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On the Shemetkov Problem for Fitting Classes*

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Abstract. Suppose that π be a set of primes and $\mathfrak{F}$ a local Fitting class. Let $K_\pi(\mathfrak{F})$ be the set of finite π-soluble groups with a Hall π-subgroup belonging to $\mathfrak{F}$. In this paper, we show that the class $K_\pi(\mathfrak{F})$ is a local Fitting class. Thus, an interesting Shemetkov question for Fitting classes will be answered positively. By using the result, the $\mathfrak{F}$-radical of a Hall π-subgroup of a finite π-soluble group is described. For an H-function f, we also give the definition and its description of f-radical of a finite π-soluble group. Some known important results follow.

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1. Introduction

In the theory of classes of finite groups, a number of classification problems and the problems of description of canonical subgroups are closed associated with the formations and Fitting classes determined by means of some properties of Hall subgroups (cf., for example, [7, IV, §16] and [4, IX, 1–4]). In this connection,

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Blessenohl [1] and Brison [3] introduced respectively the following two classes of groups $K^\pi(\mathfrak{F})$ and $K_\pi(\mathfrak{F})$ in class of all soluble groups.

If $\mathfrak{F}$ is a formation of finite groups and π a set of prime numbers, then Blessenohl defines

$$K^\pi(\mathfrak{F}) = \{G : G \text{ is a finite group and a Hall } \pi\textit{-subgroup } H \text{ of } G \text{ is in } \mathfrak{F}\}.$$

If $\mathfrak{F}$ is a Fitting class of finite groups and π a set of prime numbers, then Brison defines

$$K_\pi(\mathfrak{F}) = \{G : G \text{ is a finite group and a Hall } \pi\textit{-subgroup } H \text{ of } G \text{ is in } \mathfrak{F}\}.$$

It is easy to check that the class of groups $K^\pi(\mathfrak{F})$ is a formation and the class of groups $K_\pi(\mathfrak{F})$ is a Fitting class. In connection with the class of groups, Shemetkov proposed the following problem.

Problem (L. A. Shemetkov [7, Problem 19]). If $\mathfrak{F}$ is a local formation of finite groups and every group in $K^\pi(\mathfrak{F})$ possesses exactly one conjugate class of Hall π-subgroups, is $K^\pi(\mathfrak{F})$ a local formation?

In the class of all soluble groups, the positive answer to this problem was obtained by Blessenohl [1]. Later on, Slepova [8] proved that under some restrictive conditions on a local formation $\mathfrak{F}$, the answer to this problem is also possible in the class of all finite groups. In connection with the above results, the following dual Shemetkov problem naturally arises:

Problem. If $\mathfrak{F}$ is a local Fitting class of finite groups and every group in $K_\pi(\mathfrak{F})$ possesses exactly one conjugate class of Hall π-subgroups, is $K_\pi(\mathfrak{F})$ a local Fitting class?

The problem has been solved in the class of all soluble groups by Zagurskij and Vorob'ev [12]. In this paper, we shall give a positive answer to this problem in the class of all π-soluble groups. By using this result, we shall give some applications. In particular, the $\mathfrak{F}$-radical of a Hall π-subgroup of a finite π-soluble group is described.

All groups considered in this paper are finite π-soluble groups, and $\mathfrak{S}^\pi$ denotes the class of all finite π-soluble groups, where π is some given subset of the set $\mathbb{P}$. All unexplained notations and terminologies are standard. The reader is referred to the text of Doerk and Hawkes [4] and Guo [6] if necessary.

2. Preliminaries

Recall that a class of groups $\mathfrak{F}$ is called a *Fitting class* provided the following two conditions are satisfied:

(i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $N \in \mathfrak{F}$,

(ii) if $N_1, N_2 \trianglelefteq G$ and $N_1, N_2 \in \mathfrak{F}$, then $N_1N_2 \in \mathfrak{F}$.

Condition (ii) in the definition says that, for every non-empty Fitting class, every group G has a unique maximal normal $\mathfrak{F}$-subgroup which is called the $\mathfrak{F}$-*radical* of G and denoted by $G_{\mathfrak{F}}$.

The *product* $\mathfrak{F}\mathfrak{H}$ of two Fitting classes $\mathfrak{F}$ and $\mathfrak{H}$ is defined as the class $(G \mid G/G_{\mathfrak{F}} \in \mathfrak{H})$. It is well known that the product of any two Fitting classes is also a Fitting class and the multiplication of Fitting classes satisfies associative law.

Let σ be a non-empty set of prime numbers and σ' the complement of σ in the set of all prime numbers $\mathbb{P}$. For a group G, let $|G|$ be the order of G and $F_\pi(G)$ the maximal normal π-nilpotent subgroup of G. $\mathfrak{X}$ denotes a class of groups and $\mathfrak{X}_\sigma$ denotes the class of all finite σ-groups lying in $\mathfrak{X}$; $\mathfrak{N}$ denotes the class of all finite nilpotent groups. $\mathfrak{N}_\sigma$ denotes the class of all finite nilpotent σ-groups. In particular, $\mathfrak{N}_p$ is the class of all p-groups.

Within the universe $\mathfrak{S}^\pi$, a function f defined by $f : \mathbb{P} \longrightarrow \{\text{Fitting classes}\}$ is called a *Hartley function* (or in brevity, H-function) (see [9]). Let $\sigma = Supp(f) = \{p \in \mathbb{P} : f(p) \neq \emptyset\}$, that is, σ is the support of the function f (see [4, p. 323]) and $LR(f) = \mathfrak{S}^\pi_\sigma \cap (\cap_{p\in\sigma} f(p)\mathfrak{N}_p\mathfrak{S}^\pi_{p'})$. A Fitting class $\mathfrak{F}$ is called *local* [5] in $\mathfrak{S}^\pi$, if there exists an H-function f such that $\mathfrak{F} = LR(f)$. In this case, we say that $\mathfrak{F}$ is local defined by f or f is an H-function of $\mathfrak{F}$.

Let f be an H-function of $\mathfrak{F}$. Then f is called

(i) *integrated* if $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$, and

(ii) *full* if $f(p) = f(p)\mathfrak{N}_p$ for all $p \in \mathbb{P}$ (cf. [10]).

The following known result is useful in the sequel.

Lemma 2.1. [11] *Every local Fitting class $\mathfrak{F}$ can be defined by a largest integrated H-function F such that $F(p)\mathfrak{N}_p = F(p)$ for all $p \in \mathbb{P}$ and each non-empty value $F(p)$ is Lockett class.*

Recall that a Fitting class $\mathfrak{F}$ is said be *Lockett class* if $\mathfrak{F} = \mathfrak{F}^*$, where $\mathfrak{F}^*$ is the smallest Fitting class containing $\mathfrak{F}$ such that the $\mathfrak{F}^*$-radical of the direct product $G \times H$ of two groups G and H is equal to the direct product of the $\mathfrak{F}^*$-radical of G and the $\mathfrak{F}^*$-radical of H, that is, $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$, for all groups G and H.

Let $\pi \subseteq \mathbb{P}$. A subgroup H of a group G is called a *Hall π-subgroup* of G if the order $|H|$ of H is a π-number and the index $|G : H|$ is a π'-number.

Definition. [4, IX, 1.24] *Let π be a set of primes and $\mathfrak{F}$ a Fitting class. Then define*

$$K_\pi(\mathfrak{F}) = (G : \text{if } H \text{ is a Hall } \pi\text{-}subgroup\ of\ G, \text{then } H \in \mathfrak{F}).$$

If $\mathfrak{F} = \emptyset$, then put $K_\pi(\mathfrak{F}) = \emptyset$. In particular, $K_\emptyset(\mathfrak{F}) = \mathfrak{S}^\pi$ and $K_{\mathbb{P}}(\mathfrak{F}) = \mathfrak{F}$.

We also need the following results which generalized [4, IX, 1.25], [4, IX, p. 574, ex. 3] and [4, IX, 1.27], the condition of solubility was weakened.

Lemma 2.2.

(a) *Let $\mathfrak{F}$ be a Fitting class. Then $K_\pi(\mathfrak{F})$ is a Fitting class for any $\pi \subseteq \mathbb{P}$.*

(b) *If $\mathfrak{F}$ is a non-empty Fitting class and H is a Hall π-subgroup of G, then* $G_{K_\pi(\mathfrak{F})} \cap H = H_{\mathfrak{F}}$.

(c) *If $\mathfrak{F}$ and $\mathfrak{H}$ are Fitting classes, then* $K_\pi(\mathfrak{F}\mathfrak{H}) = K_\pi(\mathfrak{F})K_\pi(\mathfrak{H})$.

Proof. (a) It is clear by the definition of $K_\pi(\mathfrak{F})$.

(b) Put $\mathfrak{R} = K_\pi(\mathfrak{F})$ and $K = G_{\mathfrak{R}}$. Since $K \trianglelefteq G$, we have $H \cap K \in \mathrm{Hall}_\pi(K)$ and $H \cap K \trianglelefteq H$. Hence $H \cap K \subseteq H_{\mathfrak{F}}$. Let $F/K = F_\pi(G/K)$, then $F/K \in \mathfrak{N}_\pi$ since $F \in K_\pi(\mathfrak{F})\mathfrak{E}_{\pi'}\mathfrak{N}_\pi = K_\pi(\mathfrak{F})\mathfrak{N}_\pi$. Therefore, $F/K \leq HK/K \in \mathrm{Hall}_\pi(G/K)$. Obviously, $H_{\mathfrak{F}} \in \mathrm{Hall}_\pi(H_{\mathfrak{F}}K)$. So $H_{\mathfrak{F}}K \in \mathfrak{R}$. On the other hand, $F \cap H_{\mathfrak{F}}K$ *sn* G by $F/K \in \mathfrak{N}$, so $F \cap H_{\mathfrak{F}}K \leq G_{\mathfrak{R}} = K$. Therefore, $[F,\ H_{\mathfrak{F}}K] \leq F \cap H_{\mathfrak{F}}K \leq K$, and consequently, $H_{\mathfrak{F}}K \leq C_G(F/K) \leq F$ (cf. [6, Theorem 1.8.19]. It follows that $H_{\mathfrak{F}} \leq H \cap F \cap H_{\mathfrak{F}}K \leq H \cap K$. Thus, (b) holds.

(c) Let H be a Hall π-subgroup of G. If $G \in K_\pi(\mathfrak{F}\mathfrak{H})$, then $H \in \mathfrak{F}\mathfrak{H}$, that is, $H/H_{\mathfrak{F}} \in \mathfrak{H}$. By (b), we know that $H_{\mathfrak{F}} = G_{K_\pi(\mathfrak{F})} \cap H$, so $H/H_{\mathfrak{F}} \simeq HG_{K_\pi(\mathfrak{F})}/G_{K_\pi(\mathfrak{F})} \in \mathfrak{H}$ and hence $G/G_{K_\pi(\mathfrak{F})} \in K_\pi(\mathfrak{H})$. This shows that $K_\pi(\mathfrak{F}\mathfrak{H}) \leq K_\pi(\mathfrak{F})K_\pi(\mathfrak{H})$. On the other hand, if $G \in K_\pi(\mathfrak{F})K_\pi(\mathfrak{H})$, then $G/G_{K_\pi(\mathfrak{F})} \in K_\pi(\mathfrak{H})$. It follows from (b) that $H/H_{\mathfrak{F}} \simeq HG_{K_\pi(\mathfrak{F})}/G_{K_\pi(\mathfrak{F})} \in \mathfrak{H}$. Hence $H \in \mathfrak{F}\mathfrak{H}$ and consequently $G \in K_\pi(\mathfrak{F}\mathfrak{H})$. Thus, (c) holds.

Remark. The statements (b) and (c) in this Lemma maybe not true in the class $\mathfrak{G}$ of all finite groups. For example, put $\mathfrak{F} = \mathfrak{N} = \mathfrak{H}$, $G = A_5$ and $\pi = \{2,\ 3\}$. Then $H \simeq A_4$ is a Hall π-subgroup of G. Clearly $G_{K_\pi(\mathfrak{F})} \cap H = G_{K_\pi(\mathfrak{F})} = 1$, but $H_{\mathfrak{F}} \neq 1$. Hence (b) is not true. Since $H \in \mathfrak{N}^2$, we know $G \in K_\pi(\mathfrak{F}\mathfrak{H})$. But $G \notin K_\pi(\mathfrak{F})K_\pi(\mathfrak{H})$ since $G_{K_\pi(\mathfrak{F})} = 1$. Hence (c) is not true.

The following lemma is evident.

Lemma 2.3. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be two Fitting classes. Then the following statements hold:*

(a) *if $\mathfrak{F} \subseteq \mathfrak{H}$, then $K_\pi(\mathfrak{F}) \subseteq K_\pi(\mathfrak{H})$.*

(b) $K_\pi(\mathfrak{F} \cap \mathfrak{H}) = K_\pi(\mathfrak{F}) \cap K_\pi(\mathfrak{H})$.

Lemma 2.4. *Let $\mathfrak{F}$ be a Fitting class. Then the following statements hold:*

(a) *if $p \in \pi$ and $\mathfrak{F} = \mathfrak{S}^\pi_{p'}$, then $K_\pi(\mathfrak{F}) = \mathfrak{F}$;*

(b) *if $\mathfrak{F}$ is a non-empty Fitting class and $\mathfrak{F}\mathfrak{N}_p = \mathfrak{F}$ for some prime p, then* $K_\pi(\mathfrak{F})\mathfrak{N}_p = K_\pi(\mathfrak{F})$.

Proof. (a) Since a subgroup of a π-soluble p'-group is a π-soluble p'-group, it is easy to see that $\mathfrak{F} \subseteq K_\pi(\mathfrak{F})$. Let $G \in K_\pi(\mathfrak{F})$ and H is a Hall π-subgroup of G. Then $H \in \mathfrak{F}$, and so $|H|$ is a p'-number. On the other hand, since $p \in \pi$, we have $\pi' \subseteq p'$. Hence $|G : H|$ is also a p'-number. It follows that $|G|$ is a p'-number and $G \in \mathfrak{F}$. Therefore $\mathfrak{F} = K_\pi(\mathfrak{F})$.

(b) Obviously, $K_\pi(\mathfrak{F}) \subseteq K_\pi(\mathfrak{F})\mathfrak{N}_p$. Now assume that $G \in K_\pi(\mathfrak{F})\mathfrak{N}_p$. Then $G/G_{K_\pi(\mathfrak{F})}$ is a p-group. Let H be a Hall π-subgroup of G. By Lemma 2.2 (b), we see that $H/H_{\mathfrak{F}} = H/H \cap G_{K_\pi(\mathfrak{F})} \simeq HG_{K_\pi(\mathfrak{F})}/G_{K_\pi(\mathfrak{F})} \leq G/G_{K_\pi(\mathfrak{F})}$ is a p-group, that is, $H/H_{\mathfrak{F}} \in \mathfrak{N}_p$. This means that $H \in \mathfrak{F}\mathfrak{N}_p = \mathfrak{F}$, that is, $G \in K_\pi(\mathfrak{F})$. Therefore, $K_\pi(\mathfrak{F})\mathfrak{N}_p = K_\pi(\mathfrak{F})$.

Lemma 2.5. *Let $\mathfrak{F}$ be a non-empty Fitting class, and π, σ be two sets of prime numbers such that $\pi \cap \sigma = \emptyset$. Then the following statements hold:*

(a) $K_\pi(\mathfrak{F})\mathfrak{S}^\pi_\sigma = K_\pi(\mathfrak{F})$. *In particular,* $K_\pi(\mathfrak{F})\mathfrak{S}^\pi_{\pi'} = K_\pi(\mathfrak{F})$;

(b) $K_\pi(\mathfrak{F})\mathfrak{N}_\sigma = K_\pi(\mathfrak{F})$.

Proof. (a) Firstly, it is clear that $K_\pi(\mathfrak{F}) \subseteq K_\pi(\mathfrak{F})\mathfrak{S}^\pi{}_\sigma$. Assume that $G \in K_\pi(\mathfrak{F})\mathfrak{S}^\pi{}_\sigma$ and H is a Hall π-subgroup of G. Then $G/G_{K_\pi(\mathfrak{F})} \in \mathfrak{S}^\pi_\sigma$ and the Hall π-subgroup $HG_{K_\pi(\mathfrak{F})}/G_{K_\pi(\mathfrak{F})}$ of $G/G_{K_\pi(\mathfrak{F})}$ is a σ-group. Since $HG_{K_\pi(\mathfrak{F})}/G_{K_\pi(\mathfrak{F})} \simeq H/H \cap G_{K_\pi(\mathfrak{F})}$, $H/H_{\mathfrak{F}}$ is a σ-group by Lemma 2.2 (b). Hence $H/H_{\mathfrak{F}} \in \mathfrak{S}^\pi{}_\pi \cap \mathfrak{S}^\pi{}_\sigma = (1)$, where (1) is the class consisting of identity groups. Consequently, $H = H_{\mathfrak{F}}$ and hence $G \in K_\pi(\mathfrak{F})$.

(b) By the statement (a) of the lemma, we see that $K_\pi(\mathfrak{F})\mathfrak{N}_\sigma \subseteq K_\pi(\mathfrak{F})\mathfrak{S}^\pi_\sigma = K_\pi(\mathfrak{F})$. Thus, the statement (b) holds.

3. Main theorem

Theorem 3.1. *For any set of primes π and any local Fitting class $\mathfrak{F}$, the Fitting class $K_\pi(\mathfrak{F})$ is a local Fitting class.*

Proof. Since $\mathfrak{F}$ is a local Fitting class, by Lemma 2.1, there exists an H-function F such that $\mathfrak{F} = LR(F)$ and $F(p)\mathfrak{N}_p = F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$ and each value $F(p)$ is a Lockett class, for every $p \in \sigma = Supp(F)$. Then, we have that

$$\mathfrak{F} = \mathfrak{S}^\pi_\sigma \cap (\cap_{p\in\sigma} F(p)\mathfrak{N}_p\mathfrak{S}^\pi_{p'}) = \mathfrak{S}^\pi_\sigma \cap (\cap_{p\in\sigma} F(p)\mathfrak{S}^\pi_{p'}). \tag{3.1}$$

If $\pi = \mathbb{P}$, then $K_\pi(\mathfrak{F}) = \mathfrak{F}$ and so the theorem holds. Assume that $\pi = \emptyset$, then $K_\pi(\mathfrak{F}) = \mathfrak{S}^\pi$. However, it is easy to see that the class of all π-soluble groups $\mathfrak{S}^\pi = LR(h)$, where h is the H-function such that $h(p) = \mathfrak{S}^\pi$, for all $p \in \mathbb{P}$. This shows that, in this case, $K_\pi(\mathfrak{F})$ is a local Fitting class.

We now assume that $\emptyset \subsetneqq \pi \subsetneqq \mathbb{P}$ and define an H-function as follows:

$$f(p) = \begin{cases} K_{\pi\cap\sigma}(F(p)), & \text{if } p \in \pi \cap \sigma, \\ K_\pi(\mathfrak{F}), & \text{if } p \in \pi', \\ \emptyset, & \text{if } p \in \pi \cap \sigma'. \end{cases}$$

Then $\omega = Supp(f) = \sigma \cup \pi'$, and so

$$LR(f) = \mathfrak{S}^\pi_{\sigma\cup\pi'} \cap ((\cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p))\mathfrak{N}_p\mathfrak{S}^\pi_{p'}) \cap (\cap_{p\in\pi'} K_\pi(\mathfrak{F})\mathfrak{N}_p\mathfrak{S}^\pi_{p'}). \tag{3.2}$$

In order to prove the theorem, we only need to ascertain that $K_\pi(\mathfrak{F}) = LR(f)$.

For this purpose, we let $\mathfrak{M} = \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p))\mathfrak{N}_p\mathfrak{S}^\pi_{p'}$. Since the H-function F is full, by Lemma 2.4 (b), we see that $\mathfrak{M} = \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p))\mathfrak{S}^\pi_{p'}$. We now prove

$$\mathfrak{M} = K_{\pi\cap\sigma}(\mathfrak{F}). \tag{3.3}$$

Indeed, by the equality (3.1), we have that $\mathfrak{F} \subseteq \cap_{p\in\pi\cap\sigma} F(p)\mathfrak{S}^{\pi}_{p'}$. Then, by Lemma 2.3, Lemma 2.2 (c) and Lemma 2.4 (a), we see that $K_{\pi\cap\sigma}(\mathfrak{F}) \subseteq K_{\pi\cap\sigma}(\cap_{p\in\pi\cap\sigma} F(p)\mathfrak{S}^{\pi}_{p'}) \subseteq \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p)\mathfrak{S}^{\pi}_{p'}) = \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p))K_{\pi\cap\sigma}(\mathfrak{S}^{\pi}_{p'}) = \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(F(p))\mathfrak{S}^{\pi}_{p'}$. Therefore, $K_{\pi\cap\sigma}(\mathfrak{F}) \subseteq \mathfrak{M}$. On the other hand, since the H-function F is integrated, by Lemma 2.3 (a), we have $K_{\pi\cap\sigma}(F(p)) \subseteq K_{\pi\cap\sigma}(\mathfrak{F})$ for every $p \in \pi\cap\sigma$. It follows that $K_{\pi\cap\sigma}(F(p))\mathfrak{S}^{\pi}_{p'} \subseteq K_{\pi\cap\sigma}(\mathfrak{F})\mathfrak{S}^{\pi}_{p'}$ for all $p \in \pi\cap\sigma$, and consequently, $\mathfrak{M} \subseteq \cap_{p\in\pi\cap\sigma} K_{\pi\cap\sigma}(\mathfrak{F})\mathfrak{S}^{\pi}_{p'} = K_{\pi\cap\sigma}(\mathfrak{F})\mathfrak{S}^{\pi}_{(\pi\cap\sigma)'}$. However, by Lemma 2.5 (a), we see that $K_{\pi\cap\sigma}(\mathfrak{F})\mathfrak{S}^{\pi}_{(\pi\cap\sigma)'} = K_{\pi\cap\sigma}(\mathfrak{F})$, thus, the equality (3.3) holds.

Let $\mathfrak{M}_1 = \cap_{p\in\pi'} K_{\pi}(\mathfrak{F})\mathfrak{N}_p\mathfrak{S}^{\pi}_{p'}$. We prove

$$\mathfrak{M}_1 = K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi}. \tag{3.4}$$

In fact, by Lemma 2.5 (b), we have $\mathfrak{M}_1 = \cap_{p\in\pi'} K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{p'} = K_{\pi}(\mathfrak{F})(\cap_{p\in\pi'}\mathfrak{S}^{\pi}_{p'}) = K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi}$. Hence the equality (3.4) holds.

Now, by the equalities (3.2), (3.3) and (3.4), we obtain that

$$LR(f) = \mathfrak{S}^{\pi}_{\sigma\cup\pi'} \cap \mathfrak{M} \cap \mathfrak{M}_1 = \mathfrak{S}^{\pi}_{\sigma\cup\pi'} \cap K_{\pi\cap\sigma}(\mathfrak{F}) \cap K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi}. \tag{3.5}$$

Let $\mathfrak{D} = \mathfrak{S}^{\pi}_{\sigma\cup\pi'} \cap K_{\pi\cap\sigma}(\mathfrak{F})$. We prove that $\mathfrak{D} = K_{\pi}(\mathfrak{F})$. Assume that $G \in K_{\pi}(\mathfrak{F})$ and H is a Hall π-subgroup of G. Then, $H \in \mathfrak{F}$. Since $\mathfrak{F} \subseteq \mathfrak{S}^{\pi}_{\sigma}$, $|H|$ is a $(\pi\cap\sigma)$-number. It follows that $|G|$ is a $(\sigma\cup\pi')$-number, that is, $G \in \mathfrak{S}^{\pi}_{\sigma\cup\pi'}$. In addition, since $\pi' \subseteq (\sigma\cap\pi)'$, we see that H is a $(\sigma\cap\pi)$-Hall subgroup of G. This shows that $G \in K_{\pi\cap\sigma}(\mathfrak{F})$, and hence $G \in \mathfrak{D}$. On the other hand, assume that $G \in \mathfrak{D}$ and H is a $(\pi\cap\sigma)$-Hall subgroup of G. Then, $|G|$ is a $(\sigma\cup\pi')$-number and $H \in \mathfrak{F}$. It is clear that the index $|G:H|$ is a $(\pi'\cup\sigma')$-number. Hence $|G:H|$ is a μ-number, where $\mu = (\pi'\cup\sigma')\cap(\sigma\cup\pi')$. Obviously, $\mu \subseteq \pi'$. Thus, H is a Hall π-subgroup of G. This means that $G \in K_{\pi}(\mathfrak{F})$. Therefore $\mathfrak{D} = K_{\pi}(\mathfrak{F})$.

Finally, by using the above results and Lemma 2.5, we have that $LR(f) = \mathfrak{D}\cap\mathfrak{M}_1 = K_{\pi}(\mathfrak{F})\cap K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi} = K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi'}\cap K_{\pi}(\mathfrak{F})\mathfrak{S}^{\pi}_{\pi} = K_{\pi}(\mathfrak{F})(\mathfrak{S}^{\pi}_{\pi'}\cap\mathfrak{S}^{\pi}_{\pi}) = K_{\pi}(\mathfrak{F})$. This completes the proof of the theorem.

4. Remark and Example

The "local" condition in the theorem is essential. We now give an example to show it.

For this purpose, we need the concept of normal Fitting class. Recall that a non-empty Fitting class $\mathfrak{F}$ is called a normal Fitting class if for every group G, the $\mathfrak{F}$-radical $G_{\mathfrak{F}}$ of G is $\mathfrak{F}$-maximal subgroup of G. In the theory of normal Fitting classes, it is well known that the intersection of any non-empty set of non-identity normal Fitting classes is still a normal Fitting class (see Blessenohl and Gaschütz [2, Theorem 6.1]). It follows that there exists a unique minimal normal Fitting class, which is denoted by $\mathfrak{S}_*$.

Now let $\pi = \mathbb{P}$ and $\mathfrak{F} = K_{\mathbb{P}}(\mathfrak{S}_*) = \mathfrak{S}_*$. We prove that $\mathfrak{S}_*$ is not a local Fitting class. Indeed, if the class $\mathfrak{S}_*$ is a local Fitting class, then by [10, Lemma 6], $\mathfrak{S}_*$ is a Lockett class, that is, $(\mathfrak{S}_*)^* = \mathfrak{S}_*$. Then, by [4, X.1.15], we have that $\mathfrak{S}_* = (\mathfrak{S}_*)^* = \mathfrak{S}^* = \mathfrak{S}$, which is impossible.

5. Applications

Let $\mathfrak{F}$ be a non-empty Fitting class. In this section, we will describe $\mathfrak{F}$-radical of Hall π-subgroup of a group by applying Theorem 3.1. Firstly, by Lemma 2.2, we know that the $\mathfrak{F}$-radical of a Hall π-subgroup H of a group G can be formed by the following equality:

$$H_{\mathfrak{F}} = G_{K_\pi(\mathfrak{F})} \cap H. \tag{5.1}$$

Now we give the following definition.

Definition 5.1. *Let $\mathfrak{F}$ be a local Fitting class defined by H-function f, and $\sigma = Supp(f)$. A subgroup S of G is called f-radical of G (denoted by G_f) if $S = \Pi_{p\in\pi(G)\cap\sigma} G_{f(p)}$, that is, $G_f = \Pi_{p\in\pi(G)\cap\sigma} G_{f(p)}$.*

Remark 5.1. It is easy to see that if $f(p) = \mathfrak{F}$ for some $p \in \sigma = Supp(f)$ and f is an integrated H-function of $\mathfrak{F}$, then $G_f = G_{\mathfrak{F}}$.

In connection with Remark 5.1, the following problems naturally arise:

1) For a local Fitting class $\mathfrak{F}$ defined by f such that $f(p) \neq \mathfrak{F}$ for all primes $p \in \mathbb{P}$ (that is, $\sigma = Supp(f) = \{p \in \mathbb{P} : \emptyset \neq f(p) \neq \mathfrak{F}\}$), is it true that $G_f = G_{\mathfrak{F}}$?
2) Can we describe the $\mathfrak{F}$-radical of a Hall subgroup H of G ? The following theorem resolved the two problems.

Theorem 5.1. *Let $\mathfrak{F}$ be a local Fitting class defined a largest integrated H-function F and Φ be the largest integrated H-function of the local Fitting class $K_\pi(\mathfrak{F})$. Then, for every group G and its Hall π-subgroup H, the following statements hold:*

(a) *if $\sigma = Supp(F) = \{p : p \in \mathbb{P}$ and $\emptyset \neq F(p) \neq \mathfrak{F}\}$, then $G_{\mathfrak{F}} = G_F$;*
(b) *$H_{\mathfrak{F}} = G_\Phi \cap H$.*

Proof. (a) Since F is an integrated H-function, $F(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$. Then, $F(p) \subset \mathfrak{F}$ for all $p \in \pi(G)\cap\sigma$ and $G_{F(p)} \subset G_{\mathfrak{F}}$. Hence $G_F = \Pi_{p\in\pi(G)\cap\sigma} G_{F(p)} \subseteq G_{\mathfrak{F}}$. Assume that $G_F \neq G_{\mathfrak{F}}$. Then, since $G_{\mathfrak{F}} \in \mathfrak{F} = \mathfrak{S}^\pi_\sigma \cap (\cap_{p\in\sigma} F(p)\mathfrak{S}^\pi_{p'})$ and

$$G_{\mathfrak{F}}/G_{F(p)}/G_F/G_{F(p)} \simeq G_{\mathfrak{F}}/G_F,$$

we see that $G_{\mathfrak{F}}/G_F$ is a p'-group for all $p \in \sigma \cap \pi(G)$. Consequently, $G_{\mathfrak{F}}/G_F \in \mathfrak{S}^\pi_{(\sigma\cap\pi(G))'}$.

On the other hand, $G_{\mathfrak{F}}/G_F \in \mathfrak{S}^\pi_\sigma \cap \mathfrak{S}^\pi_{\pi(G)} = \mathfrak{S}^\pi_{\sigma\cap\pi(G)}$. This induces that $G_{\mathfrak{F}}/G_F = 1$ and hence $G_{\mathfrak{F}} = G_F$.

(b) By using Theorem 3.1 and its proof, we know that $K_\pi(\mathfrak{F})$ is a local Fitting class and $K_\pi(\mathfrak{F}) = LR(f)$, where f is the H-function such that

$$f(p) = \begin{cases} K_{\pi\cap\sigma}(F(p)), & \text{if } p \in \pi \cap \sigma, \\ K_\pi(\mathfrak{F}), & \text{if } p \in \pi', \\ \emptyset, & \text{if } p \in \pi \cap \sigma'. \end{cases} \tag{5.2}$$

By [10, Lemma 1] (also see [5, Lemma 6]), it is easy to see that $K_\pi(\mathfrak{F})$ is defined by a full integrated H-function Φ such that $\Phi(p) = (f(p) \cap K_\pi(\mathfrak{F}))\mathfrak{N}_p$ for all $p \in \mathbb{P}$.

We now prove that Φ is the largest integrated H-function of the class $K_\pi(\mathfrak{F})$. Indeed, by the equality (5.2), we have $f(p) = K_{\pi\cap\sigma}(F(p))$ for all $p \in \pi \cap \sigma$. Let G_1 and G_2 be π-soluble groups, $H_1 \in \mathrm{Hall}_{\pi\cap\sigma}(G_1)$ and $H_2 \in \mathrm{Hall}_{\pi\cap\sigma}(G_2)$. Then, by Lemma 2.2 (b), we have that $(G_1 \times G_2)_{f(p)} \cap (H_1 \times H_2) = (H_1 \times H_2)_{F(p)}$. By the hypotheses and Lemma 2.1, $F(p)$ is a Lockett class. Hence $(H_1 \times H_2)_{F(p)} = (H_1)_{F(p)} \times (H_2)_{F(p)}$. Now, by Lemma 2.2 (b) again, $(H)_i)_{F(p)} = (G_i)_{f(p)} \cap H_i$. Thus, $(H_1)_{F(p)} \times (H_2)_{F(p)} = ((G_1)_{f(p)} \cap H_1) \times ((G_2)_{f(p)} \cap H_2) = ((G_1)_{f(p)} \times (G_2)_{f(p)}) \cap (H_1 \times H_2)$. Therefore $(G_1 \times G_2)_{f(p)}/((G_1)_{f(p)} \times (G_2)_{f(p)})$ is a $(\pi \cap \sigma)'$-group. But, obviously, $O_{(\pi\cap\sigma)'}(G_i/(G_i)_{f(p)}) = 1$, $i = 1, 2$, so $(G_1 \times G_2)_{f(p)} = (G_1)_{f(p)} \times (G_2)_{f(p)}$. Hence $f(p)$ is a Lockett class, for all $p \in \pi \cap \sigma$.

If $p \in \pi'$, then, by the equality (5.2), $f(p) = K_\pi(\mathfrak{F})$. By using our Theorem 3.1, $K_\pi(\mathfrak{F})$ is a local Fitting class. Since every local Fitting class is a Lockett class (cf. [10, Lemma 5]), $f(p)$ is a Lockett class.

The above reasoning shows that the class $f(p)$ is a Lockett class for all $p \in Supp(\Phi)$. It follows from [4, X, 1.13] that the intersection $f(p) \cap K_\pi(\mathfrak{F})$ is still a Lockett class, and consequently, the product of the Lockett class $f(p) \cap K_\pi(\mathfrak{F})$ and the local Fitting class $\mathfrak{N}_p$ is also a Lockett class by [10, Lemma 5] and [4, Theorem X.1.26 (b)]. This shows that every non-empty value $\Phi(p)$ is a Lockett class. Thus, by Lemma 2.1, we obtain that Φ is the largest integrated H-function of the class $K_\pi(\mathfrak{F})$.

Now, by the equality (5.1), we have that $H_{\mathfrak{F}} = G_{K_\pi(\mathfrak{F})} \cap H$. Therefore, we now only need show that $G_\Phi = G_{K_\pi(\mathfrak{F})}$ in order to prove (b).

Let $\mu = Supp(\Phi)$. If there exists a prime $p \in \mu$ such that $\Phi(p) = K_\pi(\mathfrak{F})$, then, $G_\Phi = G_{K_\pi(\mathfrak{F})}$ by Remark 5.1. If $\Phi(p) \neq K_\pi(\mathfrak{F})$ for all $p \in \mu$, then, by (a), we also have that $G_\Phi = G_{K_\pi(\mathfrak{F})}$. Thus, the proof is completed.

Corollary 5.2. *Let $\mathfrak{F} = LR(F)$, for the largest integrated H-function F and $\mathfrak{F} \supseteq \mathfrak{N}$. Let H be a Hall π-subgroup of a group G. Then*

$$H_{\mathfrak{F}} = \Pi_{p\in\pi} H_{F(p)}.$$

Proof. Since $\mathfrak{F} \supseteq \mathfrak{N}$, we have that $\sigma = Supp(F) = \mathbb{P}$, and $\pi \cap \sigma = \pi$. Let Φ be that largest integrated H-function of $K_\pi(\mathfrak{F})$. Then, as we have seen in the above Theorem 5.1 and its proof, $\Phi(p) = (f(p) \cap K_\pi(\mathfrak{F}))\mathfrak{N}_p = (K_\pi(F(p)) \cap K_\pi(\mathfrak{F}))\mathfrak{N}_p$, for all $p \in \pi$. Because F is a largest integrated H-function of $\mathfrak{F}$, we have $F(p) = F(p)\mathfrak{N}_p \subseteq \mathfrak{F}$. Hence, by Lemma 2.3 and Lemma 2.4, we see that $\Phi(p) = K_\pi(F(p))\mathfrak{N}_p = K_\pi(F(p))$, for all $p \in \pi$. It follows from Theorem 5.1 (b) that

$$H_{\mathfrak{F}} = G_\Phi \cap H = \Pi_{p\in\pi} G_{\Phi(p)} \cap H = (\Pi_{p\in\pi} G_{K_\pi(F(p))}) \cap H.$$

Hence, $H_{\mathfrak{F}} = \Pi_{p\in\pi}(G_{K_\pi(F(p))} \cap H)$ (cf. [4, Lemma I.3.2 (d)]). Now, by using the equality (5.1), we obtain that $H_{\mathfrak{F}} = \Pi_{p\in\pi} H_{F(p)}$. This completed the proof.

In conclusion, we consider a simple application of Theorem 5.1 and Corollary 5.2. Let $\mathfrak{F} = \mathfrak{N}$, the class of all finite nilpotent groups. Since $\mathfrak{F}$ has a largest integrated

H-function F such that $F(p) = \mathfrak{N}_p$ for all $p \in \mathbb{P}$, by Theorem 5.1 and Corollary 5.2, we immediately obtain that $F(G) = \Pi_{p\in\pi(G)}O_p(G)$ and $F(H) = \Pi_{p\in\pi}O_p(H)$, for every group G and its Hall π-subgroup H.

References

[1] Blessenohl, D.: *Über Formationen und Halluntergruppen endlicher auflösbarer Gruppen*. Math.Z. **142** (1975), 299–300. Zbl 0291.20021

[2] Blessenohl, D.; Gaschütz, W.: *Über normale Schunck- und Fittingklassen*. Math. Z. **118** (1970), 1–8. Zbl 0208.03301

[3] Brison, O. J.: *Hall operators for Fitting classes*. Arch. Math. **33** (1979/80), 1–9. Zbl 0405.20022

[4] Doerk, K.; Hawkes, T.: *Finite Soluble Groups*. Walter de Gruyter, Berlin-New York 1992. Zbl 0753.20001

[5] Grytczuk, A.; Vorob'ev, N. T.: *On Lockett's conjecture for finite groups*. Tsukuba J. Math. **18**(1) (1994), 63–67. Zbl 0822.20015

[6] Guo, W.: *The Theory of Classes of Groups*. Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London 2000. Zbl 1005.20016

[7] Shemetkov, L. A.: *Formations of finite groups*. Nauka, Moscow 1978. Zbl 0496.20014

[8] Slepova, L. M.: *On formations of $E^{\mathfrak{F}}$-groups*. Dokl. Akad. Nauk BSSR, **21** (1977), 587–589. Zbl 0359.20017

[9] Shemetkov, L. A.; Skiba, A. N.: *Multiply ω-local formations and Fitting classes of finite groups*. Sib. Adv. Math. **10**(2) (2000), 112–141. Zbl 0969.20015

[10] Vorob'ev, N. T.: *Radical classes of finite groups with the Lockett conditions*. Mat. Zametki **43** (1988), 161–168; translated in Math. Notes **43** (1988), 91–94. Zbl 0681.20010

[11] Vorob'ev, N. T.: *On largest integrated of Hartley's function*. Proc. Gomel University, Problems in Algebra **1** (2000), 8–13.

[12] Zagurskij, V. N.; Vorob'ev, N. T.: *Fitting classes with the given given properties of Hall subgroups*. Mat. Zametki **78**(2) (2005), 234–240. Zbl 1084.20015

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