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The Densest Geodesic Ball Packing by a Type of Nil Lattices

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Abstract. W. Heisenberg's famous real matrix group provides a non-commutative translation group of an affine 3-space. The **Nil** geometry which is one of the eight homogeneous Thurston 3-geometries, can be derived from this matrix group. E. Molnár proved in [2], that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbb{R})$. In our work we will use this projective model of the **Nil** geometry.
In this paper we investigate the geodesic balls of the **Nil** space and compute their volume (see (2.4)), introduce the notion of the **Nil** lattice, **Nil** parallelepiped (see Section 3) and the density of the lattice-like ball packing. Moreover, we determine the densest lattice-like geodesic ball packing (Theorem 4.2). The density of this densest packing is ≈ 0.78085, may be surprising enough in comparison with the Euclidean result $\frac{\pi}{\sqrt{18}} \approx 0.74048$. The kissing number of the balls in this packing is 14.

1. On the Nil geometry

The **Nil** geometry can be derived from the famous real matrix group $\mathbf{L}(\mathbf{R})$ discovered by Werner Heisenberg. The left (row-column) multiplication of Heisenberg

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matrices

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+x & c+xb+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{pmatrix} \tag{1.1}$$

defines translations $\mathbf{L}(\mathbf{R}) = \{(x,y,z) : x,\ y,\ z \in \mathbf{R}\}$ on the points of the space $\mathbf{Nil} = \{(a,b,c) : a,\ b,\ c \in \mathbf{R}\}$. These translations are not commutative in general. The matrices $\mathbf{K}(z) \lhd \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix}1 & 0 & z\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \mapsto (0,0,z) \tag{1.2}$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called *fibre translations*. **Nil** geometry of the Heisenberg group can be projectively (affinely) interpreted by the right translations on points as the matrix formula

$$(1;a,b,c) \to (1;a,b,c)\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix} = (1; x+a, y+b, z+bx+c) \tag{1.3}$$

shows, according to (1.1). Here we consider $\mathbf{L}$ as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$, $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}$, $(\{\mathbf{e}^i\} \subset \boldsymbol{V}_4)$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0) = E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v} = \boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u}) = (\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0 = \mathbf{x}\boldsymbol{u} = \mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x}) = (\mathbf{y})$ and plane $(\boldsymbol{u}) = (\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus **NIL** can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well [4].

In this context E. Molnár [2] has derived the well-known infinitesimal arc-length square at any point of **Nil** as follows

$$\begin{gathered}(dx)^2 + (dy)^2 + (-xdy + dz)^2 = \\ (dx)^2 + (1+x^2)(dy)^2 - 2x(dy)(dz) + (dz)^2 =: (ds)^2\end{gathered} \tag{1.4}$$

Hence we get the symmetric metric tensor field g on **Nil** by components, furthermore its inverse:

$$g_{ij} := \begin{pmatrix}1 & 0 & 0\\ 0 & 1+x^2 & -x\\ 0 & -x & 1\end{pmatrix}, \quad g^{ij} := \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & x & 1+x^2\end{pmatrix}. \tag{1.5}$$

The translation group $\mathbf{L}$ defined by formula (1.3) can be extended to a larger group $\mathbf{G}$ of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of **Nil**. In [3] E. Molnár has shown that a rotation trough angle ω about the z-axis at the origin, as isometry of **Nil**, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z-image $\overline{z}$ as follows:

$$\begin{gathered}\mathbf{r}(O,\omega):(1;x,y,z)\to(1;\overline{x},\overline{y},\overline{z});\\ \overline{x}=x\cos\omega-y\sin\omega,\quad \overline{y}=x\sin\omega+y\cos\omega,\\ \overline{z}=z-\frac{1}{2}xy+\frac{1}{4}(x^2-y^2)\sin 2\omega+\frac{1}{2}xy\cos 2\omega.\end{gathered}\tag{1.6}$$

$$\begin{gathered}x\to x'=x,\quad y\to y'=y,\quad z\to z'=z-\frac{1}{2}xy\ \text{ to}\\ (1;x',y',z')\to(1;x',y',z')\begin{pmatrix}1&0&0&0\\0&\cos\omega&\sin\omega&0\\0&-\sin\omega&\cos\omega&0\\0&0&0&1\end{pmatrix}=(1;x",y",z"),\\ \text{with }\ x"\to\overline{x}=x",\quad y"\to\overline{y}=y",\quad z"\to\overline{z}=z"+\frac{1}{2}x"y",\end{gathered}\tag{1.7}$$

i.e. to the linear rotation formula. This quadratic conjugacy modifies the **Nil** translations in (1.3), as well. We shall use the following important classification theorem.

Theorem 1.1. (E. Molnár [3]) (1) *Any group of Nil isometries, containing a* 3*-dimensional translation lattice, is conjugate by the quadratic mapping in* (1.7) *to an affine group of the affine (or Euclidean) space* $\mathbf{A}^3=\mathbf{E}^3$ *whose projection onto the* (x,y) *plane is an isometry group of* $\mathbf{E}^2$*. Such an affine group preserves a plane* $\to$ *point polarity of signature* $(0,0,\pm 0,+)$*.*

(2) *Of course, the involutive line reflection about the* y *axis*

$$(1;x,y,z)\to(1;-x,y,-z),$$

preserving the Riemann metric in (1.5)*, and its conjugates by the above isometries in* (1) *(those of the identity component) are also* **Nil** *isometries. There does not exist orientation reversing* **Nil** *isometry.*

The geodesic curves of the **Nil** geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $g(x(t),y(t),z(t))$ in our model can be determined by the general theory of Riemann geometry (see [4]): We can assume, that the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (1.1);

$$\begin{gathered}x(0)=y(0)=z(0)=0;\quad \dot{x}(0)=c\cos\alpha,\ \dot{y}(0)=c\sin\alpha,\\ \dot{z}(0)=w;\ -\pi\le\alpha\le\pi.\end{gathered}$$

The arc length parameter s is introduced by

$$s = \sqrt{c^2 + w^2} \cdot t, \text{ where } w = \sin\theta,\ c = \cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

i.e. unit velocity can be assumed.

Remark 1.1. Thus we have harmonized the scales along the coordinate axes (see also formula (3.2) and Remark 3.3).

The equation systems of a helix-like geodesic curve $g(x(t), y(t), z(t))$ if $0 < |w| < 1$ [4]:

$$\begin{aligned} x(t) &= \frac{2c}{w}\sin\frac{wt}{2}\cos\Big(\frac{wt}{2}+\alpha\Big), \quad y(t) = \frac{2c}{w}\sin\frac{wt}{2}\sin\Big(\frac{wt}{2}+\alpha\Big),\\ z(t) &= wt\cdot\Big\{1+\frac{c^2}{2w^2}\Big[\Big(1-\frac{\sin(2wt+2\alpha)-\sin 2\alpha}{2wt}\Big)\\ &+\Big(1-\frac{\sin(2wt)}{wt}\Big)-\Big(1-\frac{\sin(wt+2\alpha)-\sin 2\alpha}{2wt}\Big)\Big]\Big\} = \\ &= wt\cdot\Big\{1+\frac{c^2}{2w^2}\Big[\Big(1-\frac{\sin(wt)}{wt}\Big)+\Big(\frac{1-\cos(2wt)}{wt}\Big)\sin(wt+2\alpha)\Big]\Big\}. \end{aligned} \tag{1.8}$$

In the cases $w = 0$ the geodesic curve is the following:

$$x(t) = c\cdot t\cos\alpha,\ y(t) = c\cdot t\sin\alpha,\ z(t) = \frac{1}{2}c^2\cdot t^2\cos\alpha\sin\alpha. \tag{1.9}$$

The cases $|w| = 1$ are trivial: $(x, y) = (0, 0),\ z = w\cdot t$. In Figure 1 it can be seen

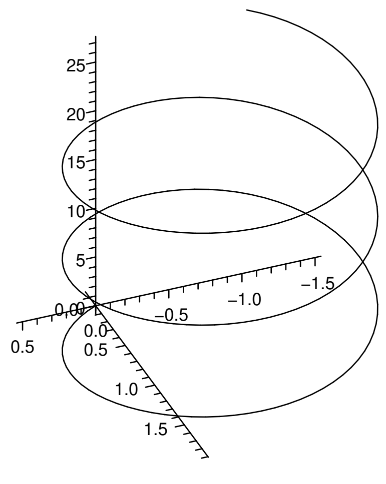

Figure 1

a **Nil** geodesic curve with parameters $\alpha = \frac{\pi}{6}$, $\theta = \frac{\pi}{4}$, $t \in [0, 8\pi]$.

Definition 1.1. *The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the shortest geodesic curve from P_1 to P_2.*

2. The geodesic ball

Definition 2.1. *The geodesic sphere of radius R with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1, P_2) = R$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in the* **Nil** *space.*

Remark 2.1. We shall see that this last condition depends on radius R.

Definition 2.2. *The body of the geodesic sphere of centre P_1 and of radius R in the* **Nil** *space is called geodesic ball, denoted by $B_{P_1}(R)$, i.e. $Q \in B_{P_1}(R)$ iff $0 \le d(P_1, Q) \le R$.*

Remark 2.2. Henceforth, typically we choose the origin as centre of the sphere and its ball, by the homogeneity of **Nil**.

Figure 2a shows a geodesic sphere of radius $R = 4$ with centre at the origin.

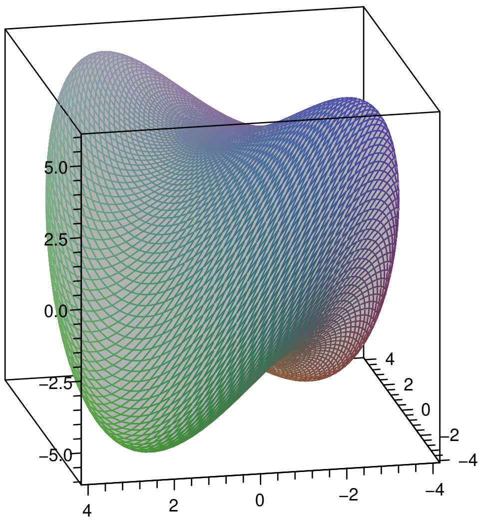

Figure 2a

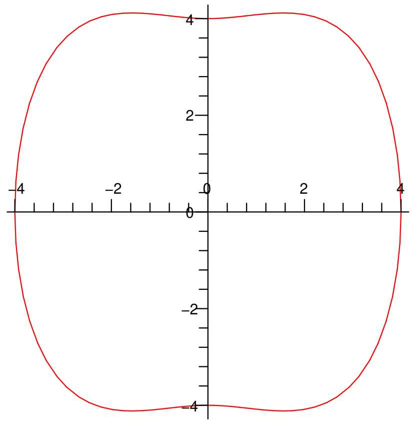

Figure 2b

We apply the quadratic mapping $\mathcal{M} : \mathbf{Nil} \longrightarrow \mathbf{A}^3$ at (1.6) to the geodesic sphere S, its $\mathcal{M}$-image is denoted by $S' = \mathcal{M}(S)$.

Consider a point $P(x(R,\theta,\alpha), y(R,\theta,\alpha), z(R,\theta,\alpha))$ lying on a sphere S of radius R with centre at the origin. The coordinates of P are given by parameters ($\alpha \in [-\pi, \pi)$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $R > 0$) (see (1.8), (1.9)), its $\mathcal{M}$-image is $P'(x'(R,\theta,\alpha), y'(R,\theta,\alpha), z'(R,\theta,\alpha)) \in S'$ where

$$\begin{gathered}
x'(R,\theta,\alpha) = \frac{2c}{w} \sin \frac{wR}{2} \cos\Big(\frac{wR}{2} + \alpha\Big), \\
y'(R,\theta,\alpha) = \frac{2c}{w} \sin \frac{wR}{2} \sin\Big(\frac{wR}{2} + \alpha\Big), \\
z'(R,\theta,\alpha) = wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR, \quad (\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}), \\
\text{if } \theta = 0 \text{ then } x'(R,0,\alpha) = R\cos\alpha, \\
y'(R,0,\alpha) = R\sin\alpha, \;\; z'(R,0,\alpha) = 0.
\end{gathered} \tag{2.1}$$

We can see from the last equations that $(x')^2 + (y')^2 = \frac{4c^2}{w^2} \sin^2 \frac{wR}{2}$ and that the z'-coordinate does not depend on the parameter α, therefore S' can be generated by rotating the following curve about the z axis (lying in the plane $[x, z]$):

$$
\begin{gathered}
X(R,\theta) = \frac{2c}{w} \sin \frac{wR}{2} = \frac{2\cos\theta}{\sin\theta} \sin \frac{R\sin\theta}{2}, \\
Z(R,\theta) = wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR = \\
R\sin\theta + \frac{R\cos^2\theta}{2\sin\theta} - \frac{\cos^2\theta}{2\sin^2\theta} \sin(R\sin\theta), \quad (\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}); \\
\text{if } \theta = 0 \text{ then } X(R,0) = R, \; Z(R,0) = 0.
\end{gathered}
\tag{2.2}
$$

Remark 2.3. From the definition of the quadratic mapping $\mathcal{M}$ at (1.6) it follows that the cross section of the spheres S and S' with the plane $[x, z]$, is the same curve (see Fig. 2b, $R = 4$) which is specified by the parametric equations (2.2).

Remark 2.4. The parametric equations of the geodesic sphere of radius R can be generated from (2.2) by **Nil** rotation (see (1.7)).

2.1. The existence of the geodesic ball

We have denoted by $B(S)$ the body of the **Nil** sphere S and by $B(S')$ the body of the sphere S', furthermore we have denoted their volumes by $Vol(B(S))$ and $Vol(B(S'))$, respectively.

From (1.8), (1.9), (2.1) and (2.2) it can be seen that a geodesic ball $B(S)$ in the **Nil** space exists if and only if its image $B(\mathcal{M}(S)) = B(S')$ exists in $\mathbf{A}^3$, therefore it is sufficient to discuss the existence of $B(S') \in \mathbf{A}^3$. From (2.2) follows that S' is a simply connected surface in $\mathbf{A}^3$ if and only if $R \in [0, 2\pi]$, because if $R > 2\pi$ then there is at least one $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$ so that $X(R, \theta) = 0$, i.e. selfintersection would occur (see (2.2)). Thus we obtain the following theorem:

Theorem 2.1. *The geodesic sphere and ball of radius R exists in the* **Nil** *space if and only if $R \in [0, 2\pi]$.*

Remark 2.5. From the above considerations follows, that the triangle inequality does not hold in the **Nil** space, in general.

2.2. The volume of the geodesic ball

The Jacobi matrix of the quadratic mapping $\mathcal{M}$ at (1.6) is

$$
J(\mathcal{M}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}y & -\frac{1}{2}x & 1 \end{pmatrix}, \text{ i.e. } \det(J(\mathcal{M})) = 1,
\tag{2.3}
$$

therefore $Vol(B(S)) = Vol(B(S'))$. Thus we obtain the volume of the geodesic ball of radius R by the following integral (see 2.2):

$$Vol(B(S)) = 2\pi \int_0^{\frac{\pi}{2}} X^2 \, \frac{\mathrm{d}\ Z}{\mathrm{d}\ \theta} \, \mathrm{d}\ \theta =$$
$$= 2\pi \int_0^{\frac{\pi}{2}} \Big(\frac{2\cos\theta}{\sin\theta} \sin\frac{(R\sin\theta)}{2}\Big)^2 \cdot \Big(-\frac{1}{2}\frac{R\cos^3\theta}{\sin^2\theta} + \frac{\cos\theta\sin(R\sin\theta)}{\sin\theta} + \frac{\cos^3\theta\sin(R\sin\theta)}{\sin^3\theta} - \frac{1}{2}\frac{R\cos^3\theta\cos(R\sin\theta)}{\sin^2\theta}\Big)\mathrm{d}\ \theta. \tag{2.4}$$

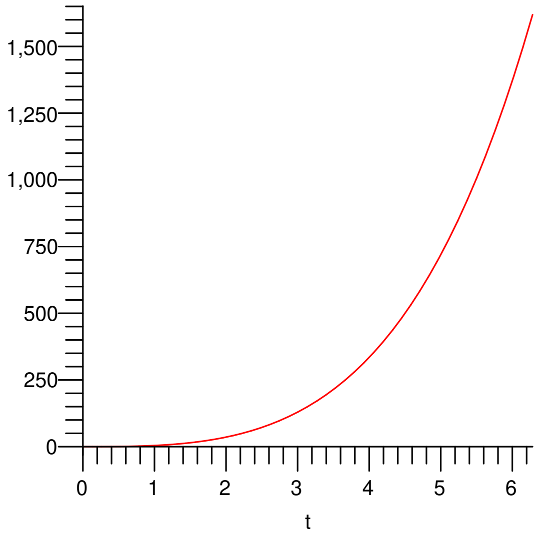

Figure 3

Figure 3 illustrates that the function $R \mapsto Vol(B(S(R)))$ is strictly increasing in the interval $[0, 2\pi]$.

2.3. The convexity of the Nil ball in the model

In this section we examine the convexity of the geodesic ball in Euclidean sense in our affine model. The **Nil** sphere of radius R is generated by the **Nil** rotation about the axis z (see the equation system (2.2) and remarks (2.1), (2.2)). The parametric equation system of the geodesic sphere $S(R)$ in our model:

$$x(R,\theta,\phi) = \frac{2c}{w}\sin\frac{wR}{2}\cdot\cos\phi = \frac{2\cos\theta}{\sin\theta}\sin\frac{R\sin\theta}{2}\cdot\cos\phi,$$
$$y(R,\theta,\phi) = \frac{2c}{w}\sin\frac{wR}{2}\cdot\sin\phi = \frac{2\cos\theta}{\sin\theta}\sin\frac{R\sin\theta}{2}\cdot\sin\phi,$$
$$z(R,\theta,\phi) = wR + \frac{c^2R}{2w} - \frac{c^2}{2w^2}\sin wR + \frac{1}{4}\Big(\frac{2c}{w}\sin\frac{wR}{2}\Big)^2\sin 2\phi =$$
$$= R\sin\theta + \frac{R\cos^2\theta}{2\sin\theta} - \frac{\cos^2\theta}{2\sin^2\theta}\sin(R\sin\theta) + \frac{1}{4}\Big(\frac{2\cos\theta}{\sin\theta}\sin R\frac{\sin\theta}{2}\Big)^2\sin 2\phi$$

$$
\begin{gathered}
-\pi < \phi \leqq \pi, \quad -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \text{ and } \theta \neq 0. \\
\text{if } \theta = 0 \text{ then } x(R,0,\phi) = R\cos\phi, \quad y(R,0,\phi) = R\sin\phi, \\
z(R,0,\phi) = \frac{1}{2}\, R^2 \cos\phi \sin\phi.
\end{gathered} \tag{2.5}
$$

We have obtained by the derivatives of these parametrically represented functions (by intensive and careful computations with *Maple* through the second fundamental form) the following theorem:

Theorem 2.2. *The geodesic* **Nil** *ball* $B(S(R))$ *is convex in affine-Euclidean sense in our model if and only if* $R \in [0, \frac{\pi}{2}]$.

3. The discrete translation group L(Z)

We consider the **Nil** translations defined in (1.1) and (1.3) and choose two arbitrary translations

$$
\tau_1 = \begin{pmatrix} 1 & t_1^1 & t_1^2 & t_1^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_1^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tau_2 = \begin{pmatrix} 1 & t_2^1 & t_2^2 & t_2^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}
$$

now with upper indices for coordinate variables. We define the translation τ_3 by the following commutator:

$$
\tau_3 = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1 = \begin{pmatrix} 1 & 0 & 0 & -t_2^1 t_1^2 + t_1^1 t_2^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}
$$

If we take integers as coefficients, their set is denoted by $\mathbb{Z}$, then we generate the discrete group $\langle \tau_1, \tau_2 \rangle$ denoted by $\mathbf{L}(\tau_1, \tau_2)$ or by $\mathbf{L}(\mathbb{Z})$. (See also Remark 1.1.)

We know (see e.g. [4]) that the orbit space $\mathbf{Nil}/\mathbf{L}(\mathbb{Z})$ is a compact manifold, i.e. a **Nil** space form.

Definition 3.1. *The* **Nil** *point lattice* $\Gamma_P(\tau_1, \tau_2)$ *is a discrete orbit of point* P *in the* **Nil** *space under the group* $\mathbf{L}(\tau_1, \tau_2) = \mathbf{L}(\mathbb{Z})$ *with an arbitrary starting point* P.

Remark 3.1. For simplicity we have chosen the origin as starting point, by the homogeneity of **Nil**.

Remark 3.2. We can assume that $t_1^2 = 0$, i.e. the image of the origin by the translation τ_1 lies on the plane $[x, z]$.

Remark 3.3. We may also assume that the centre $\mathbf{K}(\mathbb{Z})$ of $\mathbf{L}(\mathbb{Z})$ is generated by $\tau_3 = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1$. τ_3 is a so-called fibre translation. (See also Remark 1.1.)

We illustrate the action of $\mathbf{L}(\mathbb{Z})$ on the **Nil** space in Figure 4. We consider a non-convex polyhedron $\mathcal{F} = OT_1T_2T_3T_{12}T_{21}T_{23}T_{213}T_{13}$, in Euclidean sense, which is determined by translations τ_1, τ_2, τ_3. This polyhedron determines a solid $\widetilde{\mathcal{F}}$ in the **Nil** space whose images under $\mathbf{L}(\mathbb{Z})$ fill the **Nil** space just once, i.e. without gap and overlap.

Analogously to the Euclidean integer lattice and parallelepiped, the solid $\widetilde{\mathcal{F}}$ can be called **Nil** *parallelepiped.*

$\widetilde{\mathcal{F}}$ is a fundamental domain of $\mathbf{L}(\mathbb{Z})$. The homogeneous coordinates of the vertices of $\widetilde{\mathcal{F}}$ can be determined in our affine model by the translations (3.1) and (3.2) with the parameters $t_i^j,\ i \in \{1,2\},\ j \in \{1,2,3\}$ (see Fig. 4 and (3.3)).

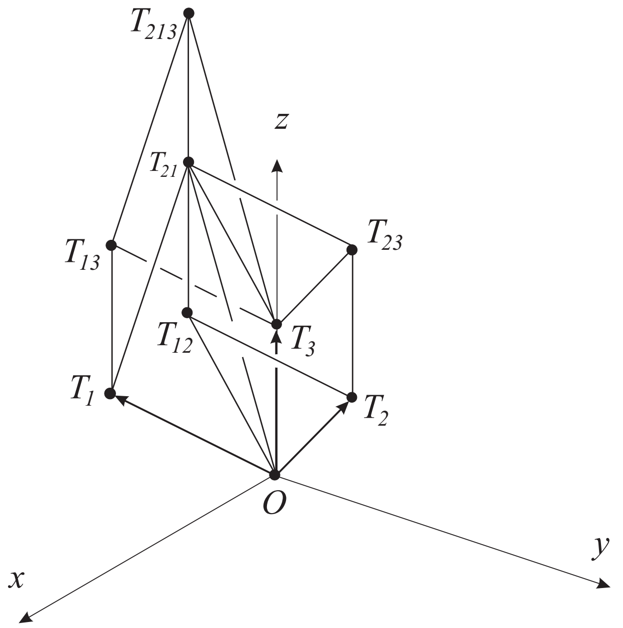

Figure 4

$$\begin{gathered} T_1(1, t_1^1, 0, t_1^3),\ T_2(1, t_2^1, t_2^2, t_2^3),\ T_3(1, 0, 0, t_1^1 t_2^2), \\ T_{13}(1, t_1^1, 0, t_1^1 t_2^2 + t_1^3),\ T_{12}(1, t_1^1 + t_2^1, t_2^2, t_2^3 + t_1^3), \\ T_{21}(1, t_1^1 + t_2^1, t_2^2, t_1^1 t_2^2 + t_1^3 + t_2^3),\ T_{23}(1, t_2^1, t_2^2, t_2^3 + t_1^1 t_2^2), \\ T_{213} = T_{231}(1, t_1^1 + t_2^1, t_2^2, 2t_1^1 t_2^2 + t_1^3 + t_2^3). \end{gathered} \tag{3.3}$$

3.1. The volume of the Nil parallelepiped $\widetilde{\mathcal{F}}$

The volume of the **Nil** solid $\widetilde{\mathcal{T}_\tau} = T_3T_{21}T_{13}T_{213}$, which is a tetrahedron in Euclidean sense (see Fig. 4), is equal to the volume of $\widetilde{\mathcal{T}} = OT_{12}T_1T_{21}$ because $\tau_3(\widetilde{\mathcal{T}}) = \widetilde{\mathcal{T}_\tau}$, thus the volume of the **Nil** parallelepiped $\widetilde{\mathcal{F}}$ is equal to the volume of the **Nil** solid $\widetilde{\mathcal{P}} = OT_1T_{12}T_2T_3T_{13}T_{21}T_{23}$. Therefore, (2.3)

$$Vol(\widetilde{\mathcal{F}}) = Vol(\widetilde{\mathcal{F}}') = Vol(\widetilde{\mathcal{P}}) = Vol(\widetilde{\mathcal{P}}'), \tag{3.4}$$

where $\widetilde{\mathcal{F}}' = \mathcal{M}(\widetilde{\mathcal{F}})$ and $\widetilde{\mathcal{P}}' = \mathcal{M}(\widetilde{\mathcal{P}}')$. Thus we obtain the volume of $\widetilde{\mathcal{F}}$ by $\det(\overrightarrow{OT}_1, \overrightarrow{OT}_2, \overrightarrow{OT}_3) = (t_1^1 \cdot t_2^2)^2$ from (3.3) or by the following integral:

$$Vol(\widetilde{\mathcal{F}}) = \int_0^{t_2^2} \int_0^{t_1^1} |t_1^1 \cdot t_2^1| \, \mathrm{d}x \mathrm{d}y = (t_1^1 \cdot t_2^2)^2. \tag{3.5}$$

From this formula it can be seen that the volume of the **Nil** parallelepiped depends on two parameters, i.e. on its projection onto the $[x, y]$ plane.

4. The lattice-like geodesic ball packings

Let $\mathcal{B}_\Gamma(R)$ denote a geodesic ball packing of **Nil** space with balls $B(R)$ of radius R where their centres give rise to a **Nil** point lattice $\Gamma(\tau_1, \tau_2)$. $\widetilde{\mathcal{F}}_0$ is an arbitrary **Nil** parallelepiped of this lattice (see (3.1),(3.2)). The images of $\widetilde{\mathcal{F}}_0$ by our discrete translation group $\mathbf{L}(\tau_1, \tau_2)$ cover the **Nil** space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $\widetilde{\mathcal{F}}_0$. Analogously to the Euclidean case it can be defined the density $\delta(R, \tau_1, \tau_2)$ of the lattice-like geodesic ball packing $\mathcal{B}_\Gamma(R)$:

Definition 4.1.

$$\delta(R, \tau_1, \tau_2) := \frac{Vol(\mathcal{B}_\Gamma(R) \cap \widetilde{\mathcal{F}}_0)}{Vol(\widetilde{\mathcal{F}}_0)}, \tag{4.1}$$

if the balls do not overlap each other.

Remark 4.1. By definition of the **Nil** lattice $\mathbf{L}(\tau_1, \tau_2)$(see Definition 3.1) the orbit space $\mathbf{Nil}/\mathbf{L}(\tau_1, \tau_2)$ is a compact **Nil** manifold, and (see Section 2),

$$Vol(\mathcal{B}_\Gamma(R) \cap \widetilde{\mathcal{F}}_0) = Vol(B(S(R))).$$

4.1. The optimal lattice-like ball packing

We look for such an arrangement $\mathcal{B}_\Gamma(R)$ of balls $B(R)$, (see Fig. 4) where the following equations hold:

$$\begin{array}{ll} \text{(a)} & d(O, T_1) = 2R = d(T_1, T_3), \\ \text{(b)} & d(O, T_2) = 2R = d(T_2, T_3), \\ \text{(c)} & d(T_1, T_2) = 2R, \\ \text{(d)} & d(O, T_3) = 2R. \end{array} \tag{4.2}$$

Here d is the distance function in the **Nil** space (see Definition 1.1). The equations (a) and (b) mean that the ball centres T_1 and T_2 lie on the equidistant geodesic surface of the points O and T_3 which is a hyperbolic paraboloid (see (1.9) and Fig. 5) in our model with equation

$$2z - xy = 2R.$$

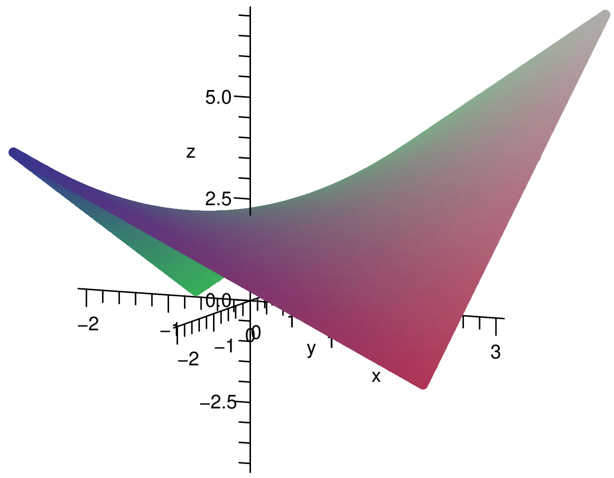

Figure 5

By continuity of the distance function it follows, that there is a (unique) solution of the equation system (4.2). We have denoted by $\mathcal{B}_\Gamma^{opt}(R_{opt})$ the geodesic ball packing of the balls $B(R_{opt})$ which satisfies the above equation system. We get the following solution by systematic approximation, where the computations were carried out by *Maple V Release 10* up to 30 decimals:

$$\begin{gathered} t_1^{1,opt} \approx 1.30633820,\ t_1^{3,opt} = R_{opt},\ R_{opt} \approx 0.73894461; \\ t_2^{1,opt} \approx 0,65316910,\ t_2^{2,opt} \approx 1,13132206,\ t_2^{3,opt} \approx 1.10841692, \\ T_1^{opt} = (1, t_1^{1,opt}, 0, t_1^{3,opt}),\quad T_2^{opt} = (1, t_2^{1,opt}, t_2^{2,opt}, t_2^{3,opt}). \end{gathered} \tag{4.3}$$

The geodesic ball packing $\mathcal{B}_\Gamma^{opt}(R_{opt})$ can be realized in **Nil** space because by Theorem 2.2 a ball of radius $R_{opt} \approx 0.73894461$ is convex in Euclidean sense and this packing can be generated by the translations $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt})$ where τ_1^{opt} and τ_2^{opt} are given by the coordinates $t_i^{j,opt}$ $i = 1, 2;\ j = 1, 2, 3$ (see (4.3) and (3.1)). Thus we obtain the neighbouring balls around an arbitrary ball of the packing $\mathcal{B}_\Gamma^{opt}(R_{opt})$ by the lattice $\Gamma(\tau_1^{opt}, \tau_2^{opt})$, the kissing number of the balls is 14. Fig. 6 shows the typical arrangement of some balls from $\mathcal{B}_\Gamma^{opt}(R_{opt})$ in our model. We have ball "columns" in z-direction and in regular hexagonal projection onto the $[x, y]$-plane.

The fundamental domain $\widetilde{\mathcal{F}}_0^{opt}$ of the discrete translation group $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt})$ is a **Nil** parallelepiped of the above determined **Nil** lattice $\Gamma(\tau_1^{opt}, \tau_2^{opt})$. By formulas (2.4), (3.5) and by Definition 4.1 we can compute the density of this ball packing:

$$\begin{gathered} Vol(\widetilde{\mathcal{F}}_0^{opt}) \approx 2.18415656, \quad Vol(\mathcal{B}_\Gamma(R_{opt}) \cap \widetilde{\mathcal{F}}_0^{opt}) \approx 1.70548775, \\ \delta(R_{opt}, \tau_1,^{opt}, \tau_2^{opt}) := \frac{Vol(\mathcal{B}_\Gamma(R_{opt}) \cap \widetilde{\mathcal{F}}_0^{opt})}{Vol(\widetilde{\mathcal{F}}_0^{opt})} \approx 0.78084501. \end{gathered} \tag{4.4}$$

Theorem 4.1. *The ball arrangement $\mathcal{B}_\Gamma^{opt}(R_{opt})$ given in formulas* (4.3), (4.4) *provides the optimal lattice-like geodesic ball packing in the* **Nil** *space.*

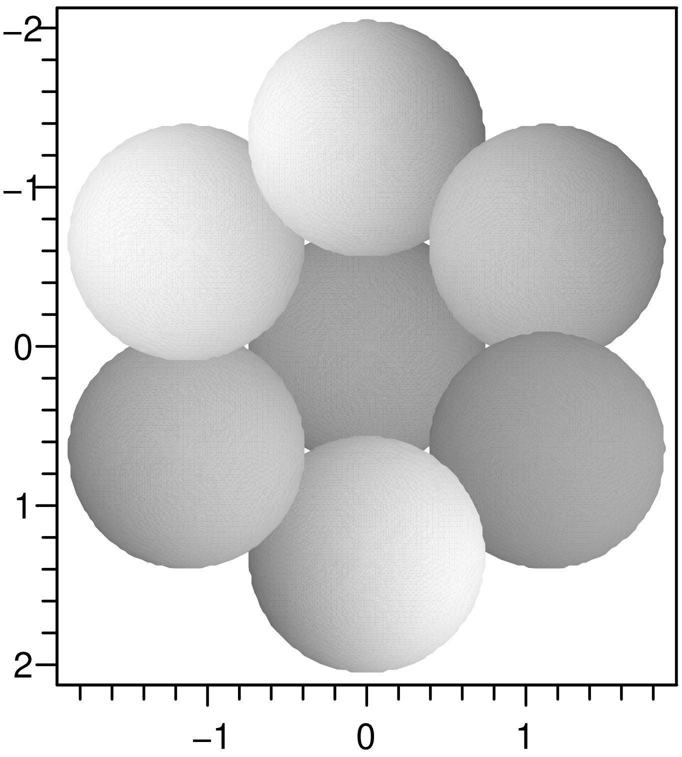

Figure 6

Proof. Let $\mathcal{B}_\Gamma(R)$ denote a geodesic ball packing of **Nil** space with balls $B(R)$ of radius R where their centres give rise to a **Nil** point lattice $\Gamma(\tau_1, \tau_2)$ (see 3.1). If we give the distance $d(O, T_3) = |t_1^1 \cdot t_2^2|$ then we fix the volume of the **Nil** parallelepiped generated by translations $\tau_1, \ \tau_2$. Thus for choosing the radius R of the balls in $\mathcal{B}_\Gamma(R)$ we have to minimalize the distance $d(O, T_3) = |t_1^1 \cdot t_2^2|$ so that we achieve the densest lattice-like geodesic ball packing. From the properties of the balls and of the **Nil** lattices we have the necessary conditions:

$$\begin{aligned} &(a) \ \ d(O,T_3) \geq 2R, \ d(T_1,T_3) \geq 2R, \\ &(b) \ \ d(O,T_2) \geq 2R, \ d(T_2,T_3) \geq 2R, \\ &(c) \ \ d(T_1,T_2) \geq 2R, \ d(O,T_1) \geq 2R. \end{aligned} \tag{4.5}$$

In our proof we consider two cases.

1. $R \in (0, R_{opt}]$

We have to minimalize the distance $d(O, T_3)$ to a given R. This distance is minimal if $d(O, T_3) = |t_1^1 \cdot t_2^2| = 2R$, i.e. the balls $B_{T_3}(R)$ and $B_O(R)$ touch each other. In these cases the volume of the **Nil** parallelepiped of the lattice $\Gamma(\tau_1, \tau_2)$ is $Vol(\widetilde{\mathcal{F}}_0) = 4R^2$. Thus we have to examine the density function (see Definition 4.1 and Remark 4.1)

$$\delta(R, \tau_1, \tau_2) := \frac{Vol(B(S(R)))}{4R^2}. \tag{4.6}$$

Figure 8 shows the increasing density function in these cases by formulas (2.4) and (4.6).

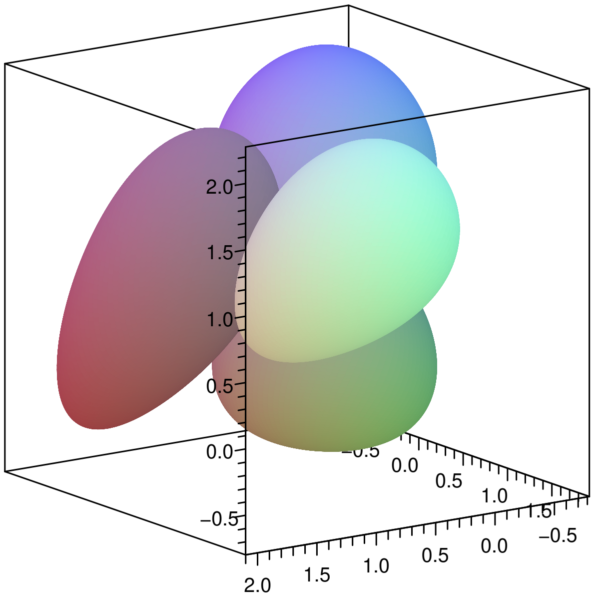

Figure 7

The following inequalities are true for parameters $t_1^1 > 0$ and $t_2^2 > 0$:

$$t_1^1 \geq R, \quad t_2^2 \geq R \Rightarrow t_1^1 \leq 2,$$

$$\text{if} \quad t_1^1 \leq 2R, \quad \text{then} \quad t_2^2 \geq 1, \quad \text{and if} \quad \frac{2R}{t_1^1} = t_2^2 \geq 2R, \quad \text{then } t_1^1 \leq 1. \tag{4.7}$$

The ball packings satisfying the above conditions can be realized in the **Nil** space, where the centers T_1 and T_2 lie on the equidistance surface of O and T_3 (at (4.2)) and the balls $B_{T_1}(R)$, $B_O(R)$ and $B_{T_3}(R)$ touch each other. Moreover, if we increase the radius of the balls in the interval $(0, R_{opt}]$, $B_{T_2}(R)$ approaches towards the above balls (see 4.7):

$$\lim_{R \to R_{opt}} T_1 = T_1^{opt}, \quad \lim_{R \to R_{opt}} T_2 = T_2^{opt}. \tag{4.8}$$

In case $R = R_{opt}$ we just obtain the ball arrangement $\mathcal{B}_\Gamma^{opt}(R_{opt})$.

2. $R' \in (R_{opt}, 2\pi]$

Similarly to the above case in 1. we have to minimalize the distance $d(O, T_3)$ to a given R'. We consider an arbitrary lattice-like ball packing $\mathcal{B}_\Gamma(R')$ of the balls $B(R')$ to a given $R' \in (R_{opt}, 2\pi]$ in the **Nil** space generated by the translations $\mathbf{L}(\tau_1, \tau_2)$ i.e. by the balls $B_{T_1}(R')$ and $B_{T_2}(R')$. By moving the balls $B_{T_1}(R')$ and $B_{T_2}(R')$ and by decreasing the distance $d(O, T_3)$ it can be achieved that the centres T_1 and T_2 lie on the equidistance surface of O and T_3 and the equations (4.2) (a), (b), (c) hold. We have denoted this ball arrangement by $\mathcal{B}_\Gamma(R)$ and its density by $\delta(R)$.

In the above cases the existence of the ball packing $\mathcal{B}_\Gamma(R)$ is not guaranteed yet, because the balls are non-convex in Euclidean sense if $R \in (\frac{\pi}{2}, 2\pi]$ (see Theorem 2.2) and the triangle inequality does not hold in general.

If the density of an arbitrary ball packing $\mathcal{B}_\Gamma(R')$ is denoted by $\delta'(R')$, then $\delta'(R') \le \delta(R)$, of course. Thus, it is sufficient to prove, that $\delta(R) \le \delta(R_{opt}, \tau_1^{opt}, \tau_2^{opt})$ if $R \in (R_{opt}, 2\pi]$.

In these cases the balls $B_O(R)$ and $B_{T_3}(R)$ do not touch each other, $d(O, T_3) > 2R$, (see (4.7) and (4.8)). The density of the arrangement $\mathcal{B}_\Gamma(R)$ of the balls $B(R)$ to a given $R \in (R_{opt}, 2\pi]$ is

$$\delta(R, \tau_1, \tau_2) := \frac{Vol(B(S(R)))}{(d(O, T_3))^2}. \tag{4.9}$$

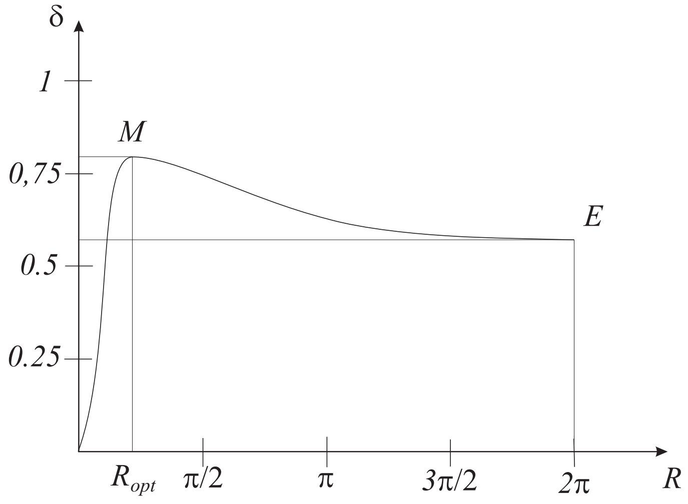

Figure 8

Considering these ball packings, we have obtained that the density function $\delta(R,\tau_1, \tau_2)$ decreases in the interval $(R_{opt}, 2\pi]$. The graph of $\delta(R, \tau_1, \tau_2)$ and its maximum point $M(R_{opt}, \delta(R_{opt}, \tau_1^{opt}, \tau_2^{opt}))$ can be seen in Figure 8.

Remark 4.2. The coordinates of the "endpoint" of the density function δ are: $E(2\pi, \approx 0.57013836)$. The volume of this ball is 1619.19850921.

If $R \in (R_{opt}, 2\pi]$ and the ball $B_{T_1}(R)$ does not touch the balls $B_O(R)$ and $B_{T_3}(R)$ then we have not obtained larger density than $\delta(R_{opt}, \tau_1^{opt}, \tau_2^{opt})$. □

In this paper we have mentioned only some problems in discrete geometry of the **Nil** space, but we hope that from these it can be seen that our projective method suits to study and solve similar problems (see [1]). Analogous questions in **Nil** geometry or, in general, in other homogeneous Thurston [7] geometries are on our program with E. Molnár and I. Prok.

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