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Envelopes – Notion and Definiteness

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Abstract. We consider a 1-parameter family of surfaces in 3-space. We use the method of Synthetic Differential Geometry to vindicate the classical but disreputed description of the characteristic curves as "limit intersection curves". The enveloping surface of the family is then synthesized from the characteristic curves.

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Introduction

We examine critically some of the existing descriptions of the envelope of a 1-parameter family of surfaces in 3-space. An old, natural, description is that the envelope is the union of the characteristic curves (defined as the "limit intersection curves" of surfaces from the family). This description fell in disrepute during the 20th century. We claim that this relegation of the limit intersection curves made some other aspect of the theory less justified, namely the definite article "the" in the phrase "the envelope". We intend to re-establish the limit intersection curves on a rigorous basis, – thereby also re-establishing the old obvious argument for the definiteness of envelopes.

The content of the present note was presented at a talk at the Logic Year at the Mittag-Leffler Institute in Stockholm in April 2001. The key formula (1) was found during discussions with Gonzalo Reyes.

1. Three descriptions of envelopes

One considers a 1-parameter family of surfaces $\{M_\tau\}$ in 3-space. Under suitable non-degeneracy conditions, this family has an enveloping surface E, classically described in three alternative ways:

1) The synthetic: E is "the" union of "the" characteristics; the characteristic C_τ is *the limit curve* of the family of curves $M_\tau \cap M_{\tau+h}$ as $h \to 0$.
2) The impredicative: E is "a" surface with the property that at each of its points, it is tangent to a unique surface from the given family. (The locus of points where E touches M_τ, we call the E-characteristic C^E_τ, for the sake of the comparison.) (Also, E should touch each M_τ.)
3) The analytic: one assumes that there is "a" function $F(x, y, z, \tau)$ such that for each τ, M_τ is the zero set of $F(-,-,-,\tau)$. Then the surface E is the union of the F-*discriminant curves*, where the F-discriminant curve C^F_τ for the parameter value τ is the solution set for the F-discriminant equations

$$F(x, y, z, \tau) = 0, \quad F_\tau(x, y, z, \tau) = 0$$

(where F_τ denotes $\frac{\partial F}{\partial \tau}$).

The two first descriptions are purely geometric. There are problems with each of the three descriptions.

ad 1) For the synthetic description, it is not clear what "limit curve" should mean: the set of (unparametrized) curves in 3-space does not carry a natural evident topology with respect to which this limit notion can refer. The assertion of Courant "If we let h tend to zero, the curve of intersection will approach a definite limiting position ..." (a similar description is in [2] p. 320 ff. or in [3] p. 249) is not supported by a rigorous notion. (We supply such notion in (1) below.)

ad 2) The impredicative description is problematic in that uniqueness is not a priori clear. (Nor is existence, and in fact the description does not imply a construction method. Note that in this description, the characteristics are described in terms of the envelope, rather than the other way round.)

ad 3) For the analytic description, one is imposing some further structure on the given geometric data, namely the function F. The problem now is that is not a priori clear that the resulting surface is independent of the choice of F.

The theory usually given, cf. e.g. [1], [7], consists in the following:

(A) Given an analytic presentation $F(x, y, z, \tau)$ for the family, as above, it is proved that the discriminant curves make up "an" (impredicative) envelope.

(B) Given an impredicative envelope E, and an analytic presentation F, for the family, the E-characteristics must be solutions of the F-discriminant equations – provided that a compatible parametrization by τ and one further parameter t exists for E (cf. [7] p. 369–370; "compatible parametrization" means that E can be parametrized smoothly by two parameters t, τ such that for each τ_0, the t-parameter curve $\tau = \tau_0$ is the E-characteristic $C^E_{\tau_0}$).

So the notion of envelope is given in terms of 2) and 3), with 1) only playing the role of heuristics. Uniqueness (definiteness) of the envelope comes about by playing the E and F out against each other in (B); and thus the uniqueness proof depends on this "interplay", as well as on existence of an auxiliary compatible parametrization of E. The reason why the sources cited hardly pay attention to this somewhat sophisticated definiteness question is presumably that they have the synthetic description 1) in mind, even though, officially, they have relegated it. Definiteness is clear, and elementary, from the synthetic description 1), i.e. from the ability to describe the characteristic curves prior to the description of the enveloping surface.

(For the case of a 1-parameter family of plane curves, the synthetic description is less problematic: two neighbour curves usually intersect in a point, and the notion of limit point, unlike limit curve, is less problematic. But even though the synthetic description thus is feasible, the sources cited want to circumvent it. Ostrowski [7] and Courant [1] even provide an example to show that it gives incorrect results (we comment on this "counterexample" in Example 1 below).)

My contention is: analytic geometry of today has not (till now, to the best of my knowledge) furnished a rigorous meaning to 1) (and thus no elementary proof of definiteness of envelopes); and further: the method of Synthetic Differential Geometry has the ability to furnish such a description, in fact, we shall present one:

Namely, the characteristic (limit intersection curve) C_τ may be described by the equation

$$C_\tau = \bigcap_{d \in D} M_{\tau+d}, \tag{1}$$

where D is the set of $d \in \mathbf{R}$ with $d^2 = 0$. (We are assuming the basic axiom of Synthetic Differential Geometry, asserting sufficiently many such real numbers d with $d^2 = 0$ – the precise meaning will be recalled in the proof below.)

Note that the description (1) is very close to the one of Courant: "This curve is often referred to in a non-rigorous but intuitive way as the intersection of "neighbouring" surfaces of the family" (loc.cit. p. 180) – except that it is rigorous.

An analogous description gives the characteristic (limit intersection point), for the case of a 1-parameter family of curves in the plane. We should stress that the characteristic, as defined by formula (1) is not always a curve, respectively a point; – see again Example 1 below.

2. Synthetic theory of characteristics

The synthetic description 1) of the characteristics depends neither on an analytic presentation, nor on an envelope; rather, the envelope is synthesized from the characteristics, as their union. The most recent text I know of, where the synthetic description is taken as the mathematical (not just heuristic) starting point for the theory of envelopes, is [3], §46 (or §12, for the case of families of curves in the plane). The characteristic is here also called *Grænseskæringskurve* ("limit intersection curve"), and the meaning of this term is the only problematic point in this otherwise very lucid exposition.

We intend here to vindicate the synthetic theory, by the description (1) of the limit intersection curve C_τ. As thus defined, it is indeed an intersection of M_τ with some of the neighbour surfaces, namely with all the $M_{\tau+d}$'s where $d^2 = 0$.

We shall prove the correctness of the definition, and make the requisite comparison.

The notion of analytic presentation F, and the resulting discriminant-curves C^F_τ, are as in the previous section.

Theorem 2.1. *Assume given an analytic presentation F of the family of surfaces. Then the F-discriminant curves agree with the limit intersection curves of* (1)*: for each τ,*

$$C^F_\tau = C_\tau$$

(and hence they are independent of the choice of F).

Proof. By assumption, $M_\tau = \{(x, y, z) \mid F(x, y, z, \tau) = 0\}$. By Taylor expansion,

$$F(x, y, z, \tau + d) = F(x, y, z, \tau) + d \cdot F_\tau(x, y, z, \tau),$$

for $d^2 = 0$. So $(x, y, z) \in \bigcap_{d\in D} M_{\tau+d}$ iff

$$\text{for all } d \in D,\ \ F(x, y, z, \tau) + d \cdot F_\tau(x, y, z, \tau) = 0.$$

Consider this expression (for fixed x, y, z, τ) as a real valued function of $d \in D$. The basic ("Kock-Lawvere"-) axiom of [4] says that such real-valued function on D is of the form $d \mapsto a + d \cdot b$, for unique numbers a and b. We conclude that $(x, y, z) \in \bigcap_{d\in D} M_{\tau+d}$ iff $F(x, y, z, \tau) = 0$ and $F_\tau(x, y, z, \tau) = 0$, which are the defining equations for the discriminant curve.

Since both the impredicative and the synthetic descriptions of envelopes are independent of analytic data, one should be able to prove that the synthetic E (constructed from the characteristics) has the property requested of an impredicative envelope. In other words, one should be able to prove that if $x \in E \cap M_\tau$, then $T_x(E) = T_x(M_\tau)$, without any involvement of analytic presentation F. In verbal formulation, translated almost literally from Hadamard's textbook [2] p. 320 ff.:

Theorem 2.2. *The [synthetic] envelope is tangent to each enveloped surface, along the corresponding characteristic.*

The method of Synthetic Differential Geometry does provide a proof not involving any F. We shall give such a proof (under a mild non-degeneracy assumption: that the envelope is the disjoint union of the characteristics. Example 1 below shows that this is not necessarily so). We shall not go into details about the foundations of the method (the reader is referred to [5] for these, or more specifically [6]). The basic notion is the notion of "first order neighbour" of a point in any manifold. (The set of first order neighbours of 0 in the real line form precisely the set D

considered above.) To say that the tangent planes of E and M_τ agree at their common point x is to say that, for any first order neighbour x' of x in $\mathbf{R}^3$, $x' \in E$ iff $x' \in M_\tau$. For dimension reasons, it suffices to prove the implication $x' \in E$ implies $x' \in M_\tau$. If $x' \in E$, there is a unique parameter value τ', so that $x' \in C_{\tau'}$. Since the property of being first-order neighbour is preserved by any mapping, it follows that τ' is a first-order neighbour of τ, in other words, $\tau' = \tau + d$ for some $d \in D$. Since $x' \in C_{\tau+d} = \bigcap_{d'} M_{\tau+d+d'}$, it follows (take $d' = -d$) that $x' \in M_\tau$.

Example 1. ([7] p. 337 or [1] p. 178.) This example deals with a parametrized family of curves in the plane, namely M_τ given by $y = (x - \tau)^3$. The envelope E is the x-axis. The point is that, in this case, the intersection of (distinct!) neighbouring curves is empty, so that the synthetic description gives "empty envelope". With the supply of nilpotent real numbers in our setting, the synthetic description does, however, give the correct result:

The characteristic on M_τ is, by (1), the set of (x, y) so that for all $d \in D$, we have the first equality sign in

$$y = (x - \tau - d)^3 = (x - \tau)^3 - 3d \cdot (x - \tau)^2,$$

the second equality sign by binomial expansion, using $d^2 = 0$. If this is to hold for all $d \in D$, it follows from the basic axiom of [4] that $(x - \tau)^2 = 0$ (and hence $(x - \tau)^3 = 0$, and hence $y = 0$). The characteristic is thus the set $\{(x, 0) \mid (x - \tau)^2 = 0\}$; the union of all these sets as τ ranges is the x-axis, which thus indeed is the synthetic envelope. But note that the characteristic in this example is not the point $(\tau, 0)$, but the slightly bigger set $(\tau + D) \times \{0\}$.

Example 2. Consider a (parametrized) space curve $\xi(\tau)$. Under suitable non-degeneracy assumptions, we may consider the family of its osculating planes M_τ. We shall prove synthetically that the characteristic curve on a given M_τ is the tangent line $T_\tau \subseteq M_\tau$ of the curve at the point $\xi(\tau)$. For dimension reasons, it suffices to see the one inclusion

$$T_\tau \subseteq \bigcap_{d \in D} M_{\tau+d}.$$

Consider a fixed d; we must prove $T_\tau \subseteq M_{\tau+d}$. But the osculating plane at a given parameter value contains all 1st order neighbour tangents (see [6] Proposition 3 for a synthetic proof of this). And T_τ is such a neigbour tangent to $T_{\tau+d} \subseteq M_{\tau+d}$, since $d \in D$.

Have we proved anything new? Consider Theorem 2.2. It is certainly not new. Geometers have known since, say, the time of Huygens. But present day analytic geometry does really not have Theorem 2.2, since limit intersection curves, and hence synthetic envelopes, have been relegated from its formalism: this formalism is too restricted to allow for them. The extension of the classical formalism, which Synthetic Differential Geometry provides, allows the Theorem to get back into mathematics.

References

[1] Courant, R.: *Differential and integral calculus, Vol II.* Blackie 1967. cf. 1936 JFM 62.1165.04

[2] Hadamard, J.: *Cours d'Analyse Tom 1.* Hermann Paris 1927. JFM 53.0180.01

[3] Jessen, B.: *Lærebog i Geometri, Vol II.* Jul. Gjellerup Copenhagen 1945. Zbl 0063.03053

[4] Kock, A.: *A simple axiomatics for differentiation*. Math. Scand. **40** (1977), 183–193. Zbl 0375.12029

[5] Kock, A.: *Synthetic Differential Geometry.* London Mathematical Society Lecture Note Series **333** Second Edition, Cambridge Univ. Press (2006). Zbl 1091.51002

[6] Kock, A.: *The osculating plane of a space curve*. Suppl. Rend. Circ. Mat. Palermo II. Serie, **64** (2000), 67–79. Zbl 0964.51013

[7] Ostrowski, A.: *Vorlesungen über Differential- und Integralrechnung, Vol II.* Birkhäuser 1961. Zbl 0094.03501

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