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Multiplication Modules and Homogeneous Idealization II

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Abstract. All rings are commutative with identity and all modules are unital. Let R be a ring, M an R-module and $R(M)$, the idealization of M. Homogenous ideals of $R(M)$ have the form $I_{(+)}N$, where I is an ideal of R and N a submodule of M such that $IM \subseteq N$. A ring $R(M)$ is called a homogeneous ring if every ideal of $R(M)$ is homogeneous. In this paper we continue our recent work on the idealization of multiplication modules and give necessary and sufficient conditions for a homogeneous ideal to be an almost (generalized, weak) multiplication, projective, finitely generated flat, pure or invertible (q-invertible). We determine when a ring $R(M)$ is a general ZPI-ring, distributive ring, quasi-valuation ring, P-ring, coherent ring or finite conductor ring. We also introduce the concept of weakly prime submodules generalizing weakly prime ideals. Various properties and characterizations of weakly prime submodules of faithful multiplication modules are considered.

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0. Introduction

Let R be a commutative ring and M an R-module. M is a *multiplication* module if every submodule N of M has the form IM for some ideal I of R. Equivalently,

$N = [N : M]M$, [12]. A submodule K of M is multiplication if and only if $N \cap K = [N : K]K$ for all submodules K of M, [30, Lemma 1.3].

Let P be a maximal ideal of R and let $T_P(M) = \{m \in M : (1-p)m = 0$ for some $p \in P\}$. Then $T_P(M)$ is a submodule of M. M is called *P-torsion* if $T_P(M) = M$. On the other hand M is called *P-cyclic* provided there exist $m \in M$ and $q \in P$ such that $(1-q)M \subseteq Rm$. El-Bast and P. F. Smith, [13, Theorem 1.2], showed that M is multiplication if and only if M is P-torsion or P-cyclic for each maximal ideal P of R. M is said to be an *almost multiplication* module if M_P is a multiplication module for each prime ideal P of R. *Weak multiplication* modules are defined to be the modules in which every prime submodule P has the form $[P : M]M$. The Z-module Q is weak multiplication but not multiplication. An R-module M is weak multiplication if and only if it is locally weak multiplication, from which it follows that a weak multiplication module over a local ring is a multiplication module. Thus the concepts of weak multiplication and multiplication coincide if the module is finitely generated. An R-module M is called a *generalized multiplication* module if for every pair of proper submodules K and N of M, $K \cap N = [K : N]N$, [20]. The $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty}$ is a generalized multiplication module which is not a multiplication module. Let N be a submodule of M and I an ideal of R. The *residual* submodule N by I is $[N :_M I] = \{m \in M : Im \subseteq N\}$, [23] and [24]. Obviously, $[N : IM]M \subseteq [N :_M I]$. The reverse inclusion is true if M is multiplication, [3]. If M is faithful multiplication, $[0 :_M I] = (\text{ann}I)M$. If M is a flat R-module and I a finitely generated ideal of R, then this property also holds. For, let $I = \sum_{i=1}^{n} Ra_i$ and $m \in [0 :_M I]$. Then $a_i m = 0$ for each $1 \leq i \leq n$. As M is flat, [24, Theorem 7.6] shows that there exist $r_i \in R$ and $k \in M$ such that $r_i a_i = 0$ and $m = r_i k$. Hence $m \in \text{ann}(a_i)M$ for each i, from which it follows that $m \in \bigcap_{i=1}^{n} \text{ann}(a_i)M = \left(\bigcap_{i=1}^{n} \text{ann}(a_i)\right) M = (\text{ann}I)M$, [24, Theorem 7.4]. Hence $(\text{ann}I)M \supseteq [0 :_M I]$ and the equality holds.

Let R be a ring and M an R-module. A submodule N of M is *pure in M* if N is a direct summand of M, [24]. Every ideal I of R is pure if and only if it is multiplication and idempotent. M is projective if and only if it is a direct summand of a free R-module. It is proved, [31, Theorems 2.1 and 2.2], that a finitely generated I is projective (resp. flat) if and only if I is multiplication and $\text{ann}I = Re$ for some idempotent e of R (resp. $\text{ann}I$ is a pure ideal of R). More generally, if M is a finitely generated multiplication such that $\text{ann}M = Re$ for some idempotent e, then M is projective, [32, Theorem 11], and multiplication modules with pure annihilator are flat, [6, Corollary 2.7] and [26, Theorem 4.1]. It is also well-known that a finitely generated ideal I of R is projective if and only if I is flat and $\text{ann}I$ is finitely generated, [33, Corollary 3.1].

Let R be a ring and M an R-module. Let S be the set of regular elements of R and R_S the total quotient ring of R. For a non-zero ideal I of R, let $I^{-1} = \{x \in R_S : xI \subseteq R\}$. I is an invertible ideal of R if $II^{-1} = R$. Let $T = \{t \in S : tm = 0$ for some $m \in M$ implies $m = 0\}$. T is a multiplicatively closed subset of S, and if M is torsion-free then $T = S$. Let N be a non-zero submodule of M and let

$N^{-1} = \{x \in R_T : xN \subseteq M\}$. N^{-1} is an R_S-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. Following [28], N is *invertible in* M if $NN^{-1} = M$. It is shown, [28, Remark 3.2 and Lemma 3.3], that if is I an invertible ideal of R then IM is invertible in M. The converse is true if M is finitely generated, faithful and multiplication.

Let R be a commutative ring with identity and M an R-module. Then $R_{(+)}M = R(M)$ with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity called *idealization of* M or the *trivial extension of* R *by* M. The idealization of a module is a well-established method to facilitate interaction between a ring on the one hand and a module over a ring on the other. The basic construction is to embed the module M as an ideal in a ring $R(M)$ which contains R as a subring. This technique was used with great success by Nagata, [25]. For a comprehensive survey on idealization, Anderson and Winders [10] and Huckaba [19, Section 25] can be consulted. An ideal H of $R(M)$ is called *homogeneous* if $H = I_{(+)}N$ where I is an ideal of R and N a submodule of M. In this case, $I_{(+)}N = (R_{(+)}M)(I_{(+)}N) = I_{(+)}(IM + N)$ gives that $IM \subseteq N$ (equivalently, $[N :_M I] = M$). In [2] we called a ring $R(M)$ is *homogeneous* if every ideal H of $R(M)$ is homogeneous. Several properties and characterizations of homogeneous ideals and homogeneous rings are considered in [1] and [2]. In this note we continue our investigation of homogeneous ideals and homogeneous rings. In Section 1 we determine when a homogeneous ideal of $R(M)$ is finitely generated flat (finitely generated projective), almost (generalized, weak) multiplication, pure or invertible. In Section 2 we investigate properties of a ring $R(M)$, especially when $R(M)$ is homogeneous. We show how properties of $R(M)$ such as SPIR, general ZPI-ring, Bezout ring, quasi-valuation ring, P-ring, coherent ring or finite conductor ring are related to those of R and M. In the last section of the paper we introduce the concept of weakly prime submodules as a generalization of weakly prime ideals. Various properties and characterizations of weakly prime submodules of faithful multiplication modules are given.

All rings are assumed to be commutative with 1 and all modules are unital. For the basic concepts used, we refer the reader to [15], [16], [19], [23], [24], [25], and [34].

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1. Homogenous ideals

Let $I_{(+)}N$ and $J_{(+)}K$ be two homogenous ideals of $R(M)$. Then the ideal $[I_{(+)}N :_{R(M)} J_{(+)}K] = [I : J] \cap [N : K]_{(+)}[N :_M J]$ is homogeneous, [2, Lemma 1]. Consequently,

$$\mathrm{ann}(I_{(+)}N) = \mathrm{ann}I \cap \mathrm{ann}N_{(+)}[0 :_M I].$$

If I is a (finitely generated) ideal and M a (flat) faithful multiplication R-module then $[0 :_M I] = (\mathrm{ann}I)M$, and hence $\mathrm{ann}(I_{(+)}N) = \mathrm{ann}I \cap \mathrm{ann}N_{(+)}(\mathrm{ann}I)M$. Let r be a regular element in R. If (r, m) is a regular element of $R(M)$ then r is regular in R from which it follows that if $I_{(+)}N$ is a regular ideal of $R(M)$ then I is a regular ideal of R, see [1, Lemma 6] and [10, Theorem 3.5]. The converse is true if M is torsion-free. For, let r be regular and $m \in M$. Let $(s, n) \in R(M)$ such that $(r, m)(s, n) = (0, 0)$. Then $rs = 0$ (and hence $s = 0$) and $rn + sm = 0$. This gives that $rn = 0$. As M is torsion-free, $n = 0$. Let $Z(R)$ be the set of zero-divisors of R and $Z(M)$ the set of zero-divisors on M, that is $Z(M) = \{r \in R : rm = 0$ for some non-zero $m \in M\}$. Then $Z(R(M)) = (Z(R) \cup Z(M))_{(+)}M$ and the set of regular elements of $R(M)$ is $S_{(+)}M$ where $S = R - (Z(R) \cup Z(M))$, [10, Theorems 3.5 and 4.3] and [19, Theorem 25.3]. If M is a torsion-free R-module then $Z(M) \subseteq Z(R)$ and hence $Z(R(M)) = Z(R)_{(+)}M$ and $S_{(+)}M = \{(r, m) : r$ is a regular element of R and $m \in M\}$.

We start this section by a result showing how properties of a submodule N of M are related to those of the ideal $0_{(+)}N$ of $R(M)$.

Proposition 1. *Let R be a ring, M an R-module and N a submodule of M.*

(1) *N is almost multiplication if and only if $0_{(+)}N$ is an almost multiplication ideal of $R(M)$.*

(2) *N is generalized multiplication if and only if $0_{(+)}N$ is a generalized multiplication ideal of $R(M)$.*

(3) *If $0_{(+)}N$ is a pure or projective ideal of $R(M)$ then $N = 0$.*

Proof. (1) This follows by the fact that N is locally cyclic if and only if $0_{(+)}N$ is a locally principal ideal of $R(M)$, [9, Theorem 3.1].

(2) Suppose N is generalized multiplication. Let H_1 and H_2 be ideals of $R(M)$ that are properly contained in $0_{(+)}N$. Then $H_1 = 0_{(+)}K$ and $H_2 = 0_{(+)}L$ for some proper submodules K and L of N. It follows that

$$\begin{aligned} H_1 \cap H_2 &= 0_{(+)}K \cap 0_{(+)}L = 0_{(+)}K \cap L = 0_{(+)}[K : L]L \\ &= ([K : L]_{(+)}M)(0_{(+)}L) = [0_{(+)}K :_{R(M)} 0_{(+)}L](0_{(+)}L) \\ &= [H_1 :_{R(M)} H_2]H_2, \end{aligned}$$

and $0_{(+)}N$ is generalized multiplication. The argument is reversible.

(3) Suppose $0_{(+)}N$ is a pure ideal of $R(M)$ then $R(M) = 0_{(+)}N \oplus H$ for some ideal H of $R(M)$. It follows that $0_{(+)}N = H(0_{(+)}N) \subseteq H \cap 0_{(+)}N = 0$, so that $0_{(+)}N = 0$ and hence $N = 0$. If $0_{(+)}N$ is projective, it follows by, [5, Theorem 3.3], that $R\,(M) = \operatorname{ann}(0_{(+)}N)^2 = \operatorname{ann}(0_{(+)}N) = \operatorname{ann}N + M$. Hence $R = \operatorname{ann}N$, and hence $N = 0$. Alternatively, by the Dual Basis Lemma there exist a family of $R\,(M)$-homomorphisms $f_i : 0_{(+)}N \to R\,(M)$ and a family of elements $\{(0, k_i)\} \subseteq 0_{(+)}N$ such that for each $(0, n) \in 0_{(+)}N$,

$$(0, n) = \sum_i f_i\,(0, n)\,(0, k_i) = \sum_i f_i\,((0, n)\,(0, k_i)) = \sum_i f_i\,(0, 0) = (0, 0)\,.$$

Hence $n = 0$, and this implies that $N = 0$. □

A. G. Naoum introduced in [27] quasi-invertible (q-invertible) ideals as a generalization of invertible ideals: An ideal I of R is called *q-invertible* for an idempotent

e if $II^{-1} = Re$ and $\mathrm{ann}I = \mathrm{ann}(e)$. If I is q-invertible for e then I is finitely generated. It is shown, [27, Theorem 1.5], that I is q-invertible for e if and only if I is a finitely generated projective ideal and $\mathrm{ann}I = \mathrm{ann}(a) = \mathrm{ann}(e)$ for some $a \in I$.

The next theorem shows that properties of an ideal I of a ring R such as multiplication, projective, flat, pure, invertible or q-invertible are usually related to those of the ideal $I_{(+)}IM$ of $R(M)$.

Theorem 2. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *I is pure if and only if $I_{(+)}IM$ is a pure ideal of $R(M)$.*

(2) *I is almost multiplication if and only if $I_{(+)}IM$ is an almost multiplication ideal of $R(M)$.*

(3) *If $I_{(+)}IM$ is an invertible ideal of $R(M)$ then I is invertible. The converse is true if M is torsion-free.*

(4) *If $I_{(+)}IM$ is a finitely generated flat ideal of $R(M)$ then I is finitely generated flat. The converse is true if M is torsion-free.*

(5) *If $I_{(+)}IM$ is a finitely generated projective ideal of $R(M)$ then I is finitely generated projective. The converse is true if M is flat.*

(6) *If M is finitely generated, faithful and multiplication and $I_{(+)}IM$ is q-invertible for $(e,0)$ then I is q-invertible for e. Conversely, if M is flat and I is q-invertible for e then $I_{(+)}IM$ is q-invertible for $(e,0)$.*

(7) *If $I_{(+)}IM$ is a generalized multiplication ideal of $R(M)$ then I is generalized multiplication.*

Proof. (1) Suppose I is a pure ideal of R. Then $R = I \oplus J$ for some ideal J of R. It follows that $R(M) = (I+J)_{(+)}(I+J)M = I_{(+)}IM + J_{(+)}JM$. Since $I \cap J = 0, IJ = 0$. As $R = I + J$, we infer that

$$IM \cap JM = I(IM \cap JM) + J(IM \cap JM) \subseteq IJM = 0,$$

so that $IM \cap JM = 0$. This implies that $I_{(+)}IM \cap J_{(+)}JM = (I \cap J)_{(+)}(IM \cap JM) = 0$, and hence $I_{(+)}IM$ is a direct summand of $R(M)$. Therefore $I_{(+)}IM$ is a pure ideal of $R(M)$. Conversely, let $I_{(+)}IM$ be pure. Then $R(M) = I_{(+)}IM \oplus H$ for some ideal H of $R(M)$. Let $H + 0_{(+)}M = J_{(+)}M$ for some ideal J of R. Then $R(M) = I_{(+)}IM + J_{(+)}M = (I_{+}J)_{(+)}M$. Hence $R = I + J$. Since $R(M) = I_{(+)}IM + H$ is a multiplication ideal of $R(M)$, we obtain from, [7, Theorem 2.1], that

$$0_{(+)}M = 0_{(+)}M + (I_{(+)}IM \cap H) = I_{(+)}M \cap H + 0_{(+)}M = I_{(+)}M \cap J_{(+)}M = (I \cap J)_{(+)}M.$$

Hence $I \cap J = 0$, and hence $R = I \oplus J$. So I is a pure ideal of R.

(2) It follows by the fact that I is locally principal if and only if $I_{(+)}IM$, is a locally principal ideal of $R(M)$, [1, Theorem 7].

(3) Suppose $I_{(+)}IM$ is an invertible ideal of $R(M)$. Then $I_{(+)}IM$ is a regular ideal of $R(M)$ and hence it contains a regular element, say (a,n). Hence a is a

regular element of I. As $I_{(+)}IM$ is multiplication, $R(M)(a,0) = H(I_{(+)}IM)$ for some ideal H of $R(M)$. Let $H + 0_{(+)}M = J_{(+)}M$ for some ideal J of R. Then

$$\begin{aligned} Ra_{(+)}IM &= R(M)(a,0) + 0_{(+)}IM = H(I_{(+)}IM) + 0_{(+)}IM \\ &= (H + 0_{(+)}M)(I_{(+)}IM) = (J_{(+)}M)(I_{(+)}IM) = JI_{(+)}IM. \end{aligned}$$

So $Ra = JI$ and hence I is an invertible ideal of R. Alternatively, as $I_{(+)}IM$ is invertible, $R(M) = (I_{(+)}IM)(I_{(+)}IM)^{-1}$. Suppose $(r,m) \in (I_{(+)}IM)^{-1}$. Then $(r,m) \in R(M)_{S(+)M} \cong R_{S(+)}M_S$, where $S = R - (Z(R) \cup Z(M))$, [10, Theorems 3.5 and 4.1]. It follows that $(r,m)(a,b) \in R(M)$ for each $(a,b) \in I_{(+)}IM$. Hence $ra \in R$ for each $a \in I$, and hence $rI \subseteq R$. This implies that $r \in I^{-1}$ and hence

$$R(M) = (I_{(+)}IM)(I_{(+)}IM)^{-1} \subseteq (I_{(+)}IM)(I^{-1}{}_{(+)}M_S) = II^{-1}{}_{(+)}(IM_S + II^{-1}M).$$

This shows that $R \subseteq II^{-1} \subseteq R$, so that $II^{-1} = R$ and I is invertible. Conversely, let M be torsion-free and I invertible. Suppose $a \in I$ is regular. Then $(a,0)$ is a regular element of $I_{(+)}IM$. Now, since I is invertible, $Ra = JI$ for some ideal J of R. Hence

$$R(M)(a,0) = Ra_{(+)}aM = JI_{(+)}JIM = (I_{(+)}IM)(J_{(+)}JM),$$

and hence $I_{(+)}IM$ is an invertible ideal of $R(M)$.

(4) An R-module M is flat if and only if it is locally flat, [15]. Also, finitely generated flat ideals are locally either zero or invertible [18, Lemma 2.1] and [33, Lemma 6]. The result follows by these two facts and (3).

(5) Let $I_{(+)}IM$ be a finitely generated projective ideal of $R(M)$. Then $I_{(+)}IM$ is finitely generated flat and by (4), I is finitely generated flat. Also, $\text{ann}(I_{(+)}IM) = \text{ann}I_{(+)}[0:_M I]$ is finitely generated from which it follows that $\text{ann}I \cong \text{ann}I_{(+)}[0:_M I]/0_{(+)}[0:_M I]$ is a finitely generated ideal of R. Hence I is finitely generated projective, [33, Corollary 3.1]. Conversely, suppose M is flat (hence torsion-free) and let I be finitely generated projective. Then I is finitely generated flat, and by (4), $I_{(+)}IM$ is finitely generated flat. Since $\text{ann}I$ is finitely generated, it follows by, [1, Theorem 7], that $\text{ann}(I_{(+)}IM) = \text{ann}I_{(+)}(\text{ann}I)M$ is finitely generated. Hence $I_{(+)}IM$ is finitely generated projective.

(6) Suppose M is finitely generated, faithful and multiplication. Let $I_{(+)}IM$ be q-invertible for $(e,0)$. Then $I_{(+)}IM$ is finitely generated projective and by (5), I is finitely generated projective. It follows by, [27, Theorem 1.5], that there exists $(a,n) \in I_{(+)}IM$ such that $\text{ann}(I_{(+)}IM) = \text{ann}(a,n) = \text{ann}(e,0)$. Hence $\text{ann}I_{(+)}(\text{ann}I)M = \text{ann}(e)_{(+)}\text{ann}(e)M$, and hence $\text{ann}I = \text{ann}(e)$. Next,

$$\text{ann}I_{(+)}(\text{ann}I)M = \text{ann}(a,n) \supseteq \text{ann}(Ra_{(+)}Rn + aM) = \text{ann}(a) \cap \text{ann}(n)_{(+)}\text{ann}(a)M.$$

This implies that $\text{ann}(a)M \subseteq (\text{ann}I)M$, and by, [32, Corollary to Theorem 9], $\text{ann}(a) \subseteq \text{ann}I \subseteq \text{ann}(a)$, and hence $\text{ann}I = \text{ann}(a)$. This shows that I is q-invertible for e. Conversely, let M be flat and I q-invertible for e. Then I is finitely generated projective and by (5), $I_{(+)}IM$ is finitely generated projective.

Now, there exists $a \in I$ such that $\text{ann}I = \text{ann}(a) = \text{ann}(e)$. Hence $\text{ann}(I_{(+)}IM) = \text{ann}(a,0) = \text{ann}(e,0)$, and therefore $I_{(+)}IM$ is q-invertible for $(e,0)$.

(7) Suppose A and B are ideals of R that are properly contained in I. Then $A_{(+)}AM$ and $B_{(+)}BM$ are ideals of $R(M)$, each of them is properly contained in $I_{(+)}IM$. As $I_{(+)}IM$ is generalized multiplication, we have that

$$A \cap B_{(+)}AM \cap BM = A_{(+)}AM \cap B_{(+)}BM = [A_{(+)}AM :_{R(M)} B_{(+)}BM]\,(B_{(+)}BM)$$
$$= ([A : B]_{(+)}[AM :_M B])\,(B_{(+)}BM) = [A : B]\,B_{(+)}\,([A : B]BM + [AM :_M B]\,B)\,.$$

Hence $A \cap B = [A : B]B$, and this shows that I is generalized multiplication. □

The next theorem shows how properties of $I_{(+)}N$ can be transferred to its components I and N.

Theorem 3. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous ideal of $R(M)$.*

(1) *If $I_{(+)}N$ is pure then I is a pure ideal of R and N a pure submodule of M.*
(2) *If $I_{(+)}N$ is invertible then I is an invertible ideal of R and N an invertible submodule of M.*
(3) *If $I_{(+)}N$ is generalized multiplication then I is a generalized multiplication ideal of R and N a generalized multiplication submodule of M.*
(4) *If $I_{(+)}N$ is almost multiplication then I is an almost multiplication ideal of R. Assuming further that M is almost multiplication then N is an almost multiplication submodule of M.*
(5) *If $I_{(+)}N$is finitely generated flat then I is a finitely generated flat ideal of R. Assuming further that M is finitely generated flat then N is a finitely generated flat submodule of M.*
(6) *If $I_{(+)}N$ is finitely generated projective then I is a finitely generated projective ideal of R. Assuming further that M is finitely generated projective then N is a finitely generated projective submodule of M.*

Proof. (1) If $I_{(+)}N$ is pure then $I_{(+)}IM = I_{(+)}IM \cap I_{(+)}N = (I_{(+)}IM)(I_{(+)}N) = I^2{}_{(+)}IN$. Hence $IM = IN$. But $I_{(+)}N$ is idempotent. Thus $I_{(+)}N = (I_{(+)}N)^2 = I^2{}_{(+)}IN$. Hence $N = IN = IM$. By Theorem 2(1), I is a pure ideal of R. Now, there exists an ideal J of R such that $R = I \oplus J$. Then $M = IM + JM$. Since $R = I + J$ and $I \cap J = 0$, we infer that $IM \cap JM = 0$ (see the proof of Theorem 2(1)). Hence $M = IM \oplus JM$, and hence $N = IM$ is pure in M.

(2) Suppose $I_{(+)}N$ is an invertible ideal of $R(M)$. Since $(I_{(+)}N)^2 = I^2{}_{(+)}IN = (I_{(+)}N)(I_{(+)}IM)$, we obtain that

$$I_{(+)}N = (I_{(+)}N)^{-1}(I_{(+)}N)^2 = (I_{(+)}N)^{-1}(I_{(+)}N)(I_{(+)}IM) = I_{(+)}IM,$$

and hence $N = IM$. It follows by Theorem 2(3) that I is an invertible ideal of R. Now, $R = II^{-1}$ gives that $M = IMI^{-1} \subseteq (IM)(IM)^{-1} \subseteq M$. Hence $M = (IM)(IM)^{-1}$, and $IM = N$ is invertible in M.

(3) [1, Theorem 9] shows that if $I_{(+)}N$ is principal then I is a principal ideal of R and if we assume that M is cyclic then N is a finitely generated multiplication submodule of M (in fact N can be generated by two elements). The result is now obvious.

(4) It is easy to check that every submodule of a generalized multiplication R-module is generalized multiplication. If $I_{(+)}N$ is a generalized multiplication ideal of $R(M)$ then so too are $0_{(+)}N$ and $I_{(+)}IM$. The result follows by Proposition 1 and Theorem 2.

(5)–(6) Let $I_{(+)}N$ be finitely generated flat (resp. finitely generated projective). It follows by, [5, Theorem 3.3], [33, Lemma 6] and [18, Lemma 2.1], that $I_{(+)}N$ is locally either zero or invertible. Hence $N = IM$ is locally true and hence globally. By Theorem 2, I is a finitely generated flat (resp. finitely generated projective) ideal of R. If M is finitely generated flat (resp. finitely generated projective) then $N = IM \cong I \otimes M$ is a finitely generated flat (resp. finitely generated projective) submodule of M. □

The next theorem gives necessary and sufficient conditions for the ideal $I_{(+)}M$ to be projective (weak multiplication).

Proposition 4. *Let R be a ring, M an R-module and I an ideal of R.*

(1) *If $I_{(+)}M$ is a projective ideal of $R(M)$ then I is projective and $M = IM$. Conversely, if I is projective and M torsion-free then $I_{(+)}IM$ is a flat ideal of $R(M)$.*

(2) *$I_{(+)}M$ is a weak multiplication ideal of $R(M)$ if and only if I is weak multiplication and for every prime ideal $P_{(+)}M \subseteq I_{(+)}M$ either $P_{(+)}M = I_{(+)}M$ or $M = IM$.*

Proof. (1) Suppose $I_{(+)}M$ is a projective ideal $R(M)$. There exists a free ideal F of $R(M)$ such that $F = I_{(+)}M \oplus H$ for some ideal H of R. Let $F + 0_{(+)}M = A_{(+)}M$ and $H + 0_{(+)}M = J_{(+)}M$ for some ideals A and J of R. It follows that $A_{(+)}M = (I + J)_{(+)}M$ from which one gets that $A = I + J$. Since $F = I_{(+)}M + H$ is a free (hence multiplication) ideal of $R(M)$, it follows by, [7, Theorem 2.1], that

$$0_{(+)}M = 0_{(+)}M + (I_{(+)}M \cap H) = I_{(+)}M \cap 0_{(+)}M + H = I_{(+)}M \cap J_{(+)}M = I \cap J_{(+)}M.$$

Hence $0 = I \cap J$, and this shows that $A = I \oplus J$. To complete the proof that I is projective, we need to show that A is a free ideal of R. By, [7, Theorem 2.1], we have that

$$\begin{aligned} F \cap 0_{(+)}M = (I_{(+)}M + H) \cap 0_{(+)}M &= I_{(+)}M \cap 0_{(+)}M + H \cap 0_{(+)}M \\ &= 0_{(+)}M + H \cap 0_{(+)}M = 0_{(+)}M. \end{aligned}$$

Hence $0_{(+)}M \subseteq F$, and therefore $F = F + 0_{(+)}M$ is a free ideal of $R(M)$. Next, $F = A_{(+)}M$ is a free ideal of $R(M)$ and by [29, Ex. 1.3], $A_{(+)}M$ is cancellation. Since

$$(A_{(+)}AM)(A_{(+)}M) = A^2{}_{(+)}A^2M = (A_{(+)}AM)^2,$$

it follows that $A_{(+)}M = A_{(+)}AM$. Assume $\{(a_\alpha, 0)\}$ is a basis of $A_{(+)}AM$. Then $A_{(+)}AM = \sum_\alpha R(M)(a_\alpha, 0)$, from which we get that $A = \sum_\alpha Ra_\alpha$. To show $\{a_\alpha\}$ is a basis of A, let $\sum_\alpha r_\alpha a_\alpha = 0$, where $r_\alpha \in R$. Then $\sum_\alpha (r_\alpha, 0)(a_\alpha, 0) = (0, 0)$, and hence $(r_\alpha, 0) = (0, 0)$ (and hence $r_\alpha = 0$). This shows that A is a free ideal of R. Next, let $I_{(+)}M$ be projective. If $I = 0$, then $0_{(+)}M$ is projective implies that $M = 0$, and hence $M = IM$. So, suppose that $I \neq 0$. Then $I_{(+)}M$ is locally free, and hence it is locally cancellation. It follows by, [29, Proposition 2.2], that $I_{(+)}M$ is cancellation and hence $M = IM$. Alternatively, $I_{(+)}M$ is locally either zero or invertible. Hence $M = IM$ is locally true and hence globally. Suppose I is projective and M torsion-free. By, [5, Theorem 3.3], I is locally either zero or invertible, and so too is $I_{(+)}IM$. This gives that $I_{(+)}IM$ is locally flat and hence $I_{(+)}IM$ is flat.

(2) Suppose $I_{(+)}M$ is weak multiplication. Let $P \subseteq I$ be a prime ideal of R. Then $P_{(+)}M \subseteq I_{(+)}M$ is a prime ideal of $R(M)$. Hence

$$\begin{aligned} P_{(+)}M = [P_{(+)}M :_{R(M)} I_{(+)}M](I_{(+)}M) &= ([P : I]_{(+)}M)(I_{(+)}M) \\ &= [P : I]I_{(+)}([P : I] + I)M. \end{aligned}$$

It follows that $P = [P : I]I$ (and hence I is a weak multiplication ideal of R) and $M = ([P : I] + I)M$. Since P is a prime ideal of R, either $P = I$ (and hence $P_{(+)}M = I_{(+)}M$) or $P = [P : I]$ and in this case $M = IM$. Conversely, let $P_{(+)}M \subseteq I_{(+)}M$ be a prime ideal of $R(M)$. If $P_{(+)}M = I_{(+)}M = R(M)(I_{(+)}M)$, then $I_{(+)}M$ is weak multiplication. Let $M = IM$. Since $P \subseteq I$ and I is weak multiplication, $P = [P : I]I$. Hence $P_{(+)}M = [P : I]I_{(+)}M = ([P : I]_{(+)}M)(I_{(+)}M)$, and this also shows that $I_{(+)}M$ is weak multiplication. □

The next result shows some conditions under which a homogeneous ideal $I_{(+)}N$ of $R(M)$ is multiplication, invertible, finitely generated flat (projective) or pure.

Theorem 5. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous ideal of $R(M)$.*

(1) *Let M be divisible. If I is a finitely generated faithful multiplication ideal of R then $I_{(+)}N$ is finitely generated multiplication.*
(2) *Let M be divisible and torsion-free. If I is an invertible ideal of R then $I_{(+)}N$ is invertible.*
(3) *Let M be divisible and torsion-free. If I is a finitely generated flat ideal of R such that $annI \subseteq annN$ then $I_{(+)}N$ is finitely generated flat.*
(4) *Let M be divisible and flat. If I is a finitely generated projective ideal of R such that $annI \subseteq annN$ then $I_{(+)}N$ is finitely generated projective.*
(5) *Let M be divisible and flat. If I is q-invertible for e such that $annI \subseteq annN$ then $I_{(+)}N$ is q-invertible for $(e, 0)$.*
(6) *Let $ann\,(I_P) \subseteq ann\,(N_P)$ for each prime ideal P of R. If I is a pure ideal of R then $I_{(+)}N$ is pure.*

(7) *If I is a multiplication ideal of R and N a multiplication submodule of M such that $\mathrm{ann}I + [IM : N] = R$ then $I_{(+)}N$ is multiplication.*

(8) *Let M be finitely generated multiplication. If I is a finitely generated idempotent ideal of R then $I_{(+)}M$ is finitely generated multiplication.*

Proof. (1) Let M be divisible and I finitely generated faithful multiplication. Let P be a prime ideal of R. Then I_P is a regular principal ideal of R_P. Since M is divisible, so too is M_P and hence $M_P = I_P M_P$. As $IM \subseteq N$, we infer that $N_P = I_P M_P$. As P is arbitrary, $N = IM$. Hence $I_{(+)}N = I_{(+)}IM$ and the result follows by, [1, Theorem 7], see also [2, Proposition 5].

(2) Suppose M is divisible and torsion-free and I invertible. Then I is finitely generated faithful multiplication, [16], and by (1), $I_{(+)}N$ is finitely generated multiplication. Since I is regular and M torsion-free, we infer that $I_{(+)}N$ is regular. Hence $I_{(+)}N$ is invertible. Alternatively, if M is divisible and torsion-free then every regular ideal H of $R(M)$ has the form $I_{(+)}IM$ where I is a regular ideal of R, [10, Theorem 3.9]. Hence the result follows by Theorem 2(3).

(3) Let M be divisible and torsion-free and I finitely generated flat. Let P be a prime ideal of R. Then $I_P = 0_P$ or I_P is invertible, [18] and [33]. If $I_P = 0_P$ and since $\mathrm{ann}I \subseteq \mathrm{ann}N$, we get that $R_P = \mathrm{ann}\,(I_P) = (\mathrm{ann}I)_P \subseteq (\mathrm{ann}N)_P \subseteq \mathrm{ann}\,(N_P)$, so that $N_P = 0_P$, and hence $N_P = I_P M_P$. If I_P is invertible and since M_P is divisible, we also have $N_P = I_P M_P$. So both cases show that $N_P = I_P M_P$. Hence $N = IM$. The result follows by Theorem 2(4).

(4) This is now clear by Theorem 2(5) and the fact that projective ideals are locally either zero or invertible, [5, Theorem 3.3].

(5) Follows by (4), Theorem 2(6) and the fact that q-invertible ideals are finitely generated projective, [27, Theorem 1.5].

(6) Suppose I is a pure ideal of R. Let $P_{(+)}M$ be a prime ideal of $R\,(M)$. Then P is a prime ideal of R. Hence $I_P = 0_P$ or $I_P = R_P$. As $\mathrm{ann}\,(I_P) \subseteq \mathrm{ann}\,(N_P)$, the first case shows that $N_P = 0_P$. Since $IM \subseteq N$, the second case gives that $N_P = M_P$. Both cases show that $N_P = I_P M_P$. Hence $N = IM$, and the result follows by Theorem 2(1).

(7) [2, Proposition 7]. However, we give here an alternative proof by using a Theorem of El-Bast and P. F. Smith, [13, Theorem 1.2], which we think that it is of some interest. Suppose $P_{(+)}M$ is a maximal ideal of $R(M)$ and let $I_{(+)}N$ be not $P_{(+)}M$- torsion. Then $0_{P_{(+)}M} \neq (I_{(+)}N)_{P_{(+)}M} \cong I_{P(+)}N_P$. It follows that P is a maximal ideal of R and either $I_P \neq 0_P$ or $N_P \neq 0_P$. We discuss two cases.

Case 1. Let $I_P \neq 0_P$. Since $R = \mathrm{ann}I + [IM : N]$, $R_P = (\mathrm{ann}I)_P + [IM : N]_P \subseteq \mathrm{ann}(I_P) + [IM : N]_P \subseteq R_P$. Since R_P is local, we have that $R_P = [IM : N]_P$. There exists $p \in P$ such that $1-p \in [IM : N]$ and hence $(1-p)N \subseteq IM$. Since I is multiplication and I not P-torsion, we get from, [13, Theorem 1.2], that there exists $q \in P$ with $(1-q)I \subseteq Ra$ for some $a \in I$. Let $1-p' = (1-p)(1-q)$. Then $p' \in P$, $(1-p')I \subseteq Ra$ and $(1-p')N \subseteq aM$. It follows that

$$((1,0)-(p',0))(I_{(+)}N) = (1-p',0)(I_{(+)}N) = (1-p')I_{(+)}(1-p')N \\ \subseteq Ra_{(+)}aM = R(M)(a,0).$$

Hence $I_{(+)}N$ is $P_{(+)}M$-principal and [13, Theorem 1.2] shows that $I_{(+)}N$ is multiplication.

Case 2. Suppose $N_P \neq 0_P$. Since N is multiplication, it follows by, [13, Theorem 1.2], that there exists $p \in P$, $n \in N$ such that $(1-p)N \subseteq Rn$. As $R_P = (\text{ann}I)_P + [IM : N]_P$, we get either $R_P = (\text{ann}I)_P$ or $R_P = [IM : N]_P$. If $R_P = (\text{ann}I)_P$, then there exists $q \in P$ such that $1-q \in \text{ann}I$, and hence $(1-q)I = 0$. Let $1-p' = (1-p)(1-q)$. Then $p' \in P$ and $(1-p\prime)I = 0$ and $(1-p')N \subseteq Rn$. This implies that

$$\begin{aligned}((1,0)-(p',0))(I_{(+)}N) = (1-p',0)(I_{(+)}N) &= (1-p')I_{(+)}(1-p')N \\ &\subseteq 0_{(+)}Rn = R(M)(0,n),\end{aligned}$$

and hence $I_{(+)}N$ is $P_{(+)}M$-principal and by [13, Theorem 1.2], $I_{(+)}N$ is multiplication. Finally, if $[IM : N]_P = R_P$, then there exists $s \in P$ such that $(1-s)N \subseteq IM$. Also $R_P = [IM : N]_P \subseteq [(IM)_P : N_P] \subseteq R_P$, that is $N_P = (IM)_P$. Since $N_P \neq 0_P$, $I_P \neq 0_P$ and Case 1 shows that $I_{(+)}N$ is multiplication.

(8) $I = Re$ for some idempotent e of R, [16, Corollary 6.3]. Since M is finitely generated multiplication (hence weak cancellation), we have that $\text{ann}I + [IM : M] = \text{ann}\,(e) + Re + \text{ann}M = R$. The result follows by (7). □

2. Homogeneous rings

An ideal H of $R(M)$ need not to have the form $I_{(+)}N$, that is, need not to be homogeneous. In [2] we called a ring $R(M)$ is a homogeneous ring if every ideal of $R(M)$ is homogeneous. It is shown, [10, Corollary 3.4], that if R is an integral domain then $R(M)$ is homogeneous if and only if M is divisible (that is $M = sM$ for every regular s in R). In this case every ideal of $R(M)$ is comparable to $0_{(+)}M$, equivalently, every ideal of $R(M)$ has the form $I_{(+)}M$ for some ideal I of R or $0_{(+)}N$ for some submodule N of M. It is also shown, [10, Theorem 3.3], that a ring $R(M)$ is homogeneous if and only if every principal ideal of $R(M)$ is homogeneous. In this case $R(M)(a,m) = Ra_{(+)}(Rm + aM) = R(M)(a,0) + R(M)(0,m)$. Since Q is a divisible $\mathbb{Z}$-module, $\mathbb{Z}_{(+)}Q$ is a homogeneous ring while $\mathbb{Z}_{4(+)}\mathbb{Z}_2$ is not homogeneous. In fact, the principal ideal of $\mathbb{Z}_{4(+)}\mathbb{Z}_2$ that is generated by $(\bar{2},\bar{1}) = \{(\bar{0},\bar{0})\},(\bar{2},\bar{1})\}$ is not homogeneous. If R is a local ring but not a domain, $R(M)$ is homogeneous if and only if $M = 0$, [10, Theorem 3.3]. In this section we study ring-theoretic constructions and properties of $R(M)$, and in particular of homogeneous rings and show how these properties of R and M relate to those of $R(M)$.

A principal ideal ring (PIR) is *special* (SPIR) if R is a local ring, not a field, whose maximal ideal is nilpotent. If every ideal of R is a finite product of prime ideals, then R is called a *general* ZPI-ring, [19]. R is a general ZPI-ring if and only if R is a finite direct sum of Dedekind domains and SPIRs, [19, p. 114], and R is a general ZPI-ring if and only if it is a Noetherian multiplication ring, [16, p. 225].

Recall that for $f = a_0 + a_1x + \cdots + a_nx^n \in R[X]$, the content A_f of f is the ideal $\sum_{i=1}^{n} Ra_i$. The set $N = \{f \in R[X] | A_f = R\}$ is a saturated multiplicative closed subset of $R[X]$, in fact $N = R[X] - \cup P[X]$ where the union runs over all maximal ideals P of R. Then $R(X) = R[X]_N$ and if M is an R-module then $M(X) = M[X]_N$ is an $R(X)$-module. It is shown, [10, Corollary 4.7], that $R(M)(X)$ is naturally isomorphic to $R(X)_{(+)}M(X)$.

R is a general ZPI-ring if and only if $R(X)$ is a general ZPI-ring if and only if $R(X)$ is a PIR, [19, Theorem 18.8]. Another fact which is also useful for the proof of our next result is that an ideal I of R is finitely generated and locally principal (equivalently, finitely generated multiplication) if and only if $IR(X)$ is principal, [19, Corollary 15.2].

A ring R which is an almost multiplication R-module is an *almost multiplication* ring, [8]. A Noetherian almost multiplication module is multiplication and a finitely generated almost multiplication module over a multiplication ring (a ring in which every ideal is multiplication) is a multiplication module [20, Theorem 2.11]. It is proved, [23, Theorems 9.23, 9.27 and 9.28], that a ring R is an almost multiplication ring if and only if R_P is a general ZPI-ring. It is also shown, [8, Theorem 8], that R is an almost multiplication ring if and only if $R(X)$ is an almost multiplication ring and $R[X]$ is an almost multiplication ring if and only if R is von Neumann regular.

An R-module M is said to be *distributive,* [34], if the lattice of its submodules is distributive, that is, $(X+Y) \cap Z = X \cap Z + Y \cap Z$ for any of its submodules X, Y and Z. Equivalently, $(X \cap Y) + Z = (X+Z) \cap (Y+Z)$ for all submodules X, Y and Z of M. Some authors call such modules *arithmetical* modules. An R- module M is arithmetical if and only if it is locally chained, that is for each prime ideal P of R, the set of (cyclic) submodules of M_P is totally ordered by inclusion. If all 2-generated submodules of M are multiplication modules, then M is distributive, [35, Proposition 2.5]. A ring R which is a distributive R-module is called a *distributive* (or an *arithmetical*) ring. R is an arithmetical ring if and only if every finitely generated ideal of R is multiplication, [21]. For properties of distributive rings and modules, consult [14], [21] and [34]. M is called a *Bezout* module if each of its finitely generated submodules is a cyclic module, [34]. Equivalently, for all $m, n \in M, Rm + Rn$ is cyclic. It is proved, [4, Proposition 1.2], that if R is an integral domain and M a faithful multiplication R-module then M is a Bezout module if and only if R is a Bezout domain.

We next characterize when a ring $R(M)$ is a SPIR, a general ZPI-ring, an almost multiplication ring, a distributive ring or a Bezout ring. Compare with [1, Theorem 11] and [10, Lemma 4.9 and Theorems 4.10 and 4.16].

Theorem 6. *Let R be a ring and M an R-module.*

(1) *If $R(M)$ is a SPIR then R is a SPIR and M a cyclic module. The converse is true if $R(M)$ is homogeneous.*

(2) *If $R(M)$ is a general ZPI-ring then R is a general ZPI-ring and M a finitely generated multiplication module. The converse is true if $R(M)$ is homogeneous.*

(3) *If $R(M)$ is an almost multiplication ring then R is an almost multiplication ring and M an almost multiplication module. The converse is true if $R(M)$*

is homogeneous.

(4) *If $R(M)$ is a distributive ring then R is a distributive ring and M a distributive module. The converse is true if $R(M)$ is homogeneous.*

(5) *If $R(M)$ is a Bezout ring then R is a Bezout ring and M a Bezout module. The converse is true if M is finitely generated and $R(M)$ is homogeneous.*

Proof. (1) Let $R(M)$ be a SPIR. Let I be an ideal of R. Then $I_{(+)}IM$ is a principal ideal of $R(M)$, say $R(M)(a,0)$ for some $a \in I$. Then $I = Ra$ is a principal ideal of R and hence R is a PIR. If $P_{(+)}M$ is the unique nilpotent maximal ideal of $R(M)$, then P is the unique nilpotent maximal ideal of R, and hence $R(M)$ is a SPIR. Since $0_{(+)}M$ is a principal ideal of $R(M)$, M is cyclic, [9, Theorem 3.1]. The converse is trivial since $M = 0$.

(2) Suppose $R(M)$ is a general ZPI-ring. It follows by, [10, Corollary 4.7] and [19, Theorem 18.8], that $R(M)\,(X) \cong R\,(X)_{(+)}M(X)$ is a PIR. Hence $R(X)$ is a PIR and this gives that R is a general ZPI-ring. Next, $(0_{(+)}M)(X) \cong 0_{(+)}M\,(X)$ is an ideal of $R\,(M)\,(X)$, hence it is principal. It follows by, [19, Corollary 15.2], that $0_{(+)}M$ is a finitely generated locally principal (equivalently finitely generated multiplication) ideal of $R(M)$. By, [9, Theorem 3.1], M is finitely generated multiplication. Conversely, suppose R is a general ZPI-ring, M finitely generated multiplication and $R(M)$ is homogeneous. Since R is a general ZPI-ring, $R(X)$ is PIR, [19, Theorem 18.8]. As M is finitely generated multiplication (hence finitely generated locally cyclic), we get from, [19, Theorem 18.8], that $M(X)$ is a cyclic $R(X)$-module. It follows by [1, Theorem 11] that $R(M)(X) \cong R(X)_{(+)}M(X)$ is a PIR and by [19, Theorem 18.8], $R(M)$ is a general ZPI-ring.

(3) Using the fact that R (resp. M) is an almost multiplication ring (resp. module) if and only if R_P (resp. M_P) is a general ZPI-ring (resp. cyclic module). The result follows by (2).

(4) Let $R(M)$ be a distributive ring. Let A, B and C be ideals of R. Then $A_{(+)}M, B_{(+)}M$ and $C_{(+)}M$ are ideals of $R(M)$. Hence

$$\begin{aligned}(A+B)\cap C_{(+)}M &= A+B_{(+)}M \cap C_{(+)}M \\ &= (A_{(+)}M + B_{(+)}M) \cap C_{(+)}M = (A_{(+)}M \cap C_{(+)}M) \\ +(B_{(+)}M \cap C_{(+)}M) &= A \cap C_{(+)}M + B \cap C_{(+)}M = (A\cap C) + (B\cap C)_{(+)}M,\end{aligned}$$

and this gives that $(A+B)\cap C = (A\cap C) + (B \cap C)$. Hence R is distributive. Now, let K, N and L be submodules of M. Then $0_{(+)}K, 0_{(+)}N$ and $0_{(+)}L$ are ideals of $R(M)$. Hence

$$\begin{aligned}0_{(+)}((K+N)\cap L) &= (0_{(+)}K + N) \cap 0_{(+)}L \\ &= (0_{(+)}K + 0_{(+)}N) \cap 0_{(+)}L = 0_{(+)}K \cap 0_{(+)}L \\ +0_{(+)}N \cap 0_{(+)}L &= 0_{(+)}K \cap L + 0_{(+)}N \cap L = 0_{(+)}(K\cap L) + (N \cap L),\end{aligned}$$

and hence $(K+N)\cap L = (K\cap L) + (N \cap L)$ and M is distributive. For the converse, let $A_{(+)}N$, $B_{(+)}K$ and $C_{(+)}L$ be ideals of $R(M)$. Since each of R and M

is distributive, we get that

$$(A_{(+)}N + B_{(+)}K) \cap C_{(+)}L = (A + B)_{(+)}(N + K) \cap C_{(+)}L$$
$$= (A + B) \cap C_{(+)}(N + K) \cap L = (A \cap C) + (B \cap C)_{(+)}(N \cap L) + (K \cap L)$$
$$= (A \cap C_{(+)}N \cap L + B \cap C_{(+)}K \cap L = A_{(+)}N \cap C_{(+)}L + B_{(+)}K \cap C_{(+)}L,$$

and $R(M)$ is a distributive ring.

(5) Suppose $R(M)$ is a Bezout ring. Let I be a finitely generated ideal of R. Then $I_{(+)}IM$ is a finitely generated ideal of $R(M)$, hence it is principal and this gives that I is principal. Hence R is Bezout. If N is a finitely generated submodule of M, then $0_{(+)}N$ is a finitely generated ideal of $R(M)$, hence principal. It follows that N is cyclic and M is Bezout. Conversely, let $I_{(+)}N$ be a finitely generated ideal of $R(M)$. Then I is a finitely generated ideal of R (hence principal). Since M is finitely generated, N is a finitely generated submodule of M (hence cyclic), see [1, Theorem 9]. Let $I = Ra$ and $N = Rn$ for some $a \in R$, $n \in M$. Since $I_{(+)}N$ is homogeneous, $IM \subseteq N$, and hence $aM \subseteq Rn$. It follows by, [10, Theorem 3.3], that

$$I_{(+)}N = Ra_{(+)}Rn = Ra_{(+)}(Rn + aM) = R(M)(a, n)$$

is a principal ideal of $R(M)$. Hence $R(M)$ is Bezout. This concludes the proof of the theorem. □

The condition that $R(M)$ is homogeneous in the above theorem is crucial. For example, the ring $\mathbb{Z}_{4(+)}\mathbb{Z}_2$ is neither SPIR nor distributive ring (and hence it is neither ZPI-ring nor Bezout ring). In fact the maximal ideal $2\mathbb{Z}_{4(+)}\mathbb{Z}_2$ is finitely generated but not principal (equivalantly, not multiplication since the ring is local).

A ring R is said to have *few zero-divisors* if $Z(R)$ is a finite union of prime ideals. A ring R is a *quasi-valuation* ring if it has few zero-divisors and for every pair of regular elements $a, b \in R$, either $Ra \subseteq Rb$ or $Rb \subseteq Ra$ and R is a P-ring if it has few zero-divisors and every finitely generated regular ideal of R is invertible, [22] and [23]. Several characterizations and properties of quasi-valuation rings and P-rings are given in [22, Theorem 5] and [23, pp. 152–155]. Among them, R is a P-ring if and only if $IJ = (I + J)(I \cap J)$ for all regular ideals I and J of R. It is also shown that R is a P-ring if and only if $R_{S(P)}$ is a quasi-valuation ring for every regular maximal ideal P of R, where $S(P)$ is the set of regular elements of R/P which is a multiplicative subset of R.

We next wish to determine when $R(M)$ is a quasi-valuation ring or a P-ring.

Theorem 7. *Let R be a ring and M a torsion-free R-module.*

(1) *$R(M)$ is a quasi-valuation ring if and only if R is a quasi-valuation ring and M divisible.*
(2) *$R(M)$ is a P-ring if and only if R is a P-ring and M divisible.*

Proof. Since M is torsion-free, $Z(M) \subseteq Z(R)$ and hence $Z(R(M)) = Z(R)_{(+)}M$. This shows that $Z(R) = \bigcup_{i=1}^{n} P_i$ if and only if $Z(R(M)) = \bigcup_{i=1}^{n} (P_{i(+)}M)$, where $\{P_i\}$

is a finite set of prime ideals of R. Hence R has a few zero-divisors if and only if $R(M)$ has.

(1) Suppose $R(M)$ is quasi-valuation. By, [23, Ex. 25(b)], $R(M)$ is integrally closed. Let $s \in S$ and $m \in M$. Then $\left(0, \frac{m}{s}\right) = \frac{(0,m)}{(s,0)} \in R(M)$. Since $\left(0, \frac{m}{s}\right)^2 = \frac{(0,m)^2}{(s,0)^2} = (0,0)$, we infer that $(0, \frac{m}{s})$ is integral over $R(M)$. Hence $(0, \frac{m}{s}) \in R(M)$ and hence $\frac{m}{s} \in M$. This shows that $m \in sM$, and hence $M \subseteq sM \subseteq M$, so that $M = sM$ and hence M is divisible, see [10, Theorem 3.9]. Let a, b be regular elements of R. Since M is torsion-free, $(a,0)$ and $(b,0)$ are regular elements of $R(M)$. It follows that $Ra_{(+)}aM = R(M)(a,0) \subseteq R(M)(b,0) = Rb_{(+)}bM$ (hence $Rb \subseteq Ra$) or $R(M)(b,0) \subseteq R(M)(a,0)$ from which it follows that $Ra \subseteq Rb$. This shows that R is a quasi-valuation ring. Conversely, let M be divisible and R quasi-valuation. Let $(a,m),(b,n) \in R(M)$ be regular elements. Since M is divisible, [10, Theorem 3.9] shows that $R(M)(a,m)$ is homogeneous and hence $R(M)(a,m) = Ra_{(+)}(Rm + aM)$. Since a is regular, $aM = M$. This implies that $R(M)(a,m) = Ra_{(+)}M$. Similarly, $R(M)(b,n) = Rb_{(+)}M$. The result is now obvious.

(2) Suppose $R(M)$ is a P-ring. Then $R(M)$ is integrally closed, [23, Theorem 10.18], and hence M is divisible. Let I and J be regular ideals of R. Since M is torsion-free, $I_{(+)}IM$ and $J_{(+)}JM$ are regular ideals of $R(M)$. It follows by, [22, Theorem 5], that

$$\begin{aligned} IJ_{(+)}IJM &= (I_{(+)}IM)(J_{(+)}JM) = (I_{(+)}IM + J_{(+)}JM)(I_{(+)}IM \cap J_{(+)}JM) \\ &= ((I+J)_{(+)}(I+J)M)((I \cap J)_{(+)}(IM \cap JM)) \\ &= (I+J)(I \cap J)_{(+)}(I \cap J)(I+J)M + (I+J)(IM \cap JM). \end{aligned}$$

This gives that $IJ = (I \cap J)(I+J)$, and hence R is a P-ring. Conversely, suppose M is divisible and R is a P-ring. Let H_1 and H_2 be regular ideals of $R(M)$. It follows by, [10, Theorem 3.9], that $H_1 = I_{(+)}M$ and $H_2 = J_{(+)}M$ for some regular ideals I and J of R. Hence

$$(H_1 \cap H_2)(H_1 + H_2) = (I \cap J_{(+)}M)(I + J_{(+)}M) = (I \cap J)(I+J)_{(+)}(I+J)M.$$

Since $I+J$ is regular and M divisible, it follows that if x is a regular element in $I+J$ then $M = xM \subseteq (I+J)M \subseteq M$, so that $M = (I+J)M$. Hence

$$(H_1 \cap H_2)(H_1 + H_2) = IJ_{(+)}M = (I_{(+)}M)(J_{(+)}M) = H_1H_2,$$

and this gives that $R(M)$ is a P-ring. $\square$

An R-module M is called a *coherent* (resp. *finite conductor*) module if for all finitely generated submodules K and N of M (resp. for all $k, n \in M$), $K \cap N$ (resp. $Rk \cap Rn$) is a finitely generated submodule of M and for each $m \in M$, $\operatorname{ann}(m)$ is a finitely generated ideal of R, [17]. A ring which is a coherent (resp. finite conductor) R-module is called a coherent (resp. finite conductor) ring, [17]. If R is a coherent ring then it is a finite conductor ring. It is shown, [17, Proposition 2.1], that an R-module M is a finite conductor module if and only if for all $k, n \in M$, $[Rk : Rn]$ is a finitely generated ideal of R. Several properties of coherent and finite conductor rings (modules) are given in [17]. The next theorem gives necessary and sufficient conditions for a ring $R(M)$ to be a coherent ring or a finite conductor ring.

Theorem 8. *Let R be a ring and M an R-module.*

(1) *If $R(M)$ is coherent then R is a coherent ring and M a coherent module. The converse is true if $R(M)$ is homogeneous and M finitely generated flat.*

(2) *If $R(M)$ is a finite conductor ring then R is a finite conductor ring and M a finite conductor module. Conversely, if $R(M)$ is homogeneous, R coherent and M finitely generated faithful multiplication finite conductor module then $R(M)$ is finite conductor.*

Proof. (1) Suppose $R(M)$ is a coherent ring. Let I and J be finitely generated ideals of R. Then $I_{(+)}IM$ and $J_{(+)}JM$ are finitely ideals of $R(M)$. Hence $(I \cap J)_{(+)}(IM \cap JM) = I_{(+)}IM \cap J_{(+)}JM$ is finitely generated and by [1, Theorem 7], $I \cap J$ is finitely generated. Let $c \in R$ then $(c, 0) \in R(M)$ and hence $\mathrm{ann}(c, 0) = \mathrm{ann}(Rc_{(+)}cM) = \mathrm{ann}(c)_{(+)}[0 :_M Rc]$ is finitely generated. It follows that $\mathrm{ann}(c)$ is a finitely generated ideal of R. Hence R is a coherent ring. Let K and N be finitely generated submodules of M. Then $0_{(+)}K$ and $0_{(+)}N$ are finitely generated ideals of $R(M)$. Hence $0_{(+)}K \cap N = 0_{(+)}K \cap 0_{(+)}N$ is finitely generated from which it follows that $K \cap N$ is finitely generated, [9, Theorem 3.1]. Let $k \in M$. Then $(0, k) \in R(M)$, and hence $\mathrm{ann}(0, k) = \mathrm{ann}(k)_{(+)}M$ is finitely generated. Hence $\mathrm{ann}(k)$ is finitely generated, [1, Theorem 9], and hence M is a coherent module. Conversely, let $R(M)$ be homogeneous, R coherent and M a finitely generated flat and coherent module. Suppose $I_{(+)}N$ and $J_{(+)}K$ are finitely generated ideals of $R(M)$. Then I, J are finitely generated ideals of R. Since M is finitely generated, K and N are finitely generated submodules of M, [1, Theorem 9]. It follows that $I_{(+)}N \cap J_{(+)}K = I \cap J_{(+)}N \cap K$ is finitely generated. Let $(r, m) \in R(M)$. Since $R(M)$ is homogeneous and M flat, we obtain that

$$\mathrm{ann}(r, m) = \mathrm{ann}(Rr_{(+)}Rm + rM) = \mathrm{ann}(r) \cap \mathrm{ann}(m)_{(+)}\mathrm{ann}(r)M.$$

As R and M are coherent, $\mathrm{ann}(r)$ and $\mathrm{ann}(m)$ are finitely generated and so too is $\mathrm{ann}(r) \cap \mathrm{ann}(m)$. Since M is finitely generated, [1, Theorem 9] implies that $\mathrm{ann}(r, m)$ is finitely generated and hence $R(M)$ is a coherent ring.

(2) Let $R(M)$ be a finite conductor ring. Let $a, b \in R$. It follows that

$$\begin{aligned}[R(M)(a, 0) :_{R(M)} R(M)(b, 0)] &= [Ra_{(+)}aM :_{R(M)} Rb_{(+)}bM] \\ &= [Ra : Rb]_{(+)}[aM :_M Rb]\end{aligned}$$

is a finitely generated ideal of $R(M)$. Hence $[Ra : Rb]$ is a finitely generated ideal of R and this shows that R is a finite conductor ring, [1, Theorem 9]. Suppose

$m, n \in M$. Then

$$[R(M)(0,m) :_{R(M)} R(M)(0,n)] = [0_{(+)}Rm :_{R(M)} 0_{(+)}Rn] = [Rm : Rn]_{(+)}M$$

is finitely generated from which it follows that $[Rm : Rn]$ is finitely generated and hence M is a finite conductor module. Conversely, let $R(M)$ be homogeneous, R coherent and M finitely generated faithful multiplication and finite conductor module. Let $(a,m),(b,n) \in R(M)$. Then

$$\begin{aligned}[R(M)(a,m) :_{R(M)} R(M)(b,n)] &= [Ra_{(+)}(Rm + aM) :_{R(M)} Rb_{(+)}(Rn + bM] \\ &= [Ra : Rb] \cap [Rm + aM : Rn + bM]_{(+)}[Rm + aM : bM]M.\end{aligned}$$

Since $Ra_{(+)}(Rm + aM)$ is principal (hence finitely generated multiplication) and M finitely generated multiplication, we infer from [1, Theorem 9] that $Rm + aM$ is finitely generated and multiplication, see also [2, Theorem 3]. Suppose $M = \sum_{i=1}^{\ell} Rk_i$. It follows by, [7, Theorem 1.2] and [32, Proposition 4], that

$$\begin{aligned}[Rm + aM : Rn + bM] &= [Rm : Rn + bM] + [aM : Rn + bM] \\ &= [Rm : Rn] \cap [Rm : bM] + [aM : Rn] \cap [aM : bM] \\ &= [Rm : Rn] \cap (\bigcap_{i=1}^{\ell}[Rm : Rbk_i]) + (\sum_{i=1}^{\ell}[Rak_i : Rn]) \cap [Ra : Rb],\end{aligned}$$

and

$$[Rm + aM : bM] = [Rm : bM] + [aM : bM] = \bigcap_{i=1}^{\ell}[Rm : Rbk_i] + [Ra : Rb].$$

Since M is finitely generated finite conductor and R coherent, one gets that $[R(M),(a,m) :_{R(M)} R(M)(b,n)]$ is finitely generated and by, [17, Proposition 2.1], $R(M)$ is a finite conductor ring. $\square$

3. Weakly prime submodules

Let R be a commutative ring with identity. D. D. Anderson and E. Smith [11] defined a proper ideal P of R to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Prime ideals are weakly prime but not conversely. Various properties and characterizations of weakly prime ideals are given in [11]. A proper submodule P of an R-module M is called *prime* if whenever $rm \in P$, where $r \in R, m \in M$, then $r \in [P : M]$ or $m \in P$. Obviously, if P is a prime submodule of M then $[P : M]$ is a prime ideal of R while the converse is true if M is faithful and multiplication, [13, Lemma 2.10]. In this section we introduce the concept of weakly prime submodules as a generalization of weakly prime ideals and prime submodules: A proper submodule P of M is *weakly prime* if $0 \neq rm \in P$, where $r \in R$ and $m \in M$, then $r \in [P : M]$ or $m \in P$. However, since 0 is weakly prime (by definition), a weakly prime submodule need not to be prime. For a less trivial example, if M is a non-zero local multiplication module with unique maximal submodule Q such that $[Q : M]Q = 0$ then every proper submodule of M is weakly prime.

We start this section by the following property of weakly prime but not prime submodules. Compare with [11, Theorem 1]. The referee noted that this result has been independently proved and will appear in Tamkang J. Math.

Proposition 9. *Let R be a ring and M an R-module. Let P be a weakly prime submodule of M. If P is not prime then $[P : M]P = 0$.*

Proof. We prove that if $[P : M]P \neq 0$, then P is a prime submodule of M. Suppose $rm \in P$ for some $r \in R$ and $m \in M$. If $0 \neq rm$, then P weakly prime gives $r \in [P : M]$ or $m \in P$. So assume $rm = 0$. Suppose first that $rP \neq 0$. Then $rm_0 \neq 0$ for some $m_0 \in P$. Hence $0 \neq r(m+m_0)$ and therefore $r \in [P : M]$ or $m + m_0 \in P$ from which it follows that $m \in P$. We can assume $rP = 0$. Since $[P : M]P \neq 0$, there exist $r_0 \in [P : M], n_0 \in P$ such that $r_0 n_0 \neq 0$. It follows that $0 \neq (r + r_0)(m + n_0)$, and hence $r + r_0 \in [P : M]$ (and hence $r \in [P : M]$) or $m + n_0 \in P$ (and hence $m \in P$). So P is a prime submodule of M. □

Let R be a ring and M an R-module. The M-radical, rad N, of a submodule N of M is defined as the intersection of all prime submodules of M containing N. If I is an ideal of R then $\sqrt{I}$ is defined as the intersection of all prime ideals of R containing I, equivalently, $\sqrt{I} = \{a \in R : a^n \in I$ for some positive integer $n\}$. If M is a faithful multiplication R-module, then M-rad $N = \sqrt{[N : M]}M$, [13, Theorem 2.12]. Hence M-rad $0 = \sqrt{0}M$. As a consequence of the above result we give the next corollary.

Corollary 10. *Let R be a ring and M a faithful multiplication R-module. Let P be a weakly prime submodule of M. Then either $P \subseteq M$-rad 0 or M-rad $0 \subseteq P$. If $P \subsetneq M$-rad 0 then P is not prime, while if M-rad $0 \subsetneq P$ then P is prime. Consequently, if R is reduced then P is weakly prime if and only if $P = 0$ or P is prime.*

Proof. Suppose P is not prime. By Proposition 9, $[P : M]P = 0$. Since M is faithful, we infer that $[P : M]^2 \subseteq [[P : M]\, P : M] = [0 : M] = 0$, so that $[P : M]^2 = 0$. If $a \in [P : M]$ then $a^2 = 0$ and hence $a \in \sqrt{0}$. So $[P : M] \subseteq \sqrt{0}$ and therefore $P = [P : M]M \subseteq \sqrt{0}M = M$-rad 0. Now suppose P is prime and let $a \in \sqrt{0}$ then $a^n = 0 \in [P : M]$ for some positive integer n. Since $[P : M]$ is a prime ideal of R, $a \in [P : M]$ and hence $\sqrt{0} \subseteq [P : M]$. This implies that M-rad $0 \subseteq P$. Finally, suppose R is reduced and P weakly prime. If $[P : M]P \neq 0$ then P is prime by Proposition 9. Suppose $[P : M]P = 0$. Then $[P : M]^2 = 0$, and hence $[P : M] = 0$. This gives that $P = 0$. □

We next give three other characterizations of weakly prime submodules.

Theorem 11. *Let R be a ring, M an R-module and P a proper submodule of M. Then the first three statements are equivalent. Assuming further that M is faithful and multiplication then (1)–(4) are equivalent.*

(1) *P is weakly prime.*

(2) *For $r \in R - [P : M]$, $[P :_M Rr] = P \cup [0 :_M Rr]$.*

(3) *For $r \in R - [P : M]$, $[P :_M Rr] = P$ or $[P :_M Rr] = [0 :_M Rr]$.*

(4) *For ideals A of R and submodules K of M with $0 \neq AK \subseteq P$ implies $A \subseteq [P : M]$ or $K \subseteq P$.*

Proof. (1)⇒(2) Let $m \in [P :_M Rr]$ where $r \notin [P : M]$. Then $rm \in P$. If $0 \neq rm$ then P weakly prime gives that $m \in P$. If $0 = rm$ then $m \in [0 :_M Rr]$ and hence $[P :_M Rr] \subseteq P \cup [0 :_M Rr]$. As the reverse containment holds for any submodule P, the equality holds.

(2)⇒(3) Obvious.

(3)⇒(1) Let $r \in R$, $m \in M$ with $0 \neq rm \in P$. Suppose $r \notin [P : M]$. Then either $[P :_M Rr] = P$ or $[P :_M Rr] = [0 :_M Rr]$. Since $rm \neq 0, m \notin [0 :_M Rr]$, and hence $m \in P$.

(1)⇒(4) Suppose P is weakly prime. Suppose A is an ideal of R and K a submodule of M such that $AK \subseteq P$ but $A \nsubseteq [P : M]$ and $K \nsubseteq P$. We show that $AK = 0$. Let $a \in A - [P : M]$. Then $aK \subseteq P$ and hence $K \subseteq [P :_M Ra]$. Since $K \nsubseteq P$, (1)⇒(3) shows that $K \subseteq [0 :_M Ra]$ and hence $aK = 0$. Next, suppose $a \in A \cap [P : M]$. Let $b \in [K : M]$. If $b \in [P : M]$ then by Proposition 9, $ab \in [P : M]^2 = 0$, so that $ab = 0$. Assume $b \in [K : M] - [P : M]$. Then $bM \subseteq K$ and hence $bAM \subseteq AK \subseteq P$, from which we obtain that $AM \subseteq [P :_M Rb]$. Since $A \nsubseteq [P : M]$, (1)⇒(3) also gives that $AM \subseteq [0 :_M Rb]$, and hence $bAM = 0$. Since M is faithful, $bA = 0$ and hence $ba = 0$. As $b \in [K : M]$ is arbitrary, $a[K : M] = 0$, and hence $aK = a[K : M]M = 0$. So $AK = 0$.

(4)⇒(1) In this part the assumption that M is faithful multiplication is not required. Suppose $0 \neq rm \in P$ for some $r \in R$, $m \in M$. Then $0 \neq RrRm \subseteq P$. Hence $Rr \subseteq [P : M]$ or $Rm \subseteq P$, and this implies $r \in [P : M]$ or $m \in P$. This concludes the proof of the theorem. □

An R-module M is said to be zero-dimensional if every prime submodule of M is maximal. Using, [13, Theorem 2.15], one can easily see that every non-zero multiplication module over a zero-dimensional ring is zero-dimensional. Another useful fact is that if M is a non-zero multiplication R-module then (M, Q) is a local module if and only if $(R, [Q : M])$ is a local ring, see also [4, Proposition 1.4].

The next two results give further properties of weakly prime submodules. The first one should be compared with [11, Theorem 4 and Corollaries 5 and 6] and the second one with [3, Proposition 2.9] and [13, Corollary 2.11].

Proposition 12. *Let R be a ring and M a faithful multiplication R-module.*

(1) *If P is a weakly prime submodule of M that is not prime then $\sqrt{0}P = 0$.*

(2) *If P and Q are weakly prime submodules of M that are not prime then $[P : M]Q = 0 = [Q : M]P$.*

(3) *Suppose (M, Q) is a zero-dimensional local module. If P is a weakly prime submodule of M then $P = Q$ or $[Q : M]P = 0$.*

Proof. (1) Let $a \in \sqrt{0}$. If $a \in [P : M]$ then $a\,[P : M] \subseteq [P : M]^2 = 0$ and hence $aP = a\,[P : M]\,M = 0$. So suppose $a \notin [P : M]$. By Theorem 11, either $[P :_M Ra] = P$ or $[P :_M Ra] = [0 :_M Ra]$. Since $P \subseteq [P :_M Ra]$, we infer from the second case that $aP \subseteq a\,[P :_M Ra] = a\,[0 :_M Ra] = 0$. Now, let $[P :_M Ra] = P$. Suppose n is the minimal positive integer such that $a^n = 0$. If $a^{n-1} \neq 0$, then M faithful gives that $a^{n-1}M \neq 0$. Since $a^n M \subseteq P$, we have that $0 \neq a^{n-1}M \subseteq [P :_M Ra] = P$. This implies that $a \in [P : M]$, a contradiction.

(2) By Corollary 10 and part (1), $P \subseteq \sqrt{0}M$ and $Q \subseteq \sqrt{0}M$. Hence $[P : M]\,Q \subseteq \sqrt{0}\,[P : M]\,M = \sqrt{0}P = 0$. Similarly, $[Q : M]\,P = 0$.

(3) Suppose $P \neq Q$. Then P is not prime. Since $(R, [Q : M])$ is a zero-dimensional ring, we get that $\sqrt{0} = [Q : M]$. It follows by (1) that $0 = \sqrt{0}P = [Q : M]\,P$. □

Proposition 13. *Let R be a ring, M a faithful multiplication R-module and I a finitely generated faithful multiplication ideal of R.*

(1) *The following statements are equivalent for a proper submodule P of M.*
   - (i) *P is weakly prime.*
   - (ii) *$[P : M]$ is a weakly prime ideal of R.*
   - (iii) *$P = QM$ for some weakly prime ideal Q of R.*

(2) *P is a weakly prime submodule of IM if and only if $[P :_M I]$ is a weakly prime submodule of M.*

Proof. (i)$\Longrightarrow$(ii) Suppose P is a weakly prime submodule of M. Let a and b be elements of R such that $0 \neq ab \in [P : M]$. Then $abM \subseteq P$. Since M is faithful, $abM \neq 0$. By Theorem 11(4), $a \in [P : M]$ or $bM \subseteq P$ (and hence $b \in [P : M]$).

(ii)$\Rightarrow$(i) Let $[P : M]$ be a weakly prime ideal of R. If $0 \neq rm \in P$, where $r \in R, m \in M$, then $r[Rm : M] \subseteq [Rrm : M] \subseteq [P : M]$. Since M is multiplication, $r[Rm : M] \neq 0$. By [13, Theorem 3(4)], $r \in [P : M]$ or $[Rm : M] \subseteq [P : M]$. The second case gives that $m \in [Rm : M]M \subseteq [P : M]M = P$.

(i)$\Rightarrow$(iii). Take $Q = [P : M]$.

(2) Suppose P is a weakly prime submodule of IM. Let $0 \neq rm \in [P :_M I]$ for some $r \in R$ and $m \in M$. Then $rIm \subseteq P$. Since M is faithful multiplication and I faithful, $0 \neq rIm$. It follows by Theorem 11 that $Im \subseteq P$ (and hence $m \in [P :_M I]$) or $r \in [P : IM]$ (and hence $rIM \subseteq P$). As I is a finitely generated faithful multiplication ideal of R, it follows by, [3, Lemma 2.4], that

$$rM = r\,[IM :_M I] \subseteq [rIM :_M I] \subseteq [P :_M I]\,,$$

and hence $r \in [[P :_M I] : M]$. This gives that $[P :_M I]$ is a prime submodule of M. Conversely, let $0 \neq AK \subseteq P$ for some ideal A of R and some submodule K of IM. Then $A\,[K :_M I] \subseteq [AK :_M I] \subseteq [P :_M I]$. We get from, [3, Lemma 2.4], that $A\,[K :_M I] \neq 0$. It follows by Theorem11 that $A \subseteq [[P :_M I] : M] \subseteq [P : IM]$ or $[K :_M I] \subseteq [P :_M I]$ from which we infer that $K = [K :_M I]\,I \subseteq [P :_M I]\,I = P$. Hence P is a weakly prime submodule of IM. □

We close by a result of weakly prime ideals (submodules) using the method of idealization. It may be compared with [11, Theorem 17].

Theorem 14. *Let R be a ring, M an R-module and $I_{(+)}N$ a homogeneous weakly prime ideal of $R(M)$.*

(1) *I is a weakly prime ideal of R and N a weakly prime submodule of M.*

(2) *For all $a, b \in R$, $ab = 0, a \notin I$, $b \notin I$ implies $a, b \in annN$ and for all $c \in R$, $k \in M$, $ck = 0$, $c \notin [N : M]$, $k \notin N$ implies $c \in annI$ and $k \in [0 :_M I]$.*

(3) *If $I_{(+)}N$ is not prime then $I_{(+)}N = I_{(+)}0 \oplus 0_{(+)}N$.*

(4) *If $I_{(+)}N$ is prime then $N = M$.*

Proof. (1) Let $a, b \in R$ with $0 \neq ab \in I$. Then $(0, 0) \neq (a, 0)(b, 0) \in I_{(+)}N$, and hence either $(a, 0) \in I_{(+)}N$ or $(b, 0) \in I_{(+)}N$ from which it follows that $a \in I$ or $b \in I$. Hence I is weakly prime. Now, let $r \in R, m \in M$ with $0 \neq rm \in N$. Then $(0, 0) \neq (r, 0)(0, m) \in I_{(+)}N$, and hence $(r, 0) \in I_{(+)}N$ or $(0, m) \in I_{(+)}N$. If $(r, 0) \in I_{(+)}N$ then $r \in I \subseteq [N : M]$. The second case shows that $m \in N$ and hence N is a weakly prime submodule of M.

(2) Let $a, b \in R$ with $ab = 0$, $a \notin I$, $b \notin I$. Suppose $a \notin \mathrm{ann}N$. Then $aN \neq 0$ and hence there exists $n \in N$ with $an \neq 0$. This implies that $(0, 0) \neq (a, 0)(b, n) = (0, an) \in I_{(+)}N$. Hence $(a, 0) \in I_{(+)}N$ (and hence $a \in I$) or $(b, n) \in I_{(+)}N$ from which we get that $b \in I$, a contradiction. Assume now that $c \in R, k \in M, ck = 0, c \notin [N : M]$ and $k \notin N$. We discuss two cases.

Case 1. Suppose $c \notin \mathrm{ann}I$. Then $cI \neq 0$, and hence there exists $a \in I$ with $ca \neq 0$. It follows that $(0, 0) \neq (ca, 0) = (c, 0)(a, k) \in I_{(+)}N$. Hence $(c, 0) \in I_{(+)}N$, and hence $c \in I \subseteq [N : M]$ or $(a, k) \in I_{(+)}N$ and this implies $k \in N$ which also contradicts our assumption.

Case 2. Let $k \notin [0 :_M I]$. Then $Ik \neq 0$. There exists $a \in I$ such that $ak \neq 0$. Since $ak \in IM \subseteq N$, we infer that $(0, 0) \neq (ac, ak) = (ac, ac+ak) = (a, k)(c, k) \in I_{(+)}N$. Hence either $(a, k) \in I_{(+)}N$ or $(c, k) \in I_{(+)}N$. This implies that $c \in I \subseteq [N : M]$ or $k \in N$, a contradiction.

(3) It follows by, [11, Theorem 4], that $(I_{(+)}N)\sqrt{0_{(+)}0} = 0$. But $\sqrt{0_{(+)}0} = \sqrt{0}_{(+)}M$, see [19, Theorem 25.1] and [10, Theorem 3.2]. Hence $IM = 0$, and the result follows.

(4) [11, Corollary 2] shows that if $I_{(+)}N$ is a prime ideal of $R(M)$ then $\sqrt{0_{(+)}0} \subseteq I_{(+)}N$. This shows that $N = M$. □

Note that the last part of the above theorem confirms that prime ideals of $R(M)$ have the form $I_{(+)}M$ where I is a prime ideal of R.

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