$w = s_{\alpha_1} \cdots s_{\alpha_k}.$$

Define

$$\gamma = \chi \cdot \chi_{\alpha_1}^{-1} \cdots \chi_{\alpha_k}^{-1}.$$

Then the shifted differential associated to γ is antidominant with respect to the point $w \cdot \mathfrak{b}$. Using the positive system $\Sigma_0^+(\chi)$ with respect to the Levi factor $\mathfrak{l}$, let V be the irreducible $\mathfrak{l}$-module with lowest weight $d\gamma$. Then V is an irreducible polarized representation for the normalizer $G_0\,[\mathfrak{p}]$ of $\mathfrak{p}$ in G_0. Let $A(\mathfrak{p}, V)$ denote the corresponding standard analytic module. Then we have the following result.

Theorem 5.3. *Assume the principal series representation $A(\chi)$ is a classifying module and let $J(\chi) \subseteq A(\chi)$ denote the corresponding irreducible subrepresentation. Suppose $\Sigma_0(\chi)$ is parabolic and define $\mathfrak{p}$ and V as above. Then there is a natural isomorphism*

$$J(\chi) \cong A(\mathfrak{p}, V).$$

Proof. Define γ and w as above. Then Proposition 5.1 and Lemma 5.2 determine a natural isomorphism

$$A(\chi) \cong A(w, \gamma).$$

Let $N \subseteq X$ be the associated manifold to the G_0-orbit of $\mathfrak{p}$. Observe that N is open in X and that the G_0-orbit of $w \cdot \mathfrak{b}$ is closed in N. Therefore we have the embedding

$$A(\mathfrak{p}, V) \subseteq A(w, \gamma)$$

given by Theorem 4.1. In addition, $A(\mathfrak{p}, V) \neq \{0\}$ by Theorem 4.3. Since the shifted differential for γ is antidominant, and since the G_0-orbit of $\mathfrak{p}$ is open, it now follows from the work in [5] that $A(\mathfrak{p}, V)$ is irreducible. $\square$

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