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Indefinite Affine Hyperspheres Admitting a Pointwise Symmetry Part 1

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Abstract. An affine hypersurface M is said to admit a pointwise symmetry, if there exists a subgroup G of $\mathrm{Aut}(T_pM)$ for all $p \in M$, which preserves (pointwise) the affine metric h, the difference tensor K and the affine shape operator S. Here, we consider 3-dimensional indefinite affine hyperspheres, i. e. $S = HId$ (and thus S is trivially preserved). First we solve an algebraic problem. We determine the non-trivial stabilizers G of a traceless cubic form on a Lorentz-Minkowski space $\mathbb{R}^3_1$ under the action of the isometry group $SO(1,2)$ and find a representative of each $SO(1,2)/G$-orbit. Since the affine cubic form is defined by h and K, this gives us the possible symmetry groups G and for each G a canonical form of K. In this first part, we show that hyperspheres admitting a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$ resp. $\mathbb{R}$-symmetry are well-known, they have constant sectional curvature and Pick invariant $J < 0$ resp. $J = 0$. The classification of affine hyperspheres admitting a pointwise G-symmetry will be continued elsewhere.

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1. Introduction

Let M^n be a connected, oriented manifold. Consider an immersed hypersurface with relative normalization, i.e., an immersion $\varphi\colon M^n \to \mathbb{R}^{n+1}$ together with a transverse vector field ξ such that $D\xi$ has its image in $\varphi_* T_x M$. Equi-affine geometry studies the properties of such immersions under equi-affine transformations, i. e. volume-preserving linear transformations ($SL(n+1,\mathbb{R})$) and translations.

In the theory of nondegenerate equi-affine hypersurfaces there exists a canonical choice of transverse vector field ξ (unique up to sign), called the affine (Blaschke) normal, which induces a connection ∇, a nondegenerate symmetric bilinear form h and a 1-1 tensor field S by

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{1}$$

$$D_X \xi = -SX, \tag{2}$$

for all $X, Y \in \mathcal{X}(M)$. The connection ∇ is called the induced affine connection, h is called the affine metric or Blaschke metric and S is called the affine shape operator. In general ∇ is not the Levi-Civita connection $\hat{\nabla}$ of h. The difference tensor K is defined as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{3}$$

for all $X, Y \in \mathcal{X}(M)$. Moreover, the form $h(K(X,Y),Z)$ is a symmetric cubic form with the property that for any fixed $X \in \mathcal{X}(M)$, $\operatorname{trace} K_X$ vanishes. This last property is called the apolarity condition. The difference tensor K, together with the affine metric h and the affine shape operator S are the most fundamental algebraic invariants for a nondegenerate affine hypersurface (more details in Section 2). We say that M^n is indefinite, definite, etc. if the affine metric h is indefinite, definite, etc. For details of the basic theory of nondegenerate affine hypersurfaces we refer to [10] and [13].

Here we will restrict ourselves to the case of affine hyperspheres, i. e. the shape operator will be a (constant) multiple of the identity ($S = H\,Id$). Geometrically this means that all affine normals pass through a fixed point or they are parallel. The abundance of affine hyperspheres dwarfs any attempts at a complete classification. Even with the restriction to locally strongly convex hyperspheres (i. e. h is positive definite) and low dimensions the class is simply too large to classify. In order to obtain detailed information one has therefore to revert to sub-classes such as the class of complete affine hyperspheres (see [10] and the references contained therein or i. e. [7] for a very recent result). Various authors have also imposed curvature conditions. In the case of constant curvature the classification is nearly finished (see [16] and the references contained therein). In analogy to Chen's work, [2], a new curvature invariant for positive definite affine hyperspheres was introduced in [14]. A lower bound was given and, for $n = 3$, the classification of the extremal class was started. This classification was completed in [15], [8] and [9]. The special (simple) form of the difference tensor K for this class is remarkable, actually it turns out that the hyperspheres admit a certain pointwise group symmetry [17].

A hypersurface is said to admit a pointwise group symmetry if at every point the affine metric, the affine shape operator and the difference tensor are preserved under the group action. Necessarily the possible groups must be subgroups of the isometry group. The study of submanifolds which admit a pointwise group symmetry was initiated by Bryant in [1] where he studied 3-dimensional Lagrangian submanifolds of $\mathbb{C}^3$, i. e. the isometry group is $SO(3)$. Because of the similar basic invariants, Vrancken transferred the problem to 3-dimensional positive definite affine hyperspheres. A classification of 3-dimensional positive definite affine hyperspheres admitting pointwise symmetries was obtained in [17] and then extended to positive definite hypersurfaces in [11] (here the affine shape operator is non-trivial and thus no longer trivially preserved by isometries). Now, for the first time, we will consider the indefinite case, namely 3-dimensional indefinite affine hyperspheres.

We can assume that the affine metric has index two, i. e. the corresponding isometry group is the (special) Lorentz group $SO(1,2)$. Our question is the following: What can we say about a three-dimensional indefinite affine hypersphere, for which there exists a non-trivial subgroup G of $SO(1,2)$ such that for every $p \in M$ and every $L \in G$:

$$K(LX_p, LY_p) = L(K(X_p, Y_p)) \quad \forall X_p, Y_p \in T_pM.$$

In Section 2 we will state the basic formulas of (equi-)affine hypersurface-theory needed in the further classification. We won't need hypersurface-theory in Section 3 and 4, were we consider the group structure of $SO(1,2)$ and its action on cubic forms. In Section 3 we show that there exist six different normalforms of elements of $SO(1,2)$, depending on the eigenvalues and eigenspaces. We can always find an oriented basis of $\mathbb{R}^3_1$ such that every $L \in SO(1,2)$ has one (and only one) of the following matrix representations:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}, t \in (0, 2\pi) \setminus \{\pi\}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Id, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

all of these with respect to an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$, $\mathbf{t}$ timelike, $\mathbf{v}, \mathbf{w}$ spacelike, or

$$\begin{pmatrix} l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{l} \end{pmatrix}, \ l \neq \pm 1, \quad \begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

with respect to a (LV)basis $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$, $\mathbf{e}, \mathbf{f}$ lightlike, $\mathbf{v}$ spacelike (Theorem 3).

Since we are interested in pointwise group symmetry, in Section 4 we study the nontrivial stabilizer of a traceless cubic form $\tilde{K}$ under the $SO(1,2)$-action $\rho(L)(\tilde{K}) = \tilde{K} \circ L$ (cp. [1] for the classification of the $SO(3)$-action). It turns out that the $SO(1,2)$-stabilizer of a nontrivial traceless cubic form is isomorphic to either $SO(2)$, $SO(1,1)$, $\mathbb{R}$, the group S_3 of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ or it is trivial (Theorem 4).

In the following we classify the indefinite affine hyperspheres which admit a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry (Section 5) resp. $\mathbb{R}$-symmetry (Section 6). We show that they are indefinite affine hyperspheres of constant sectional curvature with negative resp. vanishing Pick invariant J. These are classified in [12] resp. [4]. More precisely we prove the following:

Theorem 1. *An affine hypersphere admits a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry if and only if it is affine equivalent to an open subset of*

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1.$$

Theorem 2. *Let M^3 be an affine hypersphere admitting a pointwise $\mathbb{R}$-symmetry. Then M^3 has constant sectional curvature $\hat{\kappa} = H$ and zero Pick invariant $J = 0$.*

Furthermore, we obtain some examples of pointwise $\mathbb{Z}_2$-symmetry. The classification of hypersurfaces admitting a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry, S_3-symmetry or $SO(1,1)$-symmetry will be presented somewhere else. It turns out that these classes are very rich, most of them are warped products of two-dimensional affine spheres ($\mathbb{Z}_3$) resp. quadrics ($SO(2)$, $SO(1,1)$), with a curve. Thus we get many new examples of 3-dimensional indefinite affine hyperspheres. Furthermore, we will show how one can construct indefinite affine hyperspheres out of two-dimensional quadrics or positive definite affine spheres.

2. Basics of affine hypersphere theory

First we recall the definition of the affine normal ξ (cp. [13]). In equi-affine hypersurface theory on the ambient space $\mathbb{R}^{n+1}$ a fixed volume form det is given. A transverse vector field ξ induces a volume form θ on M by $\theta(X_1, \dots, X_n) = \det(\varphi_* X_1, \dots, \varphi_* X_n, \xi)$. Also the affine metric h defines a volume form ω_h on M, namely $\omega_h = |\det h|^{1/2}$. Now the affine normal ξ is uniquely determined (up to sign) by the conditions that $D\xi$ is everywhere tangential (which is equivalent to $\nabla\theta = 0$) and that

$$\theta = \omega_h. \tag{4}$$

Since we only consider 3-dimensional indefinite hyperspheres, i. e.

$$S = H\,Id, \quad H = \text{const.} \tag{5}$$

we can fix the orientation of the affine normal ξ such that the affine metric has signature one. Then the sign of H in the definition of an affine hypersphere is an invariant.

Next we state some of the fundamental equations, which a nondegenerate hypersurface has to satisfy, see also [13] or [10]. These equations relate S and K with amongst others the curvature tensor R of the induced connection ∇ and the curvature tensor $\hat{R}$ of the Levi-Civita connection $\widehat{\nabla}$ of the affine metric h. There are the Gauss equation for ∇, which states that:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$

and the Codazzi equation

$$(\nabla_X S)Y = (\nabla_Y S)X.$$

Also we have the total symmetry of the affine cubic form

$$C(X,Y,Z) = (\nabla_X h)(Y,Z) = -2h(K(X,Y),Z). \tag{6}$$

The fundamental existence and uniqueness theorem, see [3] or [5], states that given h, ∇ and S such that the difference tensor is symmetric and traceless with respect to h, on a simply connected manifold M an affine immersion of M exists if and only if the above Gauss equation and Codazzi equation are satisfied.

From the Gauss equation and Codazzi equation above the Codazzi equation for K and the Gauss equation for $\widehat{\nabla}$ follow:

$$\begin{aligned}(\widehat{\nabla}_X K)(Y,Z) - (\widehat{\nabla}_Y K)(X,Z) =& \tfrac{1}{2}(h(Y,Z)SX - h(X,Z)SY \\ & - h(SY,Z)X + h(SX,Z)Y),\end{aligned}$$

and

$$\begin{aligned}\hat{R}(X,Y)Z =& \tfrac{1}{2}(h(Y,Z)SX - h(X,Z)SY \\ & + h(SY,Z)X - h(SX,Z)Y) - [K_X, K_Y]Z.\end{aligned}$$

If we define the Ricci tensor of the Levi-Civita connection $\widehat{\nabla}$ by:

$$\widehat{\mathrm{Ric}}(X,Y) = \mathrm{trace}\{Z \mapsto \hat{R}(Z,X)Y\}. \tag{7}$$

and the Pick invariant by:

$$J = \frac{1}{n(n-1)} h(K,K), \tag{8}$$

then from the Gauss equation we immediately get for the scalar curvature $\hat{\kappa} = \frac{1}{n(n-1)}(\sum_{i,j} h^{ij}\widehat{\mathrm{Ric}}_{ij})$:

$$\hat{\kappa} = H + J. \tag{9}$$

For an affine hypersphere the Gauss and Codazzi equations have the form:

$$R(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y), \tag{10}$$

$$(\nabla_X H)Y = (\nabla_Y H)X, \quad \text{i. e.} \quad H = const., \tag{11}$$

$$(\widehat{\nabla}_X K)(Y,Z) = (\widehat{\nabla}_Y K)(X,Z), \tag{12}$$

$$\hat{R}(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z. \tag{13}$$

Since H is constant, we can rescale φ such that $H \in \{-1, 0, 1\}$.

3. Normalforms in $SO(1,2)$

We denote[1] by $\mathbb{R}^3_1$ the pseudo-Euclidean vector space in which a non-degenerate indefinite bilinear form of index two is given. The bilinear form is called the inner product and denoted by $\langle\ ,\ \rangle$. A basis $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ is called orthonormal (ONB) if

$$\begin{gathered}\langle \mathbf{t}, \mathbf{t}\rangle = -1, \quad \langle \mathbf{v}, \mathbf{v}\rangle = 1 = \langle \mathbf{w}, \mathbf{w}\rangle, \\ 0 = \langle \mathbf{t}, \mathbf{v}\rangle = \langle \mathbf{t}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle.\end{gathered} \tag{14}$$

[1] for the notation cp. [6]

For a chosen ONB the inner product of two vectors is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_t y_t + x_v y_v + x_w y_w, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3_1. \tag{15}$$

A basis $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ is called a light-vector basis (LVB) if

$$\begin{aligned} \langle \mathbf{e}, \mathbf{e} \rangle = 0 = \langle \mathbf{f}, \mathbf{f} \rangle, \langle \mathbf{e}, \mathbf{f} \rangle = 1, \\ \langle \mathbf{e}, \mathbf{v} \rangle = 0 = \langle \mathbf{f}, \mathbf{v} \rangle, \langle \mathbf{v}, \mathbf{v} \rangle = 1. \end{aligned} \tag{16}$$

For a chosen LVB the inner product of two vectors is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_e y_f + x_f y_e + x_v y_v, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3_1. \tag{17}$$

We want to consider the special pseudo-Euclidean rotations $SO(1,2)$, i. e. the linear transformations L of $\mathbb{R}^3_1$ which preserve the inner product and have determinant equal to one:

$$\langle L\mathbf{x}, L\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3_1, \quad \det L = 1.$$

Depending on the eigenvalues and eigenspaces we get the following normalforms of the elements of $SO(1,2)$.

Theorem 3. *There exists a choice of an oriented basis of $\mathbb{R}^3_1$ such that every $L \in SO(1,2)$ is of one (and only one) of the following types:*

1. (a) $A_t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$, $t \in [0, 2\pi)$, $t \neq 0, \pi$,
   for an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$, $\mathbf{t}$ *timelike,* $\mathbf{v}, \mathbf{w}$ *spacelike,*
   eigenvalues: $\lambda_1 = 1$, *eigenspaces* $E(1) = \operatorname{span}\{t\}$ *timelike.*

   (b) $A_\pi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, *for an ONB* $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$, *as above,*
   eigenvalues: $\lambda_1 = 1$, $\lambda_{2,3} = -1$,
   eigenspaces $E(1) = \operatorname{span}\{\mathbf{t}\}$ *timelike,* $E(-1) = \operatorname{span}\{\mathbf{v}, \mathbf{w}\}$, *spacelike.*

2. (a) $A_0 := Id$, *for an ONB* $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ *as above or an LVB* $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$,
   eigenvalues: $\lambda_{1,2,3} = 1$, *eigenspaces* $E(1) = \mathbb{R}^3_1$.

   (b) $B := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, *for an ONB* $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$, *as above, or an LVB* $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$,
   eigenvalues: $\lambda_{1,3} = -1$, $\lambda_2 = 1$,
   eigenspaces $E(-1) = \operatorname{span}\{\mathbf{t}, \mathbf{w}\} = \operatorname{span}\{\mathbf{e}, \mathbf{f}\}$ *timelike,*
   $E(1) = \operatorname{span}\{\mathbf{v}\}$, *spacelike.*

3. (a) $C_l := \begin{pmatrix} l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{l} \end{pmatrix}$, $l \neq \pm 1$, *for an LVB* $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$,
   eigenvalues: $\lambda_1 = l$, $\lambda_2 = 1$, $\lambda_3 = \frac{1}{l}$,
   eigenspaces $E(l) = \operatorname{span}\{\mathbf{e}\}$ *lightlike,* $E(1) = \operatorname{span}\{\mathbf{v}\}$, *spacelike,*
   $E(\frac{1}{l}) = \operatorname{span}\{\mathbf{f}\}$ *lightlike.*

   (b) $C_1 := \begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, *for an LVB* $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$,
   eigenvalues: $\lambda_{1,2,3} = 1$, *eigenspaces* $E(1) = \operatorname{span}\{\mathbf{e}\}$ *lightlike.*

For the proof we will use the following

Lemma 1. *Let $L \in SO(1,2)$ with $L(\mathbf{e}) = l\mathbf{e}$, $l \neq 0$, for an LVB $\{\mathbf{e},\mathbf{v},\mathbf{f}\}$. Since under L an LVB will be mapped to an LVB, the corresponding matrix must have the following form :*

$$C_{l,m} = \begin{pmatrix} l & -lm & -l\frac{m^2}{2} \\ 0 & 1 & m \\ 0 & 0 & \frac{1}{l} \end{pmatrix}.$$

Proof. We will use the notation: $\mathbf{e}' = L(\mathbf{e})$, $\mathbf{v}' = L(\mathbf{v})$, $\mathbf{f}' = L(\mathbf{f})$, and compute (16), using (17):

$$\begin{aligned} 0 = \langle \mathbf{e}', \mathbf{v}' \rangle &= l v'_f \Rightarrow v'_f = 0, \\ 1 = \langle \mathbf{v}', \mathbf{v}' \rangle &= (v'_v)^2 \Rightarrow v'_v = \varepsilon, \quad \varepsilon^2 = 1, \\ 1 = \langle \mathbf{e}', \mathbf{f}' \rangle &= l f'_f \Rightarrow f'_f = \frac{1}{l}, \\ 1 = \det L &= l\varepsilon\frac{1}{l} \Rightarrow \varepsilon = 1, \\ 0 = \langle \mathbf{f}', \mathbf{v}' \rangle &= v'_e\frac{1}{l} + f'_v \Rightarrow f'_v = -\frac{v'_e}{l}, \\ 0 = \langle \mathbf{f}', \mathbf{f}' \rangle &= 2f'_e\frac{1}{l} + \frac{(v'_e)^2}{2} \Rightarrow f'_e = -\frac{1}{l}\frac{(v'_e)^2}{2}. \end{aligned}$$

With $v'_e = -lm$ we obtain $C_{l,m}$. □

Proof of Theorem 3. Every $L \in SO(1,2)$ must have a real eigenvalue since it is an automorphism of a three dimensional vector space. A corresponding eigenvector will be either timelike, spacelike or lightlike.

Let us consider first the case that we have (at least) a timelike eigenvector. We can choose an ONB $\{\mathbf{t},\mathbf{v},\mathbf{w}\}$, such that $\mathbf{t}$ is the timelike eigenvector with eigenvalue $\varepsilon = \pm 1$. Then $\mathbf{v},\mathbf{w}$ are spacelike and L restricted to $\mathbf{t}^\perp = \text{span}\{\mathbf{v},\mathbf{w}\}$ is an isometry of $\mathbb{R}^2$, i. e. an Euclidean rotation.

If $\varepsilon = 1$, then L restricted to $\mathbf{t}^\perp$ is in $SO(2)$ (proper Euclidean rotation): In general we get no more real eigenvalues (case 1.(a)). If the restriction is a rotation by an angle of π, then we get the second eigenvalue -1 of multiplicity two. The eigenspace is $\mathbf{t}^\perp$ (and thus spacelike) (case 1.(b)). Finally, if the restriction (and thus L) is the identity map *Id*, every vector is an eigenvector and we also can choose an LVB (case 2.(a)).

If $\varepsilon = -1$, then L restricted to $\mathbf{t}^\perp$ is an improper Euclidean rotation of $\mathbb{R}^2$, thus it has eigenvalues 1 and -1. We get the eigenvalue 1 with eigenspace span$\{\mathbf{v}\}$ (spacelike) and the eigenvalue -1 of multiplicity two with eigenspace span$\{\mathbf{t},\mathbf{w}\}$ (timelike). Since L restricted to span$\{\mathbf{t},\mathbf{w}\}$ is equal to $-Id$, we also can choose a basis of two lightlike eigenvectors (case 2.(b)).

Next we will consider the case that we have (at least) a lightlike eigenvector. We can choose an LVB $\{\mathbf{e},\mathbf{v},\mathbf{f}\}$, such that $\mathbf{e}$ is this lightlike eigenvector with eigenvalue $l \in \mathbb{R}$, $l \neq 0$. Since under L an LVB will be mapped to an LVB, the corresponding matrix must have the following form (cp. Lemma 1):

$$C_{l,m} = \begin{pmatrix} l & -lm & -l\frac{m^2}{2} \\ 0 & 1 & m \\ 0 & 0 & \frac{1}{l} \end{pmatrix}.$$

$C_{l,m}$ has the eigenvalues $\lambda_1 = l$, $\lambda_2 = 1$ and $\lambda_3 = \frac{1}{l}$. If $l \neq \pm 1$, we get three distinct real eigenvalues and the corresponding (1-dim.) eigenspaces are either lightlike (Eig(l) and Eig($\frac{1}{l}$)) or spacelike (Eig(1)) (case 3.(a)). (Since we take an eigenvector basis, the LVB is up to the length of the lightlike eigenvectors uniquely determined.) If $l = -1$, $\lambda_1 = \lambda_3 = -1$, thus we have an eigenvalue of multiplicity two with eigenspace span$\{\mathbf{e}, \mathbf{f}\}$ (case 2.(b)). The case $l = 1$ is left, i. e. only one eigenvalue of multiplicity three: For $m = 0$ we obtain the identity map (case 2.(a)). For $m \neq 0$, the eigenspace Eig(1) = span$\{\mathbf{e}\}$ only is one-dimensional. To get a normalform of L we compute how $C_{1,m}$ changes if we choose another LVB $\{\mathbf{e}', \mathbf{v}', \mathbf{f}'\}$ (with the same orientation) where $\mathbf{e}'$ is an eigenvector of L ($\mathbf{e}' = a\mathbf{e}$). If we express $\{\mathbf{e}', \mathbf{v}', \mathbf{f}'\}$ in terms of $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ and compute (16), we obtain (cp. Lemma 1):

$$\begin{aligned} \mathbf{e}' &= a\mathbf{e}, \\ \mathbf{v}' &= v'_e\mathbf{e} + \mathbf{v}, \\ \mathbf{f}' &= -\frac{(v'_e)^2}{2a}\mathbf{e} - \frac{v'_e}{a}\mathbf{v} + \frac{1}{a}\mathbf{f}. \end{aligned}$$

The matrix representing L with respect to the new LVB, $C'_{1,m}$, has the form $C'_{1,m} = \begin{pmatrix} 1 & -\frac{m}{a} & -\frac{m^2}{2}\frac{1}{a^2} \\ 0 & 1 & \frac{m}{a} \\ 0 & 0 & 1 \end{pmatrix}$. Thus $C'_{1,m} = C_{1,\frac{m}{a}}$, and we can choose an LVB such that $\frac{m}{a} = 1$ (case 3.(b)). From the above computations we see that this LVB still is not completly determined, we can choose v'_e arbitrary.

Finally the case is left that we have (at least) a spacelike eigenvector $\mathbf{v}$. The corresponding eigenvalue must be $\varepsilon = \pm 1$ and L restricted to $\mathbf{v}^{\perp}$ is an isometry of $\mathbb{R}^2_1$, i. e. a pseudo-Euclidean rotation (boost). Thus it always has two real eigenvalues with one-dimensional eigenspaces. We can choose an eigenvector basis, which will be either an ONB or an LVB, and we get one of the following cases: 1.(b), 2.(a), 2.(b) or 3.(a) (cp. [6], p. 273). □

Remark. The choice of basis for the above normalforms is unique up to:

1. (a) the ONB is unique up to $\mathbf{t} \to \varepsilon\mathbf{t}, \mathbf{v}, \mathbf{w}$ up to a proper ($\varepsilon = 1$) or improper ($\varepsilon = -1$) Euclidean rotation in $\mathbb{R}^2$.

   (b) the ONB is unique up to $\mathbf{t} \to \varepsilon\mathbf{t}, \mathbf{v}, \mathbf{w}$ up to a proper ($\varepsilon = 1$) or improper ($\varepsilon = -1$) Euclidean rotation in $\mathbb{R}^2$.

2. (a) every ONB or LVB.

(b) the ONB is unique up to $\mathbf{v} \to \varepsilon\mathbf{v}, \mathbf{t}, \mathbf{w}$ up to a proper ($\varepsilon = 1$) or improper ($\varepsilon = -1$) Pseudo-Euclidean rotation in $\mathbb{R}^2_1$,

the LVB is unique up to $\begin{cases} \mathbf{v} \to & \mathbf{v}, \\ \mathbf{e} \to & a\mathbf{e}, \\ \mathbf{f} \to & \frac{1}{a}\mathbf{f}, \end{cases}$ or $\begin{cases} \mathbf{v} \to & -\mathbf{v}, \\ \mathbf{e} \to & a\mathbf{f}, \\ \mathbf{f} \to & \frac{1}{a}\mathbf{e}, \end{cases}$ $a \in \mathbb{R}$,

3. (a) the LVB is unique up to $\begin{cases} \mathbf{e} \to & a\mathbf{e}, \\ \mathbf{f} \to & \frac{1}{a}\mathbf{f}, \end{cases}$ $a \in \mathbb{R}$.

Under $\begin{cases} \mathbf{v} \to & -\mathbf{v}, \\ \mathbf{e} \to & a\mathbf{f}, \\ \mathbf{f} \to & \frac{1}{a}\mathbf{e}, \end{cases}$, C_λ goes to $C_{\frac{1}{\lambda}}$.

(b) the LVB is unique up to $\begin{cases} \mathbf{e} \to & \mathbf{e}, \\ \mathbf{v} \to & b\mathbf{e} + \mathbf{v}, \\ \mathbf{f} \to & -\frac{b^2}{2}\mathbf{e} - b\mathbf{v} + \mathbf{f}, \end{cases}$ $b \in \mathbb{R}$.

Under $\begin{cases} \mathbf{e} \to & a\mathbf{e}, \\ \mathbf{v} \to & b\mathbf{e} + \mathbf{v}, \\ \mathbf{f} \to & -\frac{b^2}{2a}\mathbf{e} - \frac{b}{a}\mathbf{v} + \frac{1}{a}\mathbf{f}, \end{cases}$, C_1 goes to $C_{1,\frac{1}{a}}$.

4. Non-trivial $SO(1,2)$-stabilizers

Since we are interested in pointwise group symmetry, we study the non-trivial stabilizer of a traceless cubic form $\tilde{K}$ under the $SO(1,2)$-action $\rho(L)(\tilde{K}) = \tilde{K} \circ L$ resp.

$$\rho(L)(h(K(\,.\,,\,.\,),\,.\,)) = h(K(L\,.\,,L\,.\,),L\,.\,))$$

(cp. [1] for the classification under the $SO(3)$-action). We will use the following notation for the coefficients of the difference tensor K with respect to an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$:

$$\begin{aligned} K_{\mathbf{t}} &= \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_1 - a_4 \end{pmatrix}, \quad K_{\mathbf{v}} = \begin{pmatrix} -a_2 & -a_4 & -a_5 \\ a_4 & a_6 & a_7 \\ a_5 & a_7 & a_2 - a_6 \end{pmatrix}, \\ K_{\mathbf{w}} &= \begin{pmatrix} -a_3 & -a_5 & -(a_1 - a_4) \\ a_5 & a_7 & a_2 - a_6 \\ a_1 - a_4 & a_2 - a_6 & a_3 - a_7 \end{pmatrix}, \end{aligned} \tag{18}$$

resp. with respect to an LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$:

$$K_{\mathbf{e}} = \begin{pmatrix} b_1 & b_4 & b_5 \\ b_2 & -2b_1 & b_4 \\ b_3 & b_2 & b_1 \end{pmatrix}, K_{\mathbf{v}} = \begin{pmatrix} b_4 & -2b_5 & b_6 \\ -2b_1 & -2b_4 & -2b_5 \\ b_2 & -2b_1 & b_4 \end{pmatrix}, K_{\mathbf{f}} = \begin{pmatrix} b_5 & b_6 & b_7 \\ b_4 & -2b_5 & b_6 \\ b_1 & b_4 & b_5 \end{pmatrix}. \tag{19}$$

We will prove the following theorem, stating not only the non-trivial stabilizers, but also give a normal form of K for each stabilizer.

Theorem 4. *Let $p \in M$ and assume that there exists a non-trivial element of $SO(1,2)$ which preserves K. Then there exists an ONB resp. an LVB of T_pM such that either*

1. *$K = 0$, and this form is preserved by every isometry, or*
2. *$a_1 = 2a_4, a_4 > 0$, and all other coefficients vanish, this form is preserved by the subgroup $\{A_t, t \in \mathbb{R}\}$, isomorphic to $SO(2)$, or*
3. *$a_6 > 0$, and all other coefficients vanish, this form is preserved by the subgroup with generators $< A_{\frac{2\pi}{3}}, B >$, isomorphic to S_3, or*
4. *$a_1 = 2a_4, a_4 > 0$, $a_6 > 0$, and all other coefficients vanish, this form is preserved by the subgroup with generator $< A_{\frac{2\pi}{3}} >$, isomorphic to $\mathbb{Z}_3$, or*
5. *$a_2, a_5 \in \mathbb{R}$, $a_6 \geq 0$, where $(a_2, a_6) \neq 0$, and all other coefficients vanish, this form is preserved by the subgroup with generator $< B >$, isomorphic to $\mathbb{Z}_2$, or*
6. *$a_5 > 0$, and all other coefficients vanish, this form is preserved by the subgroup with generators $< A_\pi, B >$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, or*
7. *$a_1 > 0$ or $a_4 > 0$, $a_1 \neq 2a_4$, and all other coefficients vanish, this form is preserved by the subgroup with generator $< A_\pi >$, isomorphic to $\mathbb{Z}_2$, or*
8. *$b_4 > 0$, and all other coefficients vanish, this form is preserved by the subgroup $\{C_l, l \in \mathbb{R} \setminus \{0\}\}$, isomorphic to $SO(1,1)$, or*
9. *$b_7 > 0$, and all other coefficients vanish, this form is preserved by the subgroup $\{\begin{pmatrix} 1 & -m & -\frac{m^2}{2} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}, m \in \mathbb{R}\}$, isomorphic to $\mathbb{R}$.*

To get ready for the proof, first we will find out what it means for K to be invariant under one element of $SO(1,2)$. Some of the computations were done with the CAS Mathematica[2].

Lemma 2. *Let $p \in M$ and assume, that K is invariant under the transformation $A_t \in SO(1,2)$, $t \in (0, 2\pi)$. Then we get for the coefficients of K with respect to the corresponding ONB of T_pM:*

1. *if $t \neq \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$, then $a_1 = 2a_4$, $a_4 \in \mathbb{R}$, and all other coefficients vanish,*
2. *if $t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$, then $a_1 = 2a_4$, $a_4, a_6, a_7 \in \mathbb{R}$, and all other coefficients vanish,*
3. *if $t = \pi$, then $a_1, a_4, a_5 \in \mathbb{R}$, and all other coefficients vanish.*

Proof. The proof is a straight forward computation, evaluating the equations $h(K(X,Y),Z) = h(K(A_tX, A_tY), A_tZ)$ for $X, Y, Z \in \{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$. The computations were done with the CAS Mathematica. For all $t \in (0, 2\pi)$ we obtain from eq2 ($X, Y = \mathbf{t}$, $Z = \mathbf{v}$) and eq3 ($X, Y = \mathbf{t}$, $Z = \mathbf{w}$) that $a_2 = 0$ and $a_3 = 0$. If $t = \pi$, then eq7 ($X, Y, Z = \mathbf{v}$) and eq8 ($X, Y = \mathbf{v}$, $Z = \mathbf{w}$) give $a_6 = 0$ and $a_7 = 0$. If $t \neq \pi$, then eq4 ($X = \mathbf{t}$, $Y, Z = \mathbf{v}$) and eq5 ($X = \mathbf{t}$, $Y = \mathbf{v}$, $Z = \mathbf{w}$) lead to $a_5 = 0$ and $a_1 = 2a_4$. Now, for $t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$, all equations are true. Otherwise, only $a_6 = 0$ and $a_7 = 0$ solve eq7 and eq8. □

[2] see Appendix or http://www.math.tu-berlin.de/~schar/IndefSym_Stabilizers.html

Lemma 3. *Let $p \in M$ and assume, that K is invariant under the transformation $B \in SO(1,2)$. Then we get for the coefficients of K with respect to the corresponding ONB of T_pM that $a_2, a_5, a_6 \in \mathbb{R}$, and all other coefficients vanish.*

Proof. The computations were done with the CAS Mathematica, too. We obtain from eq1 ($X, Y, Z = \mathbf{t}$), eq3 ($X, Y = \mathbf{t}$, $Z = \mathbf{w}$), eq4 ($X = \mathbf{t}$, $Y, Z = \mathbf{v}$) and eq8 ($X, Y = \mathbf{v}$, $Z = \mathbf{w}$) that $a_1 = 0$, $a_3 = 0$, $a_4 = 0$ and $a_7 = 0$. □

Lemma 4. *Let $p \in M$ and assume, that K is invariant under the transformation $C_l \in SO(1,2)$, $l \in \mathbb{R} \setminus \{0, 1\}$. Then we get for the coefficients of K with respect to the corresponding LVB of T_pM:*

1. *if $l \neq -1$, then $b_4 \in \mathbb{R}$, and all other coefficients vanish,*
2. *if $l = -1$, then $b_2, b_4, b_6 \in \mathbb{R}$, and all other coefficients vanish.*

Proof. The computations were done with the CAS Mathematica, too. We obtain from eq1 ($X, Y, Z = \mathbf{e}$), eq3 ($X, Y = \mathbf{e}$, $Z = \mathbf{f}$), eq6 ($X = \mathbf{e}$, $Y, Z = \mathbf{f}$) and eq10 ($X, Y, Z = \mathbf{f}$) that $b_1 = 0$, $b_3 = 0$, $b_5 = 0$ and $b_7 = 0$. If $l \neq -1$, then eq2 ($X, Y = \mathbf{e}$, $Z = \mathbf{v}$) and eq9 ($X = \mathbf{v}$, $Y, Z = \mathbf{f}$) additionally give that $b_2 = 0$ and $b_6 = 0$. □

Lemma 5. *Let $p \in M$ and assume, that K is invariant under the transformation $C_{1,m} \in SO(1,2)$, $m \in \mathbb{R} \setminus \{0\}$. Then we get for the coefficients of K with respect to the corresponding LVB of T_pM that $b_7 \in \mathbb{R}$, and all other coefficients vanish.*

Proof. The computations were done with the CAS Mathematica, too. We obtain successively from eq2 ($X, Y = \mathbf{e}$, $Z = \mathbf{v}$), eq3 ($X, Y = \mathbf{e}$, $Z = \mathbf{f}$), eq5 ($X = \mathbf{e}$, $Y = \mathbf{v}$, $Z = \mathbf{f}$), eq6 ($X = \mathbf{e}$, $Y, Z = \mathbf{f}$), eq9 ($X = \mathbf{v}$, $Y, Z = \mathbf{f}$) and eq10 ($X, Y, Z = \mathbf{f}$) that $b_3 = 0$, $b_2 = 0$, $b_1 = 0$, $b_4 = 0$, $b_5 = 0$ and $b_6 = 0$. □

In the following U denotes an arbitrary subgroup of $SO(1,2)$, which leaves K invariant. We want to find out to which extend K determines the properties of the elements of U.

Lemma 6. *If there exists $t \in (0, 2\pi)$, $t \neq \pi$, with $A_t \in U$, and $K \neq 0$, then we get for the timelike eigenvector $\mathbf{t}$ of A_t:*

1. *for $t \neq \frac{2\pi}{3}$ and $t \neq \frac{4\pi}{3}$: $M\mathbf{t} = \mathbf{t}$ for all $M \in U$,*
2. *for $t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$: $M\mathbf{t} = \varepsilon\mathbf{t}$ for all $M \in U$.*

Proof. Let $M \in U$. From Lemma 2 we know that

$$h(K(M\mathbf{t}, M\mathbf{t}), MY) = h(K(\mathbf{t}, \mathbf{t}), Y) = \begin{cases} -2a_4, & Y = \mathbf{t}, \\ 0, & Y = \mathbf{v} \text{ or } Y = \mathbf{w}. \end{cases}$$

Thus $K(M\mathbf{t}, M\mathbf{t}) = -2a_4 M\mathbf{t}$, furthermore $h(M\mathbf{t}, M\mathbf{t}) = h(\mathbf{t}, \mathbf{t}) = -1$. Now assume that $X = x\mathbf{t} + y\mathbf{v} + z\mathbf{w} \in T_pM$ has the same properties ($K(X, X) = -2a_4 X$

and $h(X,X)=-1$). This is equivalent to (cp. Lemma 2):

$$\begin{aligned}
(-2x^2-y^2-z^2)a_4 &= -2a_4x, && (20)\\
2xya_4+(y^2-z^2)a_6+2yza_7 &= -2a_4y, && (21)\\
2xza_4-2yza_6+(y^2-z^2)a_7 &= -2a_4z, && (22)\\
-x^2+y^2+z^2 &= -1. && (23)
\end{aligned}$$

If $a_4\neq 0$, then (20) and (23) imply that $3x^2-2x-1=0$ and $x^2\geq 1$, this means $x=1$ and $y=z=0$. Thus $X=\mathbf{t}$ and $M\mathbf{t}=\mathbf{t}$.

If $a_4=0$, then (21) and (22) imply that

$$\begin{aligned}
(y^2-z^2)a_6+2yza_7 &= 0,\\
-2yza_6+(y^2-z^2)a_7 &= 0.
\end{aligned}$$

The two equations, linear in a_6 and a_7, only have a non-trivial solution if $y=z=0$. With (23) we obtain that $X=\varepsilon\mathbf{t}$ and thus $M\mathbf{t}=\varepsilon\mathbf{t}$. □

Lemma 7. *If there exists $l\in\mathbb{R}$, $l\neq 0,\pm1$, with $C_l\in U$, and $K\neq 0$, then we get for the spacelike eigenvector $\mathbf{v}$ of C_l: $M\mathbf{v}=\mathbf{v}$ for all $M\in U$.*

Proof. Let $M\in U$. From Lemma 4 we know that

$$h(K(M\mathbf{v},M\mathbf{v}),MY)=h(K(\mathbf{v},\mathbf{v}),Y)=\begin{cases}-2a_4, & Y=\mathbf{v},\\ 0, & Y=\mathbf{e}\text{ or }Y=\mathbf{f}.\end{cases}$$

Thus $K(M\mathbf{v},M\mathbf{v})=-2a_4M\mathbf{v}$, furthermore $h(M\mathbf{v},M\mathbf{v})=h(\mathbf{v},\mathbf{v})=1$. Now assume that $X=x\mathbf{t}+y\mathbf{v}+z\mathbf{w}\in T_pM$ has the same properties ($K(X,X)=-2a_4X$ and $h(X,X)=1$). This is equivalent to (cp. Lemma 4):

$$\begin{aligned}
2xyb_4 &= -2b_4x, && (24)\\
2(-y^2+xz)b_4 &= -2b_4y, && (25)\\
2yzb_4 &= -2b_4z, && (26)\\
2xz+y^2 &= 1. && (27)
\end{aligned}$$

Since $b_4\neq 0$, (24) is equivalent to $x=0$ or $y=-1$, and (26) is equivalent to $z=0$ or $y=-1$. Now $y=-1$ in (25) gives $xy=2$, which is a contradiction to (27). Thus $x=0$ and $z=0$. With (25) we obtain that $X=\mathbf{v}$ and thus $M\mathbf{v}=\mathbf{v}$. □

Lemma 8. *If there exists $m\in\mathbb{R}$, $m\neq 0$, with $C_{1,m}\in U$, and $K\neq 0$, then we get for the lightlike eigenvector $\mathbf{e}$ of $C_{1,m}$: $M\mathbf{e}=\mathbf{e}$ for all $M\in U$.*

Proof. Let $M \in U$. From Lemma 5 we know that

$$K(X,X) = b_7 z^2 \mathbf{e} \text{ for all } X = x\mathbf{t} + y\mathbf{v} + z\mathbf{w} \in T_pM,$$

i. e. $\mathbf{e}$ is determined by K up to length: $\mathbf{e} = \frac{1}{h(K(\mathbf{f},\mathbf{f}),\mathbf{f})} K(\mathbf{f},\mathbf{f})$. From the invariance of K under M it follows that $M\mathbf{e} = \frac{1}{h(K(\mathbf{f},\mathbf{f}),\mathbf{f})} K(M\mathbf{f}, M\mathbf{f}) = (h(M\mathbf{f},\mathbf{e}))^2\mathbf{e}$. Since $1 = h(M\mathbf{f}, M\mathbf{e})$, we get now: $1 = h(M\mathbf{f}, h(M\mathbf{f},\mathbf{e})^2\mathbf{e}) = h(M\mathbf{f},\mathbf{e})^3$, thus $h(M\mathbf{f},\mathbf{e}) = 1$. □

Now we are ready for the proof of Theorem 4.

Proof of Theorem 4. For the proof we will consider several cases which are supposed to be exclusive. Let U be a maximal subgroup of $SO(1,2)$ which leaves K invariant.

1. *Case* We assume that there exists $t \in (0, 2\pi)$, $t \neq \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$, with $A_t \in U$. Thus there exists an ONB $\{\mathbf{t},\mathbf{v},\mathbf{w}\}$ such that K has the form (Lemma 2): $a_1 = 2a_4$, $a_4 \in \mathbb{R}$ and all other coefficients vanish. If $a_4 = 0$, then $K = 0$ and $U = SO(1,2)$. If $a_4 \neq 0$, then we know that $M\mathbf{t} = \mathbf{t}$ for all $M \in U$ (Lemma 6). We have seen in Section 3 that $M \in U$ must be of type 1.(a), 1.(b) or 2.(a). Now let $N \in SO(1,2)$ be of type 1.(a), 1.(b) or 2.(a) with eigenvector $\mathbf{t}$. We can normalize simultaneously (i. e. find an ONB such that both A_t and N have normalform) and we see that N leaves K invariant (Lemma 2). Thus $U = \{A_t, t \in \mathbb{R}\}$. Finally, if $a_4 < 0$, we can change the ONB $\{\mathbf{t},\mathbf{v},\mathbf{w}\}$ and take instead$\{-\mathbf{t},\mathbf{w},\mathbf{v}\}$.

2. *Case* We assume that $A_{\frac{2\pi}{3}} \in U$ or $A_{\frac{4\pi}{3}} \in U$. Thus there exists an ONB $\{\mathbf{t},\mathbf{v},\mathbf{w}\}$ such that K has the form (Lemma 2): $a_1 = 2a_4$, $a_4, a_6, a_7 \in \mathbb{R}$ and all other coefficients vanish. Without loss of generality a_6 and a_7 will not vanish both. Furthermore we know that $M\mathbf{t} = \varepsilon\mathbf{t}$ for all $M \in U$ (Lemma 6). We have seen in Section 3 that $M \in U$ must be of type 1.(a), 1.(b), 2.(a) or 2.(b). Now let $N \in SO(1,2)$ be of type 1.(a), 1.(b), 2.(a) or 2.(b) with eigenvector $\mathbf{t}$. We can normalize simultaneously.

If $N = A_{\frac{2\pi}{3}}$, $N = A_{\frac{4\pi}{3}}$ or $N = Id$ (type 1.(a) or 2.(a)), then it leaves K invariant (Lemma 2). If $N = A_\pi$ (type 1.(b)), then by Lemma 2 $a_6 = 0 = a_7$, which gives a contradiction. If $N = B$ (type 2.(b)), then by Lemma 3 $a_4 = 0 = a_7$ and a_6 is the only non-vanishing coefficient of K.

We get two possibilities for K and the corresponding maximal subgroup U. Either $a_6 \in \mathbb{R} \setminus \{0\}$ and all other coefficients of K vanish, and $U =< A_{\frac{2\pi}{3}}, B >$, if necessary by a change of basis ($\{-\mathbf{t},-\mathbf{v},\mathbf{w}\}$) we can make sure that $a_6 > 0$. Or $a_1 = 2a_4$, $a_4, a_6, a_7 \in \mathbb{R}$ and all other coefficients vanish, and $U =< A_{\frac{2\pi}{3}} >$. As before we can choose $\mathbf{t}$ such that $a_4 \geq 0$. A computation gives that under a change of ONB $\{\mathbf{t}^*,\mathbf{v}^*,\mathbf{w}^*\} = \{\mathbf{t}, \cos s\mathbf{v} + \sin s\mathbf{w}, -\sin s\mathbf{v} + \cos s\mathbf{w}\}$ we obtain for K (cp. (18)): $a_6^* = a_6\cos(3s) + a_7\sin(3s)$, $a_7^* = a_7\cos(3s) - a_6\sin(3s)$, i. e. there exists $s \in \mathbb{R}$ such that $a_7^* = 0$. We can change the sign of a_6 by switching from $\{\mathbf{t},\mathbf{v},\mathbf{w}\}$ to $\{\mathbf{t},-\mathbf{v},-\mathbf{w}\}$. Finally we see that $a_4 \neq 0$.

3. *Case* We assume that there exists $l \neq 0, \pm 1$, with $C_l \in U$. There exists an LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ such that K has the form (Lemma 4): $b_4 \in \mathbb{R}$ and all other coefficients vanish. If $b_4 = 0$, then $K = 0$ and $U = SO(1,2)$.

If $b_4 \neq 0$, then we know that $M\mathbf{v} = \mathbf{v}$ for all $M \in U$ (Lemma 7). We have seen in Theorem 3 that $M \in U$ must be of type 2.(a), 2.(b) or 3.(a). Now let $N \in SO(1,2)$ be of type 2.(a), 2.(b) or 3.(a) with eigenvector $\mathbf{v}$. We can normalize simultaneously and we see that N leaves K invariant (Lemma 4, $B = C_{-1}$). Thus $U = \{C_l, l \in \mathbb{R} \setminus \{0\}\}$. Finally, if $b_4 < 0$, we can change the LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ to $\{\mathbf{f}, -\mathbf{v}, \mathbf{e}\}$ (cp. Remark 3).

4. *Case* We assume that $C_1 \in U$. There exists an LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ such that K has the form (Lemma 5): $b_7 \in \mathbb{R}$ and all other coefficients vanish. If $b_7 = 0$, then $K = 0$ and $U = SO(1,2)$.

If $b_7 \neq 0$, then we know that $M\mathbf{e} = \mathbf{e}$ for all $M \in U$ (Lemma 8). We have seen in Theorem 3 that $M \in U$ must be of type 3.(b). Now let $N \in SO(1,2)$ be of type 3.(b) with eigenvector $\mathbf{e}$, i. e. N has the form $C_{1,m}$, $m \in \mathbb{R}$. We see that N leaves K invariant (Lemma 5). Thus $U = \langle C_1 \rangle$ $(C_{1,m}C_{1,n} = C_{1,m+n})$. Finally, if $b_7 < 0$, we can change the LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ to $\{-\mathbf{e}, \mathbf{v}, -\mathbf{f}\}$ (cp. Remark 3).

4. *Case* We assume that $A_\pi \in U$. There exists an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that K has the form (Lemma 2): $a_1, a_4, a_5 \in \mathbb{R}$ and all other coefficients vanish. Every $M \in U$ must be of type 1.(b) or 2.(b), otherwise we are in one of the foregoing cases.

a) Let $N \in SO(1,2)$ be of type 1.(b). There exists a spacelike eigenvector $\mathbf{w}$ such that $A_\pi \mathbf{w} = -\mathbf{w}$ and $N\mathbf{w} = -\mathbf{w}$, and we can choose an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that A_π has normalform and $N = \begin{pmatrix} \cosh s & -\sinh s & 0 \\ \sinh s & -\cosh s & 0 \\ 0 & 0 & -1 \end{pmatrix}$. If we assume that K is invariant under N then we obtain for $s \neq 0$ that $K = 0$. The computations were done with the CAS Mathematica, too. We obtain from eq3 ($X, Y = \mathbf{t}$, $Z = \mathbf{w}$) and eq9 ($X = \mathbf{v}$, $Y, Z = \mathbf{w}$) that $a_5 = 0$ and $a_1 = a_4$, and from eq1 ($X, Y, Z = \mathbf{t}$) that $a_4 = 0$.

b) Let $N \in SO(1,2)$ be of type 2.(b). There exists a spacelike eigenvector $\mathbf{w}$ such that $A_\pi \mathbf{w} = -\mathbf{w}$ and $N\mathbf{w} = -\mathbf{w}$, and we can choose an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that A_π has normalform and $N = \begin{pmatrix} -\cosh s & -\sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & -1 \end{pmatrix}$. If we assume that K is invariant under N then we obtain that $a_1 = 0 = a_4$. If $s \neq 0$, also $a_5 = 0$, i. e. $K = 0$. The computations were done with the CAS Mathematica, too. If $s \neq 0$, we get from eq3 ($X, Y = \mathbf{t}$, $Z = \mathbf{w}$) and eq9 ($X = \mathbf{v}$, $Y, Z = \mathbf{w}$) that $a_5 = 0$ and $a_1 = a_4$, and from eq1 ($X, Y, Z = \mathbf{t}$) that $a_4 = 0$. If $s = 0$, we get from eq1 ($X, Y, Z = \mathbf{t}$) and eq4 ($X = \mathbf{t}$, $Y, Z = \mathbf{v}$) that $a_1 = 0 = a_4$.

Summarized we got two different forms of K with corresponding maximal subgroups U: a) Either there exists an ONB such that $a_1, a_4, a_5 \in \mathbb{R}$, where $a_1 \neq 2a_4$ or $a_5 \neq 0$, and all other coefficients vanish, this form is preserved by $U =< A_\pi >$. Since $K_\mathbf{t}$ is a symmetric operator on the positive definite space $\mathbf{t}^\perp$, we can diagonalize, then $a_5 = 0$. If necessary, we still can take $\{-\mathbf{t}, -\mathbf{v}, \mathbf{w}\}$ to get $a_1 > 0$ or $a_4 > 0$. b) In the other case there exists an ONB such that $a_5 \in \mathbb{R}$, $a_5 \neq 0$, and all other coefficients vanish, this form is preserved by $U =< A_\pi, B >$.

If $a_5 < 0$, we switch to the ONB $\{-\mathbf{t}, \mathbf{w}, \mathbf{v}\}$.

6. *Case* We assume that $B \in U$. There exists an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that K has the form (Lemma 3): $a_2, a_5, a_6 \in \mathbb{R}$ and all other coefficients vanish. Every $M \in U$ must be of type 2.(b), otherwise we are in one of the foregoing cases. Now let $N \in SO(1,2)$ be of type 2.(b). We canot normalize simultaneously. We only know that B and N both have two-dimensional timelike eigenspaces, which intersect in a line. This line g can be space-, time- or lightlike.

a) If g is spacelike, we can choose an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that B has normalform and $N = \begin{pmatrix} -\cosh s & -\sinh s & 0 \\ \sinh s & \cosh s & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (cp. case 5). If we assume that K is invariant under N then we obtain for $s \neq 0$ that $K = 0$. The computations were done with the CAS Mathematica, too. If $s \neq 0$, we get from eq3 ($X, Y = \mathbf{t}$, $Z = \mathbf{w}$) and eq9 ($X = \mathbf{v}$, $Y, Z = \mathbf{w}$) that $a_5 = 0$ and $a_2 = a_6$, and from eq1 ($X, Y, Z = \mathbf{t}$) that $a_6 = 0$.

b) If g is timelike, we can choose an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ such that B has normalform and $N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & \sin s & -\cos s \end{pmatrix}$. If we assume that K is invariant under N then we obtain for $s \neq 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ that $K = 0$. For $s = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ we are in one of the foregoing cases. The computations were done with the CAS Mathematica, too. If $s \neq 0, \pi$, we get from eq2 ($X, Y = \mathbf{t}$, $Z = \mathbf{v}$) and eq4 ($X = \mathbf{t}$, $Y, Z = \mathbf{v}$) that $a_2 = 0$ and $a_5 = 0$. If also $s \neq \frac{2\pi}{3}, \frac{4\pi}{3}$, then we get from eq10 ($X, Y, Z = \mathbf{w}$) that $a_6 = 0$.

c) If g is lightlike, we can choose an LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ such that B has normalform and $N = \begin{pmatrix} -1 & -2m & 2m^2 \\ 0 & 1 & -2m \\ 0 & 0 & -1 \end{pmatrix}$ (use Lemma 1). If we assume that K is invariant under N then we obtain for $m \neq 0$ that $K = 0$. The computations were done with the CAS Mathematica, too. If $m \neq 0$, we get from eq5 ($X = \mathbf{t}$, $Y = \mathbf{v}$, $Z = \mathbf{w}$) and eq2 ($X, Y = \mathbf{t}$, $Z = \mathbf{v}$) that $b_6 = 0$ and $b_4 = 0$, and from eq1 ($X, Y, Z = \mathbf{t}$) that $b_2 = 0$.

Therefore we have that $U = \langle B \rangle$. If $a_6 < 0$, we can switch to $\{-\mathbf{t}, -\mathbf{v}, \mathbf{w}\}$. Since $K_{\mathbf{v}}$ is a symmetric operator on an indefinite space we cannot always diagonalize. Thus we cannot simplify K in general. □

Remark. In the proof we only have used multilinear algebra. Thus the theorem stays true for an arbitrary $(1,2)$-tensor K on $\mathbb{R}^3_1$ with $\langle K(X,Y), Z\rangle$ totally symmetric and vanishing trace K_X.

5. Pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry

Let M^3 be a hypersphere admitting a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry. According to Theorem 4, there exists for every $p \in M^3$ an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ of T_pM^3 such that

$$K(\mathbf{t}, \mathbf{t}) = 0, \qquad K(\mathbf{t}, \mathbf{v}) = a_5\mathbf{w}, \qquad K(\mathbf{t}, \mathbf{w}) = a_5\mathbf{v}, \tag{28}$$

$$K(\mathbf{v}, \mathbf{v}) = 0, \qquad K(\mathbf{v}, \mathbf{w}) = -a_5\mathbf{t}, \qquad K(\mathbf{w}, \mathbf{w}) = 0. \tag{29}$$

Substituting this in equation (13), we obtain

$$\hat{R}(X,Y)Z = (H - a_5^2)(h(Y,Z)X - h(X,Z)Y). \tag{30}$$

Schur's Lemma implies that M^3 has constant sectional curvature, by the affine theorema egregium (9) we obtain that $\hat{\kappa} = H - a_5^2$ and $J = -a_5^2 < 0$. Affine hyperspheres with constant affine sectional curvature and nonzero Pick invariant were classified by Magid and Ryan [12]. They show in their main theorem that an affine hypersphere with Lorentz metric of constant curvature and nonzero Pick invariant is equivalent to an open subset of either $(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1$ or $(x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1$. In both cases $\hat{\kappa} = 0$, i. e. $H = -J$. In the proof of the main theorem they explicitly show that only $(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1$ has negative Pick invariant and that K has normalform. This proves:

Theorem 1. *An affine hypersphere admits a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry if and only if it is affine equivalent to an open subset of*

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1.$$

For $(x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1$ they compute that the only non-vanishing coefficient of K is a_2. Thus it follows (Theorem 4):

Remark. The affine hypersphere $(x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1$ admits a pointwise $\mathbb{Z}_2$-symmetry.

6. Pointwise $\mathbb{R}$-symmetry

Let M^3 be a hypersphere admitting a pointwise $\mathbb{R}$-symmetry. According to Theorem 4, there exists for every $p \in M^3$ a LVB $\{\mathbf{e}, \mathbf{v}, \mathbf{f}\}$ of T_pM^3 such that

$$K(\mathbf{e},\mathbf{e}) = 0, \qquad K(\mathbf{e},\mathbf{v}) = 0, \qquad K(\mathbf{e},\mathbf{f}) = 0, \tag{31}$$

$$K(\mathbf{v},\mathbf{v}) = 0, \qquad K(\mathbf{v},\mathbf{f}) = 0, \qquad K(\mathbf{f},\mathbf{f}) = b_7\mathbf{e}. \tag{32}$$

Substituting this in equation (13), we obtain

$$\hat{R}(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y). \tag{33}$$

Schur's Lemma implies that M^3 has constant sectional curvature, by the affine theorema egregium (9) we obtain that $\hat{\kappa} = H$ and $J = 0$. Affine hyperspheres with constant affine sectional curvature and zero Pick invariant were classified in [4] (see Theorem 6.2 ($H = 0$), Theorem 7.2 ($H = 1$) and Theorem 8.2 ($H = -1$)). They are determined by a null curve in resp. $\mathbb{R}_1^3$, S_1^3, H_1^3, and a function along this curve (note that in the notion of [4] (2) holds).

Theorem 2. *Let M^3 be an affine hypersphere admitting a pointwise $\mathbb{R}$-symmetry. Then M^3 has constant sectional curvature $\hat{\kappa} = H$ and zero Pick invariant $J = 0$.*

Remark. A study of [4] shows that an affine hypersphere admits a pointwise $\mathbb{R}$-symmetry if and only if (2) holds (in their notations).

If (3) holds for an affine hypersphere with constant sectional curvature and zero Pick invariant, then it admits a pointwise $\mathbb{Z}_2$-symmetry (cp. Theorem 4).

Appendix: Non-trivial $SO(1,2)$-stabilizers

In[1]:= $e[1] := \{1,0,0\}; e[2] := \{0,1,0\}; e[3] := \{0,0,1\};$

ONB of SO(1,2), e[1]=t, e[2]=v, e[3]=w

Affine metric h

In[2]:= $\mathrm{honb}[y\_, z\_] := -(y[[1]] * z[[1]]) + y[[2]] * z[[2]] + y[[3]] * z[[3]];$

Difference tensor K

In[3]:= $\mathrm{Konb}[y\_, z\_] := \mathrm{Sum}[y[[i]] * z[[j]] * \mathrm{konb}[i,j], \{i,1,3\}, \{j,1,3\}];$
$\mathrm{konb}[1,1] := \{-a1, a2, a3\}; \mathrm{konb}[1,2] := \{-a2, a4, a5\}; \mathrm{konb}[1,3] := \{-a3, a5, a1-a4\};$
$\mathrm{konb}[2,1] := \{-a2, a4, a5\}; \mathrm{konb}[2,2] := \{-a4, a6, a7\}; \mathrm{konb}[2,3] := \{-a5, a7, a2-a6\};$
$\mathrm{konb}[3,1] := \{-a3, a5, a1-a4\}; \mathrm{konb}[3,2] := \{-a5, a7, a2-a6\};$
$\mathrm{konb}[3,3] := \{-(a1-a4), a2-a6, a3-a7\};$

K invariant under a transformation f[t]?

In[4]:= $\mathrm{eq1}[t\_] := \mathrm{honb}[\mathrm{konb}[1,1], e[1]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[1][t]], f[1][t]];$
$\mathrm{eq2}[t\_] := \mathrm{honb}[\mathrm{konb}[1,1], e[2]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[1][t]], f[2][t]];$
$\mathrm{eq3}[t\_] := \mathrm{honb}[\mathrm{konb}[1,1], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[1][t]], f[3][t]];$
$\mathrm{eq4}[t\_] := \mathrm{honb}[\mathrm{konb}[1,2], e[2]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[2][t]], f[2][t]];$
$\mathrm{eq5}[t\_] := \mathrm{honb}[\mathrm{konb}[1,2], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[2][t]], f[3][t]];$
$\mathrm{eq6}[t\_] := \mathrm{honb}[\mathrm{konb}[1,3], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[1][t], f[3][t]], f[3][t]];$
$\mathrm{eq7}[t\_] := \mathrm{honb}[\mathrm{konb}[2,2], e[2]] - \mathrm{honb}[\mathrm{Konb}[f[2][t], f[2][t]], f[2][t]];$
$\mathrm{eq8}[t\_] := \mathrm{honb}[\mathrm{konb}[2,2], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[2][t], f[2][t]], f[3][t]];$
$\mathrm{eq9}[t\_] := \mathrm{honb}[\mathrm{konb}[2,3], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[2][t], f[3][t]], f[3][t]];$
$\mathrm{eq10}[t\_] := \mathrm{honb}[\mathrm{konb}[3,3], e[3]] - \mathrm{honb}[\mathrm{Konb}[f[3][t], f[3][t]], f[3][t]];$

In[5]:= $\mathrm{eq}[t\_] :=$
$\mathrm{FullSimplify}[\{\mathrm{eq1}[t] == 0, \mathrm{eq2}[t] == 0, \mathrm{eq3}[t] == 0, \mathrm{eq4}[t] == 0, \mathrm{eq5}[t] == 0,$
$\mathrm{eq6}[t] == 0, \mathrm{eq7}[t] == 0, \mathrm{eq8}[t] == 0, \mathrm{eq9}[t] == 0, \mathrm{eq10}[t] == 0\}];$

Lemma 2

Transformations A_t

In[6]:= $f[1][t\_] := \{1,0,0\}; f[2][t\_] := \{0, \cos[t], \sin[t]\}; f[3][t\_] := \{0, -\sin[t], \cos[t]\};$

In[7]:= $eq[t]$

eq2 and eq3 give

In[8]:= $a2 = 0;\ \ a3 = 0;\ \ \mathrm{eq}[t]$

t=$\boldsymbol{\pi}$, thus sin[t]=0

In[9]:= $\mathrm{eq}[\pi]$

eq7 and eq8 give that $a6 = 0$ and $a7 = 0$

t≠π: eq4 and eq5 give

In[10]:= $a5 = 0;\ a1 := 2a4;\ eq[t]$

t=2π/3 or 4π/3, thus cos[3t]=1

In[11]:= $eq[2\pi/3]$

In[12]:= $eq[4\pi/3]$

In[13]:= $\{a1, a2, a3, a4, a5, a6, a7\}$

t≠2π/3 or 4π/3, thus cos[3t]≠1: eq7 and eq8 give

In[14]:= $a6 = 0;\ a7 = 0;$

In[15]:= $\{a1, a2, a3, a4, a5, a6, a7\}$

Result: $\{a1, a2, a3, a4, a5, a6, a7\} = \{2\ a4, 0, 0, a4, 0, 0, 0\}$

In[16]:= $\mathrm{Clear}[a1, a2, a3, a4, a5, a6, a7]$

Lemma 3

Transformations B

In[17]:= $f[1][t\_] := \{-1, 0, 0\};\ f[2][t\_] := \{0, 1, 0\};\ f[3][t\_] := \{0, 0, -1\};$

In[18]:= $eq[t]$

eq1, eq3, eq4 and eq8 give

In[19]:= $a1 = 0;\ a3 = 0;\ a4 = 0;\ a7 = 0;\ eq[t]$

In[20]:= $\{a1, a2, a3, a4, a5, a6, a7\}$

Result: $\{a1, a2, a3, a4, a5, a6, a7\} = \{0, a2, 0, 0, a5, a6, 0\}$

In[21]:= $\mathrm{Clear}[a1, a2, a3, a4, a5, a6, a7]$

LVB of SO(1,2), e[1]=e, e[2]=v, e[3]=f

Affine metric h

In[22]:= $hl[y\_, z\_] := y[[1]]z[[3]] + y[[3]]z[[1]] + y[[2]]z[[2]];$

Difference tensor K

In[23]:= $Kl[y\_, z\_] := \mathrm{Sum}[y[[i]] * z[[j]] * \mathrm{kl}[i, j], \{i, 1, 3\}, \{j, 1, 3\}];$
$\mathrm{kl}[1, 1] := \{b1, b2, b3\}; \mathrm{kl}[1, 2] := \{b4, -2b1, b2\}; \mathrm{kl}[1, 3] := \{b5, b4, b1\};$
$\mathrm{kl}[2, 1] := \{b4, -2\ b1, b2\}; \mathrm{kl}[2, 2] := \{-2\ b5, -2\ b4, -2\ b1\}; \mathrm{kl}[2, 3] := \{b6, -2\ b5, b4\};$
$\mathrm{kl}[3, 1] := \{b5, b4, b1\}; \mathrm{kl}[3, 2] := \{b6, -2\ b5, b4\}; \mathrm{kl}[3, 3] := \{b7, b6, b5\};$

K invariant under a transformation f[l,m]?

In[2]:= $\mathrm{eq1}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,1], e[1]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[1][l,m]], f[1][l,m]];$
$\mathrm{eq2}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,1], e[2]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[1][l,m]], f[2][l,m]];$
$\mathrm{eq3}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,1], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[1][l,m]], f[3][l,m]];$
$\mathrm{eq4}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,2], e[2]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[2][l,m]], f[2][l,m]];$
$\mathrm{eq5}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,2], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[2][l,m]], f[3][l,m]];$
$\mathrm{eq6}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[1,3], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[1][l,m], f[3][l,m]], f[3][l,m]];$
$\mathrm{eq7}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[2,2], e[2]] - \mathrm{hl}[\mathrm{Kl}[f[2][l,m], f[2][l,m]], f[2][l,m]];$
$\mathrm{eq8}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[2,2], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[2][l,m], f[2][l,m]], f[3][l,m]];$
$\mathrm{eq9}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[2,3], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[2][l,m], f[3][l,m]], f[3][l,m]];$
$\mathrm{eq10}[l\_, m\_] := \mathrm{hl}[\mathrm{kl}[3,3], e[3]] - \mathrm{hl}[\mathrm{Kl}[f[3][l,m], f[3][l,m]], f[3][l,m]];$

Transformations C_l and C_1,m

In[25]:= $f[1][l\_, m\_] := \{l, 0, 0\};\quad f[2][l\_, m\_] := \{-l\ m\ , 1, 0\};\ f[3][l\_, m\_] := \{-l\ m\hat{}2/2, m, 1/l\};$

In[26]:= $\mathrm{eq}[l\_, m\_] :=$
$\mathrm{FullSimplify}[\{\mathrm{eq1}[l,m] == 0, \mathrm{eq2}[l,m] == 0, \mathrm{eq3}[l,m] == 0, \mathrm{eq4}[l,m] == 0,$
$\mathrm{eq5}[l,m] == 0, \mathrm{eq6}[l,m] == 0, \mathrm{eq7}[l,m] == 0, \mathrm{eq8}[l,m] == 0, \mathrm{eq9}[l,m] == 0,$
$\mathrm{eq10}[l,m] == 0\}];$
$\mathrm{eq}[l,m]$

Lemma 4

C_l, i.e. m=0

In[27]:= $\mathrm{eq}[l, 0]$

eq1, eq3, eq6 and eq10 give (l≠ 1)

In[28]:= $b3 = 0;\ b1 = 0;\ b5 = 0;\ b7 = 0;$

l=-1 (C_(-1)=B)

In[29]:= $\mathrm{eq}[-1, 0]$

In[30]:= $\{b1, b2, b3, b4, b5, b6, b7\}$

Result: $\{b1, b2, b3, b4, b5, b6, b7\} = \{0, b2, 0, b4, 0, b6, 0\}$

l≠-1

eq2 and eq9 give (l≠-1)

In[31]:= $b2 = 0;\ b6 = 0;\ \{b1, b2, b3, b4, b5, b6, b7\}$

Result: $\{b1, b2, b3, b4, b5, b6, b7\} = \{0, 0, 0, b4, 0, 0, 0\}$

In[32]:= $\mathrm{Clear}[b1, b2, b3, b4, b5, b6, b7]$

Lemma 5

C_1, i.e. l=1 (m≠0)

In[33]:= $\mathrm{eq}[1, m]$

eq2 gives

In[34]:= $b3 = 0;\ \mathrm{eq}[1, m]$

eq3 gives

In[35]:= $b2 = 0;\ \mathrm{eq}[1, m]$

eq5 gives

In[36]:= $b1 = 0;\ \mathrm{eq}[1, m]$

eq6 gives

In[37]:= $b4 = 0;\ \mathrm{eq}[1, m]$

eq9 gives

In[38]:= $b5 = 0;\ \mathrm{eq}[1, m]$

eq10 gives

In[39]:= $b6 = 0;\ \mathrm{eq}[1, m]$

In[40]:= $\{b1, b2, b3, b4, b5, b6, b7\}$

Result: $\{b1, b2, b3, b4, b5, b6, b7\} = \{0, 0, 0, 0, 0, 0, b7\}$

In[41]:= $\mathrm{Clear}[b1, b2, b3, b4, b5, b6, b7]$

Proof of Theorem 2

5. Case

In[42]:= $a2 = 0;\ a3 = 0;\ a6 = 0;\ a7 = 0;$

a) transformations N

In[43]:= $f[1][t\_] := \{\cosh[t], -\sinh[t], 0\};\ f[2][t\_] := \{\sinh[t], -\cosh[t], 0\};\ f[3][t\_] := \{0, 0, -1\};$

In[44]:= $\mathrm{eq}[t]$

eq5 and eq9 give

In[45]:= $a5 = 0; a1 = a4;\ \mathrm{eq}[t]$

In[46]:= $a4 = 0;\ \{a1, a2, a3, a4, a5, a6, a7\}$

Result: K= 0

In[47]:= $\mathrm{Clear}[a1, a4, a5]$

b) transformations N

In[48]:= $f[1][t\_] := \{-\cosh[t], -\sinh[t], 0\}; f[2][t\_] := \{\sinh[t], \cosh[t], 0\};\ f[3][t\_] := \{0, 0, -1\};$

In[49]:= $\mathrm{eq}[t]$

s≠0: eq3 and eq9 give

In[50]:= $a5 = 0;\ a1 = a4;\ \mathrm{eq}[t]$

In[51]:= $a4 = 0;\ \{a1, a2, a3, a4, a5, a6, a7\}$

Result: s≠0: K= 0

In[52]:= $\mathrm{Clear}[a1, a4, a5]$

s=0

In[53]:= $\mathrm{eq}[0]$

In[54]:= $a1 = 0;\ a4 = 0;\ \{a1, a2, a3, a4, a5, a6, a7\}$

Result: s=0: $\{a1, a2, a3, a4, a5, a6, a7\} = \{0, 0, 0, 0, a5, 0, 0\}$

In[55]:= $\mathrm{Clear}[a1, a2, a3, a4, a5, a6, a7]$

6. Case, ONB

In[56]:= $a1 = 0;\ a3 = 0;\ a4 = 0;\ a7 = 0;$

a) transformations N

In[57]:= $f[1][t\_] := \{-\cosh[t], -\sinh[t], 0\};\ f[2][t\_] := \{\sinh[t], \cosh[t], 0\};\ f[3][t\_] := \{0, 0, -1\};$

In[58]:= $\mathrm{eq}[t]$

s≠0: eq3 and eq9 give

In[59]:= $a5 = 0;\ a2 = a6;\ \mathrm{eq}[t]$

In[60]:= $a6 = 0;\ \{a1, a2, a3, a4, a5, a6, a7\}$

Result: s≠0: K= 0

In[61]:= $\mathrm{Clear}[a2, a5, a6]$

b) transformations N

In[62]:= $f[1][t\_] := \{-1, 0, 0\};\ f[2][t\_] := \{0,\ \cos[t], \sin[t]\};\ f[3][t\_] := \{0, \sin[t], -\cos[t]\};$

In[63]:= $\mathrm{eq}[t]$

s≠0,π (s=0 or π, then f=B): eq2 and eq4 give

In[64]:= $a2 = 0;\ a5 = 0;$

in addition s≠2π/3,4π/3 (s=2π/3 or 4π/3, then f=A_{2π/3}. B)

In[65]:= $\{a1, a2, a3, a4, a5, a6, a7\}$

eq10 gives

In[66]:= $a6 = 0;\ \{a1, a2, a3, a4, a5, a6, a7\}$

In[67]:= $\mathrm{Clear}[a1, a2, a3, a4, a5, a6, a7]$

6. Case, LVB

In[68]:= $b1 = 0;\ b3 = 0;\ b5 = 0;\ b7 = 0;$

c) transformations N

In[69]:= $f[1][l\_, m\_] := \{-1, 2\ m, 2\ m\hat{\ }2\}; f[2][l\_, m\_] := \{0, 1, 2\ m\};\ f[3][l\_, m\_] := \{0, 0, -1\};$

In[70]:= $\mathrm{eq}[l, m]$

m≠0: eq5 and eq2 give

In[71]:= $b6 = 0;\ b4 = 0;\ \mathrm{eq}[l, m]$

In[72]:= $b2 = 0;\ \{b1, b2, b3, b4, b5, b6, b7\}$

Result: K= 0

In[73]:= $\mathrm{Clear}[b1, b2, b3, b4, b5, b6, b7]$

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