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A Projective Characterization of Cyclicity

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Abstract. In this note we obtain a new cyclicity criterion for four points in the Euclidean plane, by using algebraic and geometric structures induced in $\mathbb{C}^2$ by the two dimensional complex projective space. We show that if four points lie on a circle in the real plane, then the type-one isotropic lines intersect z_2-axis in four points of real cross ratio.

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1. Introduction

In Euclidean geometry or complex analysis the cyclicity is related to the study of a geometric quantity similar to the cross ratio (see e.g. [3], p. 49, or [8], p. 260). In the present note we explore the underlying projective content of this geometric quantity. This exploration leads to a new cyclicity criterion for four points in the Euclidean plane. This criterion, presented below as Theorem 1, has a complex projective nature. To obtain this result, some algebraic properties of the complex two dimensional projective space are converted into geometric properties of certain special configurations in the Euclidean plane.

We recall a few well-known facts in projective geometry (see e.g. [5, 7]). Let A, B, C, and D be four points, in this order, on the line d in the Euclidean plane. Consider a system of coordinates on d such that A, B, C, and D correspond to

x_1, x_2, x_3, and x_4, respectively. The cross ratio of four ordered points on a line d, A, B, C, D is by definition (see for example [1], pp. 161–164, or [4], p. 77, or [6], p. 248):

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{x_3 - x_1}{x_3 - x_2} : \frac{x_4 - x_1}{x_4 - x_2}. \tag{1}$$

This definition may be extended to a pencil consisting of four ordered lines d_1, d_2, d_3, d_4. By definition, the cross ratio of four ordered lines is the cross ratio determined by the points of intersection with a line d. Therefore,

$$(d_1 d_2 d_3 d_4) = \frac{A_1A_3}{A_2A_3} : \frac{A_1A_4}{A_2A_4}$$

where $\{A_i\} = d \cap d_i$. It is known that the above definition is independent of d, since

$$(d_1 d_2 d_3 d_4) = \frac{\sin(\alpha + \beta)}{\sin \beta} : \frac{\sin(\alpha + \beta + \gamma)}{\sin(\beta + \gamma)} \quad . \tag{2}$$

Furthermore, the cross ratio may be extended to four points on a circle. Denote by A_1, A_2, A_3, and A_4 four points on a circle $\mathcal{C}$. Then $(A_1A_2A_3A_4)_{\mathcal{C}} := (d_1 d_2 d_3 d_4)$ where $d_i = MA_i,\ M \in \mathcal{C}$. As formula (2) shows, the definition does not depend on M. A projectivity on a line d is a map $f : d \to d$ such that, for any four points and their images, the cross ratio is preserved, that is $(A_1A_2A_3A_4) = (B_1B_2B_3B_4)$ where $B_i = f(A_i),\ i = \overline{1,4}$. The points A_i and B_i are called homologous points of the projectivity on d, and the relation $B_i = f(A_i)$ is denoted $A_i \to B_i$. It is well-known that a projectivity on d is determined by three pairs of homologous points.

Proposition 1. *Denoting by x and y the coordinates of the homologous points corresponding through a projectivity on* d*, we have:*

$$y = \frac{mx + n}{px + q}\ ,\ mq - np \neq 0,$$

where $m, n, p, q \in \mathbb{R}$.

Proof. Note that one of the parameters m, n, p, or q has to be different than zero. Therefore, the above expression depends only of the remaining three parameters, which would be determined by three pairs of homologous points.

Conversely, if $x_i,\ i = \overline{1,4}$ are the coordinates of four arbitrary points on d and $y_i = \frac{mx_i+n}{px_i+q}$ are their images then it is easy to see that

$$\frac{x_3 - x_1}{x_3 - x_2} : \frac{x_4 - x_1}{x_4 - x_2} = \frac{y_3 - y_1}{y_3 - y_2} : \frac{y_4 - y_1}{y_4 - y_2}. \qquad \square$$

We can extend the definition of projectivity to pencils of lines. Let A and B be two points and d be a line in the plane. Consider M and N on d, and the families of lines $\{AM/M \in d\}$, $\{BN/N \in d\}$, which are called pencils of lines determined by A and respectively by B. The two pencils of lines are called *projective* if M and N are homologous points in a projectivity on d. The homologous lines AM and BN are called homologous rays. According to (2) this definition does not depend on the line d.

2. The cyclicity theorem

The two dimensional complex projective space is defined by

$$\mathbb{CP}(2) = \{(z_1, z_2, z_3) \in \mathbb{C}^3; (z_1, z_2, z_3) \sim (z_1', z_2', z_3'), \\ \Leftrightarrow \exists \alpha \in \mathbb{C}^* : z_i = \alpha z_i', i = 1, 2, 3\}$$

where we use the natural inclusion $\mathbb{R}^2 \subset \mathbb{C}^2 \subset \mathbb{CP}(2)$. (See e.g. [7], pp. 158–160.) The geometric objects from $\mathbb{R}^2$ can be regarded as objects in $\mathbb{C}^2$, since we may use complex coordinates; in particular, any real number x can be written in the form $x + i0$ in $\mathbb{C}$. Any pair (z_1, z_2) from $\mathbb{C}^2$ can be regarded as the triple $(z_1, z_2, 1)$ from $\mathbb{CP}(2)$. However, this triple is the same as (az_1, az_2, a) and is the same as any triple (z_1', z_2', z_3') with the property that $\frac{z_1'}{z_3'} = z_1$, $\frac{z_2'}{z_3'} = z_2$, with $z_3' \neq 0$.

A line $ax + by + c = 0$ in $\mathbb{R}^2$ can be viewed also as a geometric object in $\mathbb{C}^2$. In that case its equation becomes $az_1 + bz_2 + c = 0$. In $\mathbb{CP}(2)$, this equation takes the form $az_1' + bz_2' + cz_3' = 0$, by replacing z_1 and z_2 by $\frac{z_1'}{z_3'}$, $\frac{z_2'}{z_3'}$, respectively. With this simple procedure we can obtain an induced line in $\mathbb{CP}(2)$ for any line from $\mathbb{R}^2$. The converse however is not true. To see why, consider the particular case when $a = b = 0$, and $c = 1$ in the equation $az_1' + bz_2' + cz_3' = 0$. We obtain a line in $\mathbb{CP}(2)$, $z_3 = 0$, which is not determined by a line in $\mathbb{R}^2$. We call this line the line at infinity of $\mathbb{CP}(2)$.

A circle in $\mathbb{R}^2$ given by $(x-a)^2 + (y-b)^2 = r^2$ can be transformed using this procedure into a figure that we call *circle in* $\mathbb{CP}(2)$, having the equation

$$(z_1 - az_3)^2 + (z_2 - bz_3)^2 = r^2 z_3^2.$$

We now observe that all real circles intersect the line at infinity in the same two points, after they are embedded in $\mathbb{C}^2$, and then in $\mathbb{CP}(2)$

Indeed, if we intersect $z_3 = 0$ and $(z_1 - az_3)^2 + (z_2 - bz_3)^2 = r^2 z_3^2$ we obtain $z_1^2 + z_2^2 = 0$. In light of $\mathbb{CP}(2)$'s definition, we can choose $z_1 = 1$ and $z_2 = \pm i$. It follows that the points $\Omega\,(1, i, 0)$ and $\Omega'\,(1, -i, 0)$ are the points of intersection between an embedded real circle and the line at infinity. These points are called *the absolute points* of $\mathbb{CP}(2)$.

Similarly, the real line of equation $y = mx + n$, regarded in $\mathbb{CP}(2)$, intersects the infinity line at $(1, m, 0)$, where m is the slope of the original line. Since we are performing only an algebraic computation, it does not matter if the line's coefficients a, b, c are real or purely complex. This allows us to extend our discussion to $\mathbb{C}^2$.

In consequence, the result above reveals two types of lines in $\mathbb{C}^2$, which correspond to the absolute points: $z_2 = iz_1 + l$ passing through Ω, and $z_2 = -iz_1 + l'$ passing through Ω'. They are called isotropic type-one lines, and isotropic type-two lines, respectively.

Two such isotropic lines intersect in $\mathbb{R}^2$ if and only if $l' = \bar{l}$. Indeed, if (a, b) is their point of intersection, then $l = b - ia$ and $l' = b + ia$. For our problem, it is important to understand the rôle played by the points $(0, b - ia)$ and $(0, b + ia)$.

These points are of interest since they are the intersection between z_2-axis and each of the lines $z_2 = iz_1 + (b - ia)$ and $z_2 = -iz_1 + (b + ia)$, respectively.

By naturally extending the real case, a complex line in $\mathbb{C}^2$ gets the form $az_1 + bz_2 + c = 0$, where the coefficients $a, b, c \in \mathbb{C}$. The cross-ratio presented above for four collinear points in the real plane $\mathbb{R}^2$ can be extended similarly to four collinear points in $\mathbb{C}^2$.

This analysis leads us to the main result of this paper. We call this result the "cyclicity theorem".

Theorem 1. *If four points lie on a circle in the real plane $\mathbb{R}^2$, then the type-one isotropic lines intersect z_2-axis in four points of real cross ratio.*

Proof. Consider four points $(x_o + r\cos\alpha_j,\ y_o + r\sin\alpha_j)$, $j = \overline{1,4}$, which belong to the circle having (x_o, y_o) as a center and r as radius. In line with the discussion above, the type-one isotropic line

$$z_2 = iz_1 + (y_o + r\sin\alpha_j - i(x_o + r\cos\alpha_j)),\ j = \overline{1,4}$$

intersect z_2-axis in

$$W_j(0, y_o + r\sin\alpha_j - i(x_o + r\cos\alpha_j)),\ j = \overline{1,4}.$$

The cross ratio ω is

$$\omega = \frac{\sin\alpha_1 - i\cos\alpha_1 - \sin\alpha_2 + i\cos\alpha_2}{\sin\alpha_1 - i\cos\alpha_1 - \sin\alpha_3 + i\cos\alpha_3} : \frac{\sin\alpha_4 - i\cos\alpha_4 - \sin\alpha_2 + i\cos\alpha_2}{\sin\alpha_4 - i\cos\alpha_4 - \sin\alpha_3 + i\cos\alpha_3}.$$

Therefore,

$$\omega = \frac{\sin\frac{\alpha_1-\alpha_2}{2}}{\sin\frac{\alpha_1-\alpha_3}{2}} \cdot e^{i\frac{\alpha_1+\alpha_2}{2} - i\frac{\alpha_1+\alpha_3}{2}} : \frac{\sin\frac{\alpha_4-\alpha_2}{2}}{\sin\frac{\alpha_4-\alpha_3}{2}} \cdot e^{i\frac{\alpha_4+\alpha_2}{2} - i\frac{\alpha_4+\alpha_3}{2}}$$

which means that

$$\omega = \frac{\sin\frac{\alpha_1-\alpha_2}{2}}{\sin\frac{\alpha_1-\alpha_3}{2}} : \frac{\sin\frac{\alpha_4-\alpha_2}{2}}{\sin\frac{\alpha_4-\alpha_3}{2}} \in \mathbb{R}.$$ □

Consider the type-two isotropic lines which correspond to the same four points of the previous circle. We obtain the equations

$$z_2 = -iz_1 + (y_o + r\sin\alpha_j + i(x_o + r\cos\alpha_j)),\ j = \overline{1,4}.$$

The points of intersection between these lines and z_2-axis are

$$\bar{W}_j(0, y_o + r\sin\alpha_j + i(x_o + r\cos\alpha_j)),\ j = \overline{1,4}.$$

Their cross ratio is also equal to ω. It means that both type-one and type-two pencils of isotropic lines ΩW_j and $\Omega' \bar{W}_j$ are projective.

If we trace backwards the arguments from this conclusion to the computations we did above, we obtain the proof of the converse of the previous assertion: the corresponding rays ΩW_j and $\Omega' \bar{W}_j$ meet on a given circle in the real plane $\mathbb{R}^2$. Thus we have the following fact.

Theorem 2. *Two pencils of isotropic conjugate lines are projective if and only if the corresponding rays intersect on a circle.*

It follows that a circle can be seen as the intersection of two projective pencils of isotropic conjugate lines. Theorem 2 is a consequence of our Theorem 1. (Theorem 2 appears, with a different proof, in [2], pp. 335–336.) Furthermore, the cyclicity theorem becomes a criterion of recognition of the projectivity of conjugate isotropic pencils, and therefore of the existence of the circle of intersection of conjugate isotropic lines that correspond in that projectivity. The above mentioned criterion of recognition is:

If four type-one isotropic lines intersect z_2-axis in four points having a real cross ratio, then each of them intersects the conjugate isotropic line in a point such that the four points lie on a circle in the real plane $\mathbb{R}^2$.

3. When can we apply the cyclicity criterion?

We prove that Theorem 1 is equivalent to the classical cyclicity condition known in Euclidean geometry or complex analysis, (see e.g. [8], p. 260) that a quadrilateral is cyclic if and only if the cross ratio of the complex numbers corresponding to its vertices is real.

Consider the quadrilateral $ABCD$ in the Euclidean plane $\mathbb{R}^2$. Denote by z_A, z_B, z_C, and z_D the complex numbers corresponding to the points A, B, C, and D respectively. Denote by A the measure of $\angle BAD$ and by C the measure of $\angle DCB$.

Then, a direct computation yields:

$$\begin{aligned} z_D - z_A &= \rho(z_B - z_A)(\cos A + i \sin A), \\ z_D - z_C &= \mu(z_B - z_C)(\cos(-C) + i \sin(-C)), \end{aligned}$$

where $\rho = \frac{|z_D - z_A|}{|z_B - z_A|}$, and $\mu = \frac{|z_D - z_C|}{|z_B - z_C|}$. In the second formula we get $-C$ due to the orientation of the rotation of the position vector $z_B - z_C$ when its image overlaps on $z_D - z_C$. Furthermore, we have

$$\frac{z_D - z_A}{z_D - z_C} : \frac{z_B - z_A}{z_B - z_C} = \frac{\rho}{\mu}(\cos(A + C) + i \sin(A + C)).$$

This proves that $ABCD$ is cyclic (in classical terms, $A + C = \pi$) is equivalent to

$$\frac{z_D - z_A}{z_B - z_A} : \frac{z_D - z_C}{z_B - z_C} = -\frac{\rho}{\mu} \in \mathbb{R}.$$

This means that we have proved cyclicity in the classical theory.

We show below that the classical criterion of cyclicity is equivalent to Theorem 1. Consider A a point in the plane and its corresponding complex number z_A. The isotropic line of first type passing through A has in $\mathbb{C}^2$ the equation $z_2 = i(z_1 - z_A)$.

Its z_2-intercept has the coordinates $(0, -iz_A)$. Computing similarly the intercepts for the points B, C, D we get that $ABCD$ is cyclic if and only if the cross ratio

$$\frac{-iz_D + iz_A}{-iz_B + iz_A} : \frac{-iz_D + iz_C}{-iz_B + iz_C} \tag{3}$$

is real, thus, after a simplification, if and only if

$$\frac{z_D - z_A}{z_B - z_A} : \frac{z_D - z_C}{z_B - z_C} \in \mathbb{R}.$$

As a matter of fact, what role would Theorem 1 play if we already have a classical theorem for cyclicity? The classical criterion is, in fact, a metric characterization, since it is related to the sum of the measures of two opposite angles in a cyclic quadrilateral. Actually, the concept of measure of an angle is a metric concept. On the other hand, bringing a different view on the same problem, the projective context is at least as suitable for a study of the cyclicity as the classical approach, since it offers an algebraic qualitative meaning of the cyclicity phenomenon. More precisely, the emphasis is on the existence of a projectivity between two conjugated isotropic pencils or, equivalently, on the existence of a real cross ratio. This discussion shows that, at least in theory, the cyclicity criterion brought by Theorem 1 could be used every time the classical criterion applies.

4. Examples

To better illustrate the cyclicity theorems discussed previously, we include here several applications.

Example 1. Let $\Omega\,(1, i, 0)$ be the first absolute point. Consider $A(1,0), B(0,1), C(-1,0), D(0,-1)$. To check if they lie on the same circle, consider the isotropic type-one lines of equations $\Omega A \;:\; z_2 = iz_1 - i$, $\Omega B \;:\; z_2 = iy_1 + 1$, $\Omega C \;:\; z_2 = iz_1 + i$, $\Omega D : z_2 = iz_1 - 1$, with their z_2-intercepts lying on the line of equation $z_1 = 0$. These intercepts are: $z'_A = -i, z'_B = 1, z'_C = i, z'_D = -1$, respectively. The cross ratio is $\frac{1+i}{1-i} : \frac{-1+i}{-1-i} = -1 \in \mathbb{R}$. Thus A, B, C, D lie on the same circle. □

Example 2. Let $ABCD$ be a cyclic quadrilateral. Prove that the centroids of triangles ABC, BCD, CDA, and DAB lie on the same circle.

For the proof, we use the classical cyclicity criterion. Denote by z_A, z_B, z_C, and z_D the complex numbers corresponding to the points A, B, C, and D respectively. The hypothesis that $ABCD$ is cyclic can be expressed in equivalent form

$$\frac{z_D - z_C}{z_D - z_A} : \frac{z_B - z_C}{z_B - z_A} \in \mathbb{R}.$$

The centroids in $\Delta ABC, \Delta BCD, \Delta ADC, \Delta ABD$ correspond to the complex numbers

$$z_{ABC} = \frac{1}{3}(z_A + z_B + z_C), \quad z_{BCD} = \frac{1}{3}(z_D + z_B + z_C),$$

$$z_{ADC} = \frac{1}{3}(z_A + z_D + z_C), \quad z_{BAD} = \frac{1}{3}(z_D + z_B + z_A).$$

A direct computation yields:

$$\frac{z_{ABD} - z_{ABC}}{z_{BCD} - z_{ABC}} : \frac{z_{ABD} - z_{ABC}}{z_{BCD} - z_{ABC}} = \frac{z_D - z_C}{z_D - z_A} : \frac{z_B - z_C}{z_B - z_A} \in \mathbb{R}.$$

Thus, the quadrilateral formed by the four centroids is cyclic. □

Remark that this proof can be interpreted also as an application of our cyclicity criterion, in the sense that the above computation expresses also the cross ratio (3).

Example 3. We use our cyclicity theorem to prove Problem 10710 from Amer. Math. Monthly, proposed by the second author in **106** (1999), pp. 68. A synthetic solution, by A. Sinefakopoulos, is in Amer. Math. Monthly **107**(6) (2000), pp. 572–573. To better serve our exposition, we slightly rephrase the problem in the following form.

Given a triangle ABC, let us denote by $D \in BC$, $E \in AB$, $F \in AC$ the contact points between the incircle and the sides of ABC and let I be the incenter. The parallel through A to BC intersects DE and DF in M and N, respectively. Let L, T be the midpoints of MD and ND. Then A, L, I and T lie on a circle.

We show that the circle from the initial problem is the intersection of two projective pencils of isotropic conjugate lines.

We will use the same notations as in the problem and let p be the semiperimeter of ΔABC.

If $A(0,a)$, $B(-b,0)$, and $C(c,0)$ then

$$I\left(\frac{c + \sqrt{a^2+b^2} - b - \sqrt{a^2+c^2}}{2}, \frac{a(b+c)}{b + c + \sqrt{a^2+b^2} + \sqrt{a^2+c^2}}\right),$$

$$T\left(\frac{c - \sqrt{a^2+c^2}}{2}, \frac{a}{2}\right) \text{ and } L\left(\frac{\sqrt{a^2+b^2} - b}{2}, \frac{a}{2}\right).$$

Obviously, $BD = p - AC$, thus $BD = \frac{b+c+\sqrt{a^2+b^2}-\sqrt{a^2+c^2}}{2}$.

The equality $BD = BO + OD$ yields $x_D = \frac{c+\sqrt{a^2+b^2}-b-\sqrt{a^2+c^2}}{2}$. The inradius r can be computed by the well-known formula $r = \frac{S}{p}$, where S is the area of the triangle ABC. Thus,

$$r = \frac{a(b+c)}{b + c + \sqrt{a^2+b^2} + \sqrt{a^2+c^2}}.$$

It follows that

$$I\left(\frac{c + \sqrt{a^2+b^2} - b - \sqrt{a^2+c^2}}{2}, \frac{a(b+c)}{b + c + \sqrt{a^2+b^2} + \sqrt{a^2+c^2}}\right).$$

Since $AM = AE = p - BC$, we get

$$AM = \frac{\sqrt{a^2+b^2}+\sqrt{a^2+c^2}-b-c}{2}.$$

Since L and T are midpoints for MD and ND, respectively, we have

$$T\left(\frac{c-\sqrt{a^2+c^2}}{2}, \frac{a}{2}\right); \qquad L\left(\frac{\sqrt{a^2+b^2}-b}{2}, \frac{a}{2}\right).$$

Consequence. The type-one isotropic lines which pass through A, L, I, and T intersect z_2-axis in A', L', I', and T', respectively, with the coordinates:

$$\begin{aligned}
z_2(A') &= a;\\
z_2(L') &= \frac{a}{2} - i\frac{\sqrt{a^2+b^2}-b}{2};\\
z_2(T') &= \frac{a}{2} - i\frac{c-\sqrt{a^2+c^2}}{2};\\
z_2(I') &= \frac{a\,(b+c)}{b+c+\sqrt{a^2+b^2}+\sqrt{a^2+c^2}} + i\frac{c+\sqrt{a^2+b^2}-b-\sqrt{a^2+c^2}}{2}.
\end{aligned}$$

Now we are ready to apply the cyclicity theorem to conclude the proof. Considering the results asserted by Theorem 2, if we compute $\omega = \frac{z_2(A')-z_2(L')}{z_2(A')-z_2(T')} : \frac{z_2(I')-z_2(L')}{z_2(I')-z_2(T')}$ we obtain $\omega \in \mathbb{R}$.

According to our cyclicity theorem, the isotropic lines $\Omega A', \Omega L', \Omega I'$, and $\Omega T'$ cut $\Omega' A'', \Omega' L'', \Omega' I''$, and $\Omega' T''$ at the points A, L, I, and T, respectively, which lie on the same circle. □

The geometric configuration from Example 3 is studied also, from a different viewpoint, in [9].

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