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On Braxtopes, a Class of Generalized Simplices

Margaret M. Bayer* **Tibor Bisztriczky†**

Department of Mathematics, University of Kansas
Lawrence KS 66045-7523 USA
e-mail: bayer@math.ku.edu

Department of Mathematics and Statistics, University of Calgary
Calgary, Alberta, T2N 1N4 Canada
e-mail: tbisztri@math.ucalgary.ca

Abstract. In a d-simplex every facet is a $(d-1)$-simplex. We consider as generalized simplices other combinatorial classes of polytopes, all of whose facets are in the class. Cubes and multiplexes are two such classes of generalized simplices. In this paper we study a new class, braxtopes, which arise as the faces of periodically-cyclic Gale polytopes. We give a geometric construction for these polytopes and various combinatorial properties.

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1. Introduction

In the study of combinatorial properties of convex polytopes, best understood are the simplicial ones. Among simplicial polytopes, cyclic polytopes have played an important role. They have the largest number of facets among all polytopes

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with given dimension and number of vertices. The combinatorial study of nonsimplicial polytopes is hampered by the difficulty of generating classes with varied combinatorial structure. The simplicial (and their duals, the simple) polytopes are, in some sense, an extremal class of polytopes; we need other extremal classes to better understand combinatorial parameters associated with polytopes.

One approach is to find nonsimplicial analogues of cyclic polytopes. In [7, 8], Bisztriczky introduced two such classes: the ordinary polytopes (of odd dimensions) and the periodically-cyclic Gale polytopes (of even dimensions). The faces of the ordinary polytopes themselves form an interesting class of polytopes, the multiplexes [6, 9]. Ordinary polytopes were studied further in [2, 3, 5, 10]. The periodically-cyclic Gale polytopes have until now been less studied. In this paper we begin a study of periodically-cyclic Gale polytopes by examining the polytopes that arise as their faces, and that are, as are multiplexes, generalizations of simplices. In [4] we study periodically-cyclic Gale polytopes further.

2. Definitions

Let Y be a set of points in $\mathbf{R}^d$, $d \geq 1$. Then $[Y]$ and $\langle Y \rangle$ denote, respectively, the convex hull and the affine hull of Y. If $Y = \{y_1, y_2, \ldots, y_s\}$ is a finite set, we set $[y_1, y_2, \ldots, y_s] = [Y]$ and $\langle y_1, y_2, \ldots, y_s \rangle = \langle Y \rangle$.

Let $V = \{x_0, x_1, \ldots, x_n\}$ be a totally ordered set of $n+1$ points in $\mathbf{R}^d$ with $x_i < x_j$ if and only if $i < j$. We say that x_j *separates* x_i and x_k if $x_i < x_j < x_k$. For $Y \subset V$, Y is a *Gale set* (in V) if any two points of $V \setminus Y$ are separated by an even number of points of Y.

Let $P \subset \mathbf{R}^d$ be a (convex) d-polytope. For $-1 \leq i \leq d$, let $\mathcal{F}_i(P)$ denote the set of i-dimensional faces of P and $f_i(P) = |\mathcal{F}_i(P)|$. For convenience, we set $\mathcal{V}(P) = \mathcal{F}_0(P)$, $\mathcal{E}(P) = \mathcal{F}_1(P)$, and $\mathcal{F}(P) = \mathcal{F}_{d-1}(P)$. We recall that $(f_{-1}(P), f_0(P), f_1(P), \ldots, f_{d-1}(P), f_d(P))$ is the *f-vector* of P. In the case that P is simplicial, the *h-vector* of P is $(h_0(P), h_1(P), \ldots, h_d(P))$ with $h_i(P) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(P)$. A chain of faces $\emptyset \subset G_1 \subset G_2 \subset \cdots \subset G_r \subset P$ is an *S-flag* if $S = \{\dim G_1, \dim G_2, \ldots, \dim G_r\}$. Writing $f_S(P)$ as the number of S-flags of P, the *flag vector* $(f_S(P))_{S \subseteq \{0,1,\ldots,d-1\}}$ of P is a vector with 2^d entries. Finally we refer to [16] for the definitions of a triangulation and a shelling of P.

Let $\mathcal{V}(P) = \{x_0, x_1, \ldots, x_n\}$, $n \geq d$. We set $x_i < x_j$ if and only if $i < j$, and call $x_0 < x_1 < \cdots < x_n$ a *vertex array* of P. Let $G \in \mathcal{F}_i(P)$, $1 \leq i \leq d-1$, such that $G \cap \mathcal{V}(P) = \{y_0, y_1, \ldots, y_m\}$ (each y_j is some x_i). Then $y_0 < y_1 < \cdots < y_m$ is the vertex array of G if it is induced by $x_0 < x_1 < \cdots < x_n$. Finally, P with $x_0 < x_1 < \cdots < x_n$ is *Gale* (with respect to the vertex array) if the vertex set of each facet of P is a Gale set.

We recall from [11] and [12] that a d-polytope P is *cyclic* if it is simplicial and Gale with respect to some vertex array. From [8], P is *periodically-cyclic* if there is a vertex array, say, $x_0 < x_1 < \cdots < x_n$ and an integer k with $n+1 \geq k \geq d+2$, such that

- $[x_{i+1}, x_{i+2}, \ldots, x_{i+k}]$ is a cyclic d-polytope with $x_{i+1} < x_{i+2} < \cdots < x_{i+k}$, for $-1 \leq i \leq n-k$, and

- $[x_{i+1}, x_{i+2}, \dots, x_{i+k+1}]$ is not cyclic for $-1 \le i \le n-k-1$.

We note that if $k = n+1$, then P is cyclic.

From [6], P is a *multiplex* if there is a vertex array, say, $x_0 < x_1 < \cdots < x_n$ such that the facets of P are $[x_{i-d+1}, \dots, x_{i-1}, x_{i+1}, \dots, x_{i+d-1}]$ for $0 \le i \le n$ under the convention: $x_t = x_0$ for $t \le 0$ and $x_t = x_n$ for $t \ge n$. Next, P is *multiplicial* if each facet of P is a multiplex with respect to the ordering induced by a fixed vertex array of P. Finally, P is *ordinary* if it is Gale and multiplicial with respect to some vertex array. We note from [7] that if $d \ge 4$ is even, then every ordinary d-polytope is cyclic.

We observe that the noncyclic polytopes mentioned above are nonsimplicial. In this spirit we introduce another class of nonsimplicial polytopes, the braxtopes.

Definition. *For $d \le 2$, a d-braxtope is a d-simplex. For $n \ge d \ge 3$, P is a d-braxtope if there is a vertex array, say, $x_0 < x_1 < \cdots < x_n$ such that the facets of P are*

$$T_i = [x_i, x_{i+1}, \dots, x_{i+d-1}] \text{ for } 0 \le i \le n-d+1$$

and

$$E_j = [x_0, x_{j-(d-2)}, \dots, x_{j-1}, x_{j+1}, \dots, x_{j+(d-2)}] \text{ for } 2 \le j \le n$$

under the convention: $x_t = x_0$ for $t \le 0$ and $x_t = x_n$ for $t \ge n$.

We note that $E_n = [x_0, x_{n-(d-2)}, \dots, x_{n-1}, x_n]$ and $|\mathcal{F}(P)| = 2n-d+1$. Finally, P is *braxial* if each proper face of P is a braxtope with respect to the ordering induced by a fixed vertex array of P.

3. Realizability and properties of braxtopes

Henceforth $Q^{d,n}$ denotes a d-braxtope with the vertex array $x_0 < x_1 < \cdots < x_n$, $n \ge d \ge 3$. For $n = d$, $Q^{d,d}$ is a d-simplex. If $d+1 \le n \le 2d-2$, then $Q^{d,n}$ is a face of a periodically-cyclic Gale $2m$-polytope when $2m \ge d+1$ [8]. This is not obvious, and this observation by the first author led to the formulation of braxtopes as a new class of polytopes.

Theorem A. *$Q^{d,n}$ is realizable in $\mathbf{R}^d$ for all $n \ge d \ge 3$.*

Proof. In view of the preceding, we may assume that $n \ge 2d-1$ and that $Q^{d,n-1} \subset \mathbf{R}^d$ exists with $x_0 < x_1 < \cdots < x_{n-1}$. Let $Q' = Q^{d,n-1}$ and $\mathcal{F}(Q') = \{T'_0, \dots, T'_{n-d}, E'_2, \dots, E'_{n-1}\}$. It is easy to check by a simple beneath-beyond argument (see [12]) that $[Q', x_n]$ is a $Q^{d,n}$ with $x_0 < x_1 < \cdots < x_{n-1} < x_n$ if $x_n \in \mathbf{R}^d$ is a point with the properties:

- $x_n \in L = \langle x_0, x_{n-d}, x_{n-d+1}, x_{n-1} \rangle = \displaystyle\bigcap_{j=n-d+2}^{n-2} E'_j$,
- x_n is beyond E'_{n-1}, and
- x_n is beneath every other facet of Q'.

Specifically, the three $(d-2)$-faces $[x_{n-d+1}, \dots, x_{n-1}]$, $[x_0, x_{n-d+1}, \dots, x_{n-2}]$, and $[x_0, x_{n-d+2}, \dots, x_{n-1}]$ of E'_{n-1} yield T_{n-d+1}, E_{n-1}, and E_n, respectively. We observe that the existence of such a point x_n is due to the fact that L is a 3-flat, $L \cap \langle E'_{n-1} \rangle = \langle x_0, x_{n-d+1}, x_{n-1} \rangle$ is a supporting plane of $L \cap Q'$, and $|\mathcal{F}(Q') \setminus \{E'_{n-d+2}, \dots, E'_{n-1}\}|$ is finite. □

Proposition 1. *Let $Q = Q^{d,n}$ be a d-braxtope with $x_0 < x_1 < \cdots < x_n$, $n \geq d \geq 3$. Then*

(i) $[x_0, x_u] \in \mathcal{E}(Q)$ *for* $1 \leq u \leq n$,

(ii) $[x_1, x_u] \in \mathcal{E}(Q)$ *if and only if* $u = 0$ *or* $2 \leq u \leq d$,

(iii) $[x_u, x_n] \in \mathcal{E}(Q)$ *if and only if* $u = 0$ *or* $n-(d-1) \leq u \leq n-1$,

(iv) *for* $2 \leq t \leq n-1$, $[x_t, x_u] \in \mathcal{E}(Q)$ *if and only if* $u = 0$, *or* $t-d+1 \leq u \leq t+d-1$, $u \neq t$,

(v) $[x_0, x_{t+1}, x_{t+k}] \in \mathcal{F}_2(Q)$ *for* $0 \leq t \leq n-k$ *and* $2 \leq k \leq d-2$,

(vi) $[x_0, x_t, x_{t+1}, x_{t+d-1}, x_{t+d}] \in \mathcal{F}_3(Q)$ *for* $1 \leq t \leq n-d$, *and*

(vii) $\{x_t, x_{t+1}, \dots, x_{t+d}\}$ *is an affinely independent set for* $0 \leq t \leq n-d$.

Proof. (i) We check that each edge $[x_0, x_u]$ is the intersection of specific facets. For example, $[x_0, x_1] = T_0 \cap \bigcap_{j=2}^{d-1} E_j$ and $[x_0, x_u] = E_{u-d+2} \cap E_{u+d-2}$ for $d \leq u \leq n-d+1$.

(ii) We note $T_1 = [x_1, x_2, \dots, x_d] \in \mathcal{F}(Q)$ and for $u \geq d+1$, at most $d-3$ facets of Q contain $[x_1, x_u]$.

(iii) and (iv) We argue as in (ii).

(v) If $t+1 \geq n-d+2$ or $t+k \leq d-1$, then $[x_0, x_{t+1}, x_{t+k}]$ is in a simplex facet (E_n or T_0), and so it is a 2-face. Assume $d-k \leq t \leq n-d$. Then

$$\begin{aligned} & E_{t-d+k+2} \cap E_{t+d-1} \cap \bigcap_{j=t+2}^{t+k-1} E_j \\ &= [x_0, x_{t+1}, x_{t+2}, \dots, x_{t+k}] \cap \bigcap_{j=t+2}^{t+k-1} E_j \\ &= [x_0, x_{t+1}, x_{t+k}]. \end{aligned}$$

So $[x_0, x_{t+1}, x_{t+k}] \in \mathcal{F}_2(Q)$.

(vi) $[x_0, x_t, x_{t+1}, x_{t+d-1}, x_{t+d}] = \bigcap_{j=t+2}^{t+d-2} E_j$, $[x_t, x_{t+d}] \notin \mathcal{E}(P)$, and $[x_0, x_t, x_{t+1}] \in \mathcal{F}_2(P)$.

(vii) This follows immediately from $\{T_0, T_1, \dots, T_{n-d}\} \subset \mathcal{F}(Q)$. □

Theorem B. *Let $Q = Q^{d,n}$ with $x_0 < x_1 < \cdots < x_n$, $n \geq d+1 \geq 4$. Then $Q' = [x_0, x_1, \dots, x_{n-1}]$ is a d-braxtope with $x_0 < x_1 < \cdots < x_{n-1}$.*

Proof. With the notation above, we observe that

$$\{T_0, \dots, T_{n-d-1}, E_2, \dots, E_{n-d+1}\} \subset \mathcal{F}(Q').$$

Let $n-d+2 \le j \le n-2$. Then $x_n \in E_j$ yields that

$$E'_j = [x_0, x_{j-d+2}, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}] \in \mathcal{F}(Q').$$

Thus, we need only to verify that $E'_{n-1} = [x_0, x_{n-d+1}, \dots, x_{n-1}] \in \mathcal{F}(Q')$. By Proposition 1(iii), x_n is a simple vertex of Q and $[x_u, x_n] \in \mathcal{E}(Q)$ for exactly $x_u \in X = \{x_0, x_{n-d+1}, \dots, x_{n-1}\}$. By Proposition 1(vii), $\langle X \rangle$ is a hyperplane of $\mathbf{R}^d$ and $[X]$ is a $(d-1)$-polytope.

Finally, $x_n \notin Q'$ implies that x_n is beyond some $F' \in \mathcal{F}(Q')$. Let x_u be a vertex of F'. Since there is an $F \in \mathcal{F}(Q)$ such that $x_u \in F$ and $x_n \notin F$, it follows that $[x_u, x_n] \in \mathcal{E}(P)$ and $x_u \in X$. Thus $|X| = d$ yields that $F' = [X] = E'_{n-1}$, and $\mathcal{F}(Q')$ contains the set of facets of a $Q^{d,n-1}$. It is well known that this implies $Q' = Q^{d,n-1}$. $\square$

Theorem C. *Let $Q = Q^{d,n}$ with $x_0 < x_1 < \dots < x_n$, $n \ge d \ge 3$. Then*

(i) *Q is a braxial d-polytope.*

(ii) *The vertex figure Q/x_0 of Q at x_0 is a $(d-1)$-multiplex with the induced ordering.*

(iii) *Let $n \le 2d-3$. Then Q is a $(2d-2-n)$-fold pyramid over an $(n-d+2)$-braxtope with the induced ordering.*

(iv) *For $-1 \le j \le d$, $f_j(Q) = \binom{d+1}{j+1} + (n-d)\left[\binom{d-1}{j} + \binom{d-2}{j-1}\right]$.*

(v) *Q is elementary; that is, $f_{\{0,2\}}(Q) - 3f_2(Q) + f_1(Q) - df_0(Q) + \binom{d+1}{2} = 0$.*

Proof. (i) We verify that each $F \in \mathcal{F}(Q)$ is a $(d-1)$-braxtope with the induced ordering. Assume F is not a $(d-1)$-simplex, and hence, $F \in \{E_3, \dots, E_{n-2}\}$ with the standard notation. If $f_0(F) = m+1$ and F is a $(d-1)$-braxtope, then we denote its $(d-2)$-faces by $T'_0, \dots, T'_{m-d+2}, E'_2, \dots, E'_m$.

If $3 \le u \le d-1$, then $E_u = [x_0, x_1, \dots, x_{u-1}, x_{u+1}, \dots, x_{u+d-2}]$, $m = u+d-3$, $T'_i = T_i \cap E_u$ for $0 \le i \le u-1 = m-d+2$, $\{E'_2, E'_3, \dots, E'_{u+d-4}\} = \{E_j \cap E_u |\ 2 \le j \le u+d-3,\ j \ne u\ \}$, and $E'_m = E_u \cap E_{u+d-1}$. A similar argument yields the claim for E_d, E_{d+1}, $\dots$, E_{n-2}.

(ii) Since $[x_0, x_u] \in \mathcal{E}(P)$ for $1 \le u \le n$, it follows from the description of $\mathcal{F}(Q)$ that the $(d-2)$-faces of $\overline{Q} = Q/x_0$ are (writing $\overline{x}_u$ for the vertex of $\overline{Q}$ corresponding to $[x_0, x_u]$), $[\overline{x}_{i-d+2}, \dots, \overline{x}_{i-1}, \overline{x}_{i+1}, \dots, \overline{x}_{i+d-2}]$ for $1 \le i \le n$, with the convention $\overline{x}_r = \overline{x}_1$ for $r \le 1$ and $\overline{x}_r = \overline{x}_n$ for $r \ge n$. These are the $(d-2)$-faces of a $(d-1)$-multiplex with n vertices.

(iii) We observe that for $n \le 2d-3$, $Q = [E_{n-d+2}, x_{n-d+2}]$; that is, Q is a pyramid over the $(d-1)$-braxtope E_{n-d+2} with apex x_{n-d+2}. We note that $E_{n-d+2} = Q^{d-1,n-1}$ with $n-1 \le 2d-4 = 2(d-1)-2$. Thus, either $n = 2d-3$ and we are done, or $n-1 \leq 2(d-1)-3$, and we repeat the argument. In summary, Q is a $(2d-2-n)$-fold pyramid over the $(n-d+2)$-braxtope $[x_0, x_1, \ldots, x_{n-d+1}, x_d, \ldots, x_n]$ with apices $x_{n-d+2}, \ldots, x_{d-2}, x_{d-1}$.

(iv) We count the faces of $Q^{d,n}$ in two groups: those faces containing the vertex x_0, and those not containing x_0. The former are intersections of the facets E_i (and T_0), and the latter are all contained in the facets T_i $(1 \leq i \leq n-d+1)$.

Recall that the vertex figure of x_0 in Q is the $(d-1)$-multiplex with n vertices. The f-vector of the multiplex is given in [6]. Thus the number of j-faces of Q containing x_0 is $\binom{d}{j} + (n-d)\binom{d-2}{j-1}$.

For $d+1 \leq \ell \leq n$, $\binom{d-1}{j}$ is the number of j-faces of $\bigcup_{i=1}^{n-d+1} T_i$ containing x_ℓ as the greatest vertex. The number of j-faces in $T_1 = [x_1, x_2, \ldots, x_d]$ is $\binom{d}{j+1}$. Thus the total number of j-faces in $\bigcup_{i=1}^{n-d+1} T_i$ is $\binom{d}{j+1} + (n-d)\binom{d-1}{j}$.

(v) In the inductive construction of the d-braxtope (proof of Theorem A), new vertices are not placed on flats spanned by 2-faces. So all 2-dimensional faces of every d-braxtope are triangles, and thus $f_{\{0,2\}}(Q) - 3f_2(Q) = 0$. Now

$$f_1(Q) - df_0(Q) + \binom{d+1}{2} = \binom{d+1}{2} + (n-d)d - d(n+1) + \binom{d+1}{2} = 0.$$

□

Remarks. If $n > d$, then the f-vector of the d-braxtope equals the f-vector of the $(d-3)$-fold pyramid over the bipyramid over an $(n-d+2)$-gon. We conjecture that this result extends to flag vectors.

Kalai [14] introduced *elementary* polytopes as d-polytopes satisfying $f_{\{0,2\}} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} = 0$. This quantity is nonnegative for all d-polytopes by a rigidity argument [13]. It may be interpreted as the difference $h_2 - h_1$ of (middle perversity) Betti numbers of the associated toric variety.

4. Triangulation and the *h*-vector of the braxtope

Earlier we defined the h-vector of a simplicial polytope by a linear transformation of the f-vector. The definitions of f-vector and h-vector extend naturally to simplicial complexes. The h-vector of a simplicial polytope is the sequence of cohomology ranks of the toric variety associated to the polytope. For nonsimplicial polytopes the middle perversity Betti numbers of the toric variety form the h-vector, but it depends on the flag vector, not just on the f-vector. A triangulation Δ of a polytope P is *shallow* if every k-face of Δ is contained in a face of P of dimension at most $2k$. The h-vector of a shallow triangulation (if one exists) may be used to compute the h-vector of the nonsimplicial polytope [1].

Theorem D. *Let $Q = Q^{d,n}$ with $x_0 < x_1 < \cdots < x_n$, $n \geq d \geq 3$.*

(i) *For* $1 \le i \le n-d+1$, *let*

$$J_i = [x_0, x_i, x_{i+1}, \dots, x_{i+d-1}].$$

Then $\{J_1, J_2, \dots, J_{n-d+1}\}$ *are the facets of a triangulation* Δ *of* Q.

(ii) Δ *is a shallow triangulation of* Q.

(iii) $h(Q) = (1, n-d+1, n-d+1, \dots, n-d+1, 1)$.

Proof. (i) Since the facets not containing x_0 are the simplices T_i $(1 \le i \le n-d+1)$, this is the triangulation resulting from pulling the vertex x_0 (see [15]).

(ii) First observe that for any j, $2 \le j \le n-d+2$, $\bigcap_{i=j}^{j+d-3} E_i = [x_0, x_{j-1}, x_{j+d-2}]$. In particular this intersection is two-dimensional, and so for any set I contained in a consecutive $(d-2)$-element subset of $\{2, 3, \dots, n\}$, $\dim \bigcap_{i \in I} E_i = d - |I|$.

Suppose σ is a face of Δ. Note that all vertices and edges of Δ are vertices and edges of Q, since $[x_0, x_i] \in \mathcal{E}(Q)$ for all i, so assume $\dim \sigma \ge 2$. If $x_0 \notin \sigma$, then $\sigma \subset T_i$ for some facet T_i $(i \ge 1)$ of Q, and σ is thus a face of Q. Now suppose $x_0 \in \sigma \subset [T_{i+1}, x_0]$. Let $\tau = [\sigma, x_{i+1}, x_{i+d}]$, and $I = \{j : i+1 \le j \le i+d,\ x_j \notin \tau\}$. Then $\sigma \subseteq \tau \subseteq \bigcap_{i \in I} E_i$, and

$$\dim \bigcap_{i \in I} E_i = d - |I| = f_0(\tau) - 1 \le f_0(\sigma) + 1 = \dim \sigma + 2 \le 2 \dim \sigma.$$

Thus σ is contained in the face $\bigcap_{i \in I} E_i$ of Q of dimension at most $2 \dim \sigma$.

(iii) The ordering $J_1, J_2, \dots, J_{n-d+1}$ of the facets of Δ forms a shelling of Δ; for $2 \le j \le n-d+1$, the unique minimal face of $J_j \setminus \bigcup_{i<j} J_i$ is $\{x_{j+d-1}\}$. So the h-vector of Δ is $(1, n-d, 0, 0, \dots, 0)$. By [1, Theorem 4] this implies that the h-vector of Q is $h(Q) = (1, n-d+1, n-d+1, \dots, n-d+1, 1)$. $\square$

Remark. The formula for $h(Q^{d,n})$ would also follow from the conjecture that the flag vector of $Q^{d,n}$ equals the flag vector of the $(d-3)$-fold pyramid over the bipyramid over the $(n-d+2)$-gon.

The colex order of the facets of $Q = Q^{d,n}$ gives a shelling of Q. Like the colex shelling of the ordinary polytopes and multiplexes [3], this shelling of Q has special properties that are important for counting faces: for each j, $F_j \setminus \cup_{i<j} F_i$ has a unique minimal face G_j, which is a simplex, and the quotient polytope F_j/G_j is a simplex.

5. Extension

Theorem C (i) says that the vertex figure of x_0 in a braxtope Q is a multiplex. The antistar of x_0 (the polytopal complex of faces of Q not containing x_0) is a triangulation of the multiplex into the simplices T_1, T_2, ..., T_{n-d}. This multiplex-braxtope relationship may be extended by adding more vertices like x_0.

Definition. *For $d \leq r+1$, an (r,d)-braxtope is a d-simplex.*

For $n \geq d \geq r+2 \geq 2$, P is an (r,d)-braxtope if there is a vertex array, say, $x_0 < x_1 < \cdots < x_n$ such that the facets of P are

$$T_{i,j} = [\{x_0, x_1, \ldots, x_{r-1}\} \setminus \{x_i\} \cup \{x_j, x_{j+1}, \ldots, x_{j+d-r}\}]$$

for $0 \leq i \leq r-1 \leq r \leq j \leq n-d+r$,

$$T_{0,0} = [x_0, x_1, \ldots, x_{d-1}],$$

and

$$E_j = [\{x_0, x_1, \ldots, x_{r-1}, x_{j-(d-r-1)}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{j+(d-r-1)}\}]$$

for $r+1 \leq j \leq n$, under the convention: $x_t = x_0$ for $t \leq 0$ and $x_t = x_n$ for $t \geq n$.

We note that a $(1,d)$-braxtope is a d-braxtope. If $r = 0$, we understand that there are no facets $T_{i,j}$ (except $T_{0,0}$) and that the set $\{x_0, x_1, \ldots, x_{r-1}\}$ is empty, so that a $(0,d)$-braxtope is a d-multiplex. The theorems in this paper have natural analogues for (r,d)-braxtopes. The $(1,d)$-braxtopes are of special interest because they arise as facets of periodically-cyclic Gale polytopes. It would be interesting to investigate polytopes, all of whose facets are (r,d)-braxtopes.

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