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Topological Criteria for *k*-formal Arrangements*

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Abstract. We prove a criterion for k-formality of arrangements, using a complex constructed from vector spaces introduced in [2]. As an application, we give a simple description of k-formality of graphic arrangements: Let G be a connected graph with no loops or multiple edges. Let Δ be the flag (clique) complex of G and let $H_\bullet(\Delta)$ be the homology of the chain complex of Δ. If $\mathcal{A}_G$ is the graphic arrangement associated to G, we will show that $\mathcal{A}_G$ is k-formal if and only if $H_i(\Delta) = 0$ for every $i = 1, \ldots, k-1$.

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1. Introduction

In [1], Falk and Randell introduced the notion of a formal arrangement. An arrangement is formal iff every linear dependency among the defining forms of the hyperplanes can be expressed as linear combination of dependencies among exactly 3 defining forms. Many interesting classes of arrangements are formal: in [1], Falk and Randell proved that $K(\pi, 1)$ arrangements and arrangements with quadratic Orlik-Solomon ideal are formal and, in [8], Yuzvinsky showed that free arrangements are also formal; and gave an example showing that formality does not depend on the intersection lattice. In [2], Brandt and Terao generalized the notion of formality to k-formality, proving that every free arrangement is k-formal.

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For this they introduced the concept of 'higher' relation spaces, which capture 'the dependencies among dependencies'.

In the first section of this paper we briefly recall the notions of relation spaces and k-formality. By rewriting the definitions, we obtain a lemma characterizing k-formality topologically. Then we apply this criteria for graphic arrangements to obtain a description of k-formality in terms of the homology of a certain chain complex. With this it is easy to produce examples of graphic arrangements which are k-formal but not $(k+1)$-formal, for any given k.

2. Preliminaries

In what follows we adopt all the notation from [2]. Let $\mathcal{A}$ be an arrangement of n hyperplanes in a vector space V over a field $\mathbb{K}$. For each $H \in \mathcal{A}$ we fix the defining form $\alpha_H \in V^*$.

Define a map $\phi : E(\mathcal{A}) := \oplus_{H\in\mathcal{A}}\mathbb{K}e_H \to V^*$, by $\phi(e_H) = \alpha_H$, where $E(\mathcal{A})$ is the vector space with basis $\{e_H\}$.

Let $F(\mathcal{A})$ be the kernel of this map. Then $\dim F(\mathcal{A}) = n - r(\mathcal{A})$ where $r(\mathcal{A})$ is the rank of $\mathcal{A}$. The vector space $F(\mathcal{A})$ describes which linear forms are linearly dependent, as well as the dependency coefficients (up to scalar multiplication). We will refer to elements of $F(\mathcal{A})$ as *relations*.

Let $F_2(\mathcal{A})$ be the subspace of $F(\mathcal{A})$ generated by the relations corresponding to dependencies of exactly 3 linear forms.

Definition 2.1. *$\mathcal{A}$ is formal iff $F(\mathcal{A}) = F_2(\mathcal{A})$.*

Definition 2.2. *For $3 \le k \le r(\mathcal{A})$, recursively define $R_k(\mathcal{A})$ to be the kernel of the map*

$$\pi_{k-1} = \pi_{k-1}(\mathcal{A}) : \bigoplus_{X\in L, r(X)=k-1} R_{k-1}(\mathcal{A}_X) \to R_{k-1}(\mathcal{A}),$$

where L is the lattice of intersections of $\mathcal{A}$ and π_{k-1} is the sum of the inclusion maps $R_{k-1}(\mathcal{A}_X) \hookrightarrow R_{k-1}(\mathcal{A})$. We identify $R_2(\mathcal{A})$ with $F(\mathcal{A})$.

To simplify notation, for $k \ge 2$ we will denote with $D_k = D_k(\mathcal{A})$ the vector space $\bigoplus_{X\in L, r(X)=k} R_k(\mathcal{A}_X)$.

Definition 2.3. *We define:*

1. *An arrangement is 2-formal if it is formal.*
2. *For $k \ge 3$, $\mathcal{A}$ is k-formal iff it is $(k-1)$-formal and the map $\pi_k : D_k \to R_k(\mathcal{A})$ is surjective.*

Lemma 2.4. *For any arrangement $\mathcal{A}$, the following sequence of vector spaces and maps form a complex:*

$$D_\bullet : 0 \longrightarrow \cdots \xrightarrow{d_3} D_2 \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0,$$

where $D_0 = V^*$, $D_1 = E(\mathcal{A})$ and for $k \ge 2$, D_k are the spaces from the notations above. Also, $d_1 = \phi$ and $d_k : D_k \to D_{k-1}, d_k = \pi_k$ for $k \ge 2$.

Proof. We have $d_k(D_k) = \pi_k(D_k) \subseteq R_k(\mathcal{A}) = \ker(\pi_{k-1}) \subseteq D_{k-1}$. So d_k is well defined. Also, $d_{k-1} \circ d_k(v) = \pi_{k-1}(\pi_k(v)) = 0$ for any $v \in D_k$, as $\pi_k(v) \in R_k(\mathcal{A}) = \ker(\pi_{k-1})$. So, indeed we have a complex. □

Lemma 2.5. *$\mathcal{A}$ is k-formal iff $H_i(D_\bullet) = 0$ for every $i = 1, \dots, k-1$.*

Proof. π_l is surjective iff $\forall w \in R_l(\mathcal{A})$ there exists $v \in D_l$ such that $\pi_l(v) = w$.

We have $R_l(\mathcal{A}) = \ker(\pi_{l-1}) = \ker(d_{l-1})$ and $w = \pi_l(v) = d_l(v) \in Im(d_l)$. So we get $\ker(d_{l-1}) \subseteq Im(d_l)$ which give us $H_{l-1}(D) = 0$. □

Example 2.6. In this example we will discuss [2], Example 5.1., in terms of the homology of the above complex. We must specify that all the computations are already done in [2], and we are just translating into topological language.

$\mathcal{A}$ is a real essential arrangement of rank 4 consisting of 10 hyperplanes, defined by the vanishing of the following linear forms: $\alpha_1 = x_3, \alpha_2 = x_3 - x_4, \alpha_3 = x_2, \alpha_4 = x_2 + x_3 - 2x_4, \alpha_5 = x_1, \alpha_6 = x_1 + x_3 - 2x_4, \alpha_7 = x_2 + 2x_3 - 2x_4, \alpha_8 = x_1 + 2x_3 - 2x_4, \alpha_9 = x_1 + x_2 + x_3 - 2x_4, \alpha_{10} = x_4$.

So $D_0 = \mathbb{R}^4$, $D_1 = \mathbb{R}^{10}$ and the map $d_1 : D_1 \longrightarrow D_0$ is just the map ϕ and has rank 4. Therefore $\ker(d_1)$ has dimension $10 - 4 = 6$.

We have 7 nondegenerate rank 2 elements in $L(\mathcal{A})$ and each is an intersection of exactly 3 hyperplanes. So we have 7 relations of length 3: $\alpha_1 - \alpha_2 - \alpha_{10} = 0, \alpha_1 + \alpha_4 - \alpha_7 = 0, \alpha_1 + \alpha_6 - \alpha_8 = 0, 2\alpha_2 + \alpha_3 - \alpha_7 = 0, 2\alpha_2 + \alpha_5 - \alpha_8 = 0, \alpha_3 + \alpha_6 - \alpha_9 = 0, \alpha_4 + \alpha_5 - \alpha_9 = 0$.

Therefore $D_2 = \mathbb{R}^7$. The matrix of the map $d_2 : D_2 \longrightarrow D_1$ is exactly the matrix in [2], page 61

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

and it has rank 6. So $\dim Im(d_2) = 6$ and $\dim\ker(d_2) = 7 - 6 = 1$.

Also in [2] we have listed all the elements of rank 3 from $L(\mathcal{A})$: $\{1,2,9,10\}$, $\{3,6,9,10\}$, $\{4,5,9,10\}$, $\{1,3,6,8,9\}$, $\{1,4,5,7,9\}$, $\{1,4,6,7,8\}$, $\{2,3,5,7,8\}$, $\{2,3,6,7,9\}$, $\{2,4,5,8,9\}$, $\{3,4,5,6,9\}$, $\{1,2,3,4,7,10\}$, $\{1,2,5,6,8,10\}$.

If X is such an element (with $r(X) = 3$), then $R_3(\mathcal{A}_X) \neq 0$ means that there is at least a relation among the relations of length 3 of elements of rank 2 in $L(\mathcal{A}_X)$. The nondegenerate rank 2 elements in $L(\mathcal{A}_X)$ are nondegenerate rank 2 elements in $L(\mathcal{A})$ and these are listed above. It is not difficult to check which are the relations of length 3 for each rank 3 element in $\mathcal{A}$. For reference, these are listed in the chart on page 62 in [2]. Also, there is no problem to check that for each $r(X) = 3$, the length 3 relations are linearly independent. Therefore we conclude that $D_3 = 0$.

So the complex we get is:

$$D_\bullet : 0 \longrightarrow \mathbb{R}^7 \longrightarrow \mathbb{R}^{10} \longrightarrow \mathbb{R}^4 \longrightarrow 0$$

with homology: $H_1(D_\bullet) = 0$ and $H_2(D_\bullet) = 1$. So $\mathcal{A}$ is formal, but not 3-formal.

Graphic arrangements are a class of arrangements possessing many nice properties (see, for example, [3], [5], [6], [7]), and for this class there is a pleasant combinatorial interpretation of Lema 2.5. In the next section, G denotes a connected graph with no loops or multiple edges. For the graphic arrangement $\mathcal{A}_G$, we will identify the complex above with the chain complex of the flag complex of G. Then, with Lemma 2.5., the statement in the abstract will be natural.

3. Graphic Arrangements

Let G be a connected graph on vertices $[n] = \{1, \dots, n\}$ with no loops or multiple edges. The *flag(clique) complex* $\Delta = \Delta(G)$ is the simplicial complex with:

- The 0-faces = the vertices of G.
- The 1-faces = the edges of G.
- For $i \geq 2$, the i-faces = the K_{i+1} (i.e., complete graph on $i+1$ vertices) subgraphs of G.

For $i \geq 0$, let a_i be the number of $i-$faces of Δ. We have the natural chain complex of Δ:

$$0 \longrightarrow \cdots \xrightarrow{f_3} C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \longrightarrow 0,$$

where $C_i = \mathbb{K}^{a_i}$ and $f_i : C_i \to C_{i-1}$ is the usual differential given in terms of generators: $f_i([n_1, \dots, n_{i+1}]) = \sum_{j=1}^{i+1} (-1)^{j-1} [n_1, \dots, \hat{n_j}, \dots, n_{i+1}]$.

The homology of this complex will be denoted by $H_\bullet(\Delta)$.

By definition, the graphic arrangement associated to G is $\mathcal{A} = \mathcal{A}_G = \{\ker\{\alpha_{ij}\} | \alpha_{ij} = x_i - x_j, i < j$ and $[ij]$ is an edge in $G\}$. Note that $\mathcal{A}$ is an arrangement in $V = \mathbb{K}^{a_0}$ of rank $a_0 - 1$ (if G is connected) and consists of a_1 (= the number of edges in G) hyperplanes.

Notice that from the beginning we fixed the defining forms α_{ij}. To be consistent with notation, $e_{ij}, i < j$ will be the symbols in $E(\mathcal{A})$ (i.e., $\phi(e_{ij}) = \alpha_{ij}$). With these, we can identify $D_1 = E(\mathcal{A})$ with C_1 by $e_{ij} \leftrightarrow [ij]$ for $i < j$.

If we fix the form of the elements in the basis of D_i's and with proper notations of those, the correspondence between the two complexes will become natural. The next lemma will do this, but before we state and prove it here is the flavor of it:

For $X \in L$, let G_X be the subgraph of G built on the edges corresponding to the hyperplanes in X.

We have $D_2 = \oplus_{X \in L_2} R_2(\mathcal{A}_X)$. Suppose for an $X \in L_2$ we have $R_2(\mathcal{A}_X) = F(\mathcal{A}_X) \neq 0$. This means that we must have a dependency (relation) among some of the linear forms corresponding to some edges in G_X. But this translates in the fact that G_X contains a cycle. If the length of this cycle is ≥ 4, then the linear forms corresponding to 3 consecutive edges in the cycle are linearly independent. This contradicts the fact that $rk(X) = 2$. So G_X contains a triangle. If we have an extra edge in G_X, beside those from the triangle, then the linear form of this extra edge and the linear forms associated to two of the edges of the triangle are linearly independent. Again we get a contradiction with the fact that $rk(X) = 2$. So G_X = a triangle. So each nonzero summand of D_2 corresponds to a triangle in G. The converse of this statement is obvious.

Lemma 3.1. (The Recursive Identification Lemma) *Let $X \in L$ with $r(X) = l, l \geq 2$. Then $R_l(\mathcal{A}_X) \neq 0$ iff G_X is a K_{l+1} subgraph of G. More,* $\dim R_l(\mathcal{A}_X) = 1$ *and if $G_X = [i_1 i_2 \cdots i_{l+1}], i_1 < i_2 < \cdots < i_{l+1}$, then we can pick a 'special' basis element of $R_l(\mathcal{A}_X)$ to be the relation on the special elements corresponding to the K_l subgraphs of G_X: $r_{i_2 \cdots i_{l+1}} - r_{i_1 i_3 \cdots i_{l+1}} + \cdots + (-1)^l r_{i_1 i_2 \cdots i_l}$. This element is denoted with $r_{i_1 i_2 \cdots i_{l+1}}$.*

Proof. Suppose $R_l(\mathcal{A}_X) \neq 0$. We will use induction on l.

For $l = 2$ we have already seen this case above.

Suppose $l \geq 3$. By definition, we have $R_l(\mathcal{A}_X) = \ker(\pi_{l-1})$, where

$$\pi_{l-1} : D_{l-1}(\mathcal{A}_X) = \bigoplus_{Y \in L(\mathcal{A}_X), r(Y) = l-1} R_{l-1}((\mathcal{A}_X)_Y) \longrightarrow R_{l-1}(\mathcal{A}_X).$$

The induction hypothesis is telling that for each $Y \in L(\mathcal{A}_X), r(Y) = l - 1$ such that $R_{l-1}((\mathcal{A}_X)_Y) \neq 0$, $G_Y = [i_1 i_2 \cdots i_l]$ is a K_l subgraph of G_X and $\dim R_{l-1}((\mathcal{A}_X)_Y) = 1$ with $r_{i_1 i_2 \cdots i_l}$ 'special' basis element of $R_{l-1}((\mathcal{A}_X)_Y)$.

From this we get first that $\dim D_{l-1}(\mathcal{A}_X)$ = the number of K_l subgraphs of G_X.

The condition $R_l(\mathcal{A}_X) \neq 0$ is telling us that, since $R_l(\mathcal{A}_X) \subseteq D_{l-1}(\mathcal{A}_X)$, G_X has at least one K_l subgraph.

If G_X has just one K_l subgraph, then $R_l(\mathcal{A}_X) = D_{l-1}(\mathcal{A}_X)$. But π_{l-1} is a sum of inclusions, and in this particular case it will be exactly an inclusion. So we get that $R_l(\mathcal{A}_X) = \ker(\pi_{l-1}) = 0$, which is a contradiction.

Therefore, G_X has at least two K_l subgraphs.

Let's take two of them K_l^1 and K_l^2, and first suppose they do not share any vertex. Let $v \in K_l^1$ and $w \in K_l^2$ be two vertices of G_X. Through v pass exactly $l - 1$ edges and the corresponding linear forms $\alpha_1, \ldots, \alpha_{l-1}$ are linearly independent. Let's take two edges $[w, w_1]$ and $[w, w_2]$ of K_l^2 and let β_1 and β_2 be the corresponding linear forms. Then, $\alpha_1, \ldots, \alpha_{l-1}, \beta_1, \beta_2$ are linearly dependent if at least one of the vertices $\{w, w_1, w_2\}$ is a vertex in K_l^1. Contradiction. Therefore, $\alpha_1, \ldots, \alpha_{l-1}, \beta_1, \beta_2$ are linearly independent. But this will contradict $r(\mathcal{A}_X) = l$.

Hence, K_l^1 and K_l^2 have at least a common vertex v. Suppose w_1, w_2 are two vertices of K_l^2 but not of K_l^1. Then, through v pass at least $l + 1$ edges: $l - 1$ from K_l^1 and $[v, w_1]$, $[v, w_2]$ from K_l^2. The corresponding linear forms are linearly independent and again we obtain a contradiction with the fact that $r(\mathcal{A}_X) = l$.

The conclusion of all of above is that any two distinct K_l subgraphs of G_X have exactly $l - 1$ vertices in common. $(*)$

Suppose G_X has exactly two K_l subgraphs: $[1,2,\dots,l-1,l]$ and $[1,2,\dots,l-1,l+1]$. Let $r \in R_l(\mathcal{A}_X), r \neq 0$. Then $r = r_{1,2,\dots,l-1,l} + br_{1,2,\dots,l-1,l+1}$ for some $b \in \mathbb{K} - \{0\}$. We have $\pi_{l-1}(r) = 0$ in $D_{l-2}(\mathcal{A}_X)$. So we get a relation on the 'special' basis elements of $D_{l-2}(\mathcal{A}_X)$:

$$0 = (r_{2,\dots,l-1,l} - r_{1,3,\dots,l-1,l} + \cdots + (-1)^{l-1} r_{1,2,\dots,l-1})$$

$$+b(r_{2,\dots,l-1,l+1} - r_{1,3,\dots,l-1,l+1} + \cdots + (-1)^{l-1} r_{1,2,\dots,l-1}).$$

Observe that this equation is impossible.

So G_X has at least three distinct K_l subgraphs: $K_l^1 = [1,2,\dots,l-1,l]$, $K_l^2 = [1,2,\dots,l-1,l+1]$ and K_l^3. If both l and $l+1$ are vertices in K_l^3, then l and $l+1$ are connected in G_X, so G_X contains a K_{l+1} subgraph. If, for example, $l \notin K_l^3$, then from $(*)$ and since $K_l^i, i = 1,2,3$ are distinct we get that $K_l^3 = [1,2,\dots,l-1,l+2]$, for some other vertex $l+2$ in G_X. Observe that through the vertex 1 pass at least $l+1$ edges of G_X: $[1,2],[1,3],\dots,[1,l-1],[1,l],[1,l+1],[1,l+2]$. The corresponding linear forms of these edges are linearly independent so we get a contradiction with the fact that $r(\mathcal{A}_X) = l$.

We can conclude that G_X contains a K_{l+1} subgraph. Now, if there exists an extra edge of G_X not on this K_{l+1}, then the corresponding linear form of this edge together with the corresponding linear forms of the edges passing through any vertex of the K_{l+1} subgraph will form a linearly independent set of $l+1$ elements. Again we get a contradiction with the fact that $r(\mathcal{A}_X) = l$. So G_X is a K_{l+1}.

With this, G_X has exactly $l+1$ K_l subgraphs. These subgraphs will give us the 'special' elements of $D_{l-1}(\mathcal{A}_X)$: $r_{2,\dots,l+1}, r_{1,3,\dots,l+1}, \dots, r_{1,2,\dots,l}$. The only relation on these elements is exactly the 'special' element in $R_l(\mathcal{A}_X)$:

$$r_{2,\dots,l+1} - r_{1,3,\dots,l+1} + \cdots + (-1)^l r_{1,2,\dots,l}.$$

We denote this element with $r_{1,2,\dots,l,l+1}$ and it is forming the basis for $R_l(\mathcal{A}_X)$.

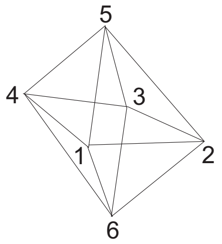
For the converse, it is obvious that if G_X is a K_{l+1}, then $R_l(\mathcal{A}_X) \neq 0$ and even more, $\dim R_l(\mathcal{A}_X) = 1$. $\square$

With this lemma we can identify easily the two complexes. The way we pick the special basis elements will give us the same matrices for the differentials of the two complexes and, hence, with Lemma 2.5., we have proved the following proposition:

Proposition 3.2. *Let G be a connected graph. $\mathcal{A}_G$ is k-formal if and only if $H_i(\Delta) = 0$ for every $i = 1,\dots,k-1$.*

Note that from this proposition we get that in the graphic arrangement case, k-formality depends only on combinatorics, contrary to the case of lines arrangements (see Yuzvinsky's example).

Example 3.3. We conclude with an easy example of a formal graphic arrangement which is not 3-formal. Consider the graph G in the figure below:



The associated flag complex Δ is the boundary complex of an octahedron on the same vertices and edges. The associated chain complex of Δ is:

$$0 \longrightarrow \mathbb{K}^8 \xrightarrow{f_2} \mathbb{K}^{12} \xrightarrow{f_1} \mathbb{K}^6 \longrightarrow 0,$$

where, if we order the basis lexicographically we have:

$$f_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

and

$$f_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since G is connected, then $\dim H_0(\Delta) = 1$. So $rk(f_1) = 6 - 1 = 5$. Therefore, $\dim \ker(f_1) = 12 - 5 = 7$.

Every 4-cycle in G is a linear combination of 3-cycles. So $\mathcal{A}_G$ is formal (2-formal). By the proposition above, $\dim H_1(\Delta) = 0$ and with this we get $rk(f_2) = 7$. Therefore, $\dim \ker(f_2) = 8 - 7 = 1$. So we get $\dim H_2(\Delta) = 1$. Hence $\mathcal{A}_G$ is not 3-formal.

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