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Sums of dth Powers in Non-commutative Rings

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Abstract. Sums of dth powers in central simple algebras and other non-commutative rings are investigated.

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Introduction

The study of sums of squares in fields or rings is a classical number theoretic problem and goes back to Diophantes, Fermat, Lagrange and Gauss who studied how to express integers as sums of squares. The classical notion of level of a field was generalized to commutative rings (see Pfister [14] and Dai, Lam and Peng [3] for lists of references), and then to non-commutative rings (e.g. to division rings and hence quaternion algebras over fields) for instance by Leep [8] and Lewis [11]. Becker [1] studied sums of $(2n)$th powers in fields and rings using higher level orderings. There does not seem to be much literature about sums of dth powers in a non-commutative ring, or even in a non-associative algebra (whereas for $d = 2$, see for instance Leep, Shapiro, Wadsworth [9], or the references in [15]). Quadratic trace forms play a role when investigating sums of squares in fields or certain types of algebras (for instance central simple ones).

There is an intimate relationship between sums of dth powers and higher trace forms of degree d. The trace form of degree d of an algebra A determines whether or not 0 can be represented as a non-trivial sum of dth powers in A. Moreover, higher trace forms provide examples of absolutely indecomposable forms of arbitrary even degree which are (even strongly) anisotropic and become isotropic under a suitable quadratic field extension. Independently of their connection to sums of dth powers, higher trace forms associated to algebras constitute an interesting class of forms of degree d, and were previously studied by O'Ryan and Shapiro [12] (for central simple algebras), by Harrison [4] (for commutative algebras only), and also by Wesolowski [20], and in [16].

One might argue that the appropriate generalization of sums of dth powers to the non-commutative case is rather sums of products of dth powers: for $d = 2$, a central division algebra D over a field k admits an ordering if and only if -1 is not a sum of products of squares in D [18], which is the analogue of a well-known characterization for a formally real field. Accordingly, one can define a dth product level, a dth product Pythagoras number and so forth (cf. Cimpric [2] for results on these "higher product levels" of non-commutative rings in a very general sense). We refrain from following this approach, since several well-known results on sums of squares can be rephrased effortlessly to sums of dth powers in non-commutative rings (and even to sums of dth powers in certain non-associative algebras), see for instance Theorem 1 or Proposition 3.

1. Preliminaries

1.1.

Let k be a field of characteristic 0 or greater than d. A *d-linear form* over k is a k-multilinear map $\theta : V \times \cdots \times V \to k$ (d-copies of V) on a finite-dimensional vector space V over k which is *symmetric*; i.e., $\theta(v_1, \ldots, v_d)$ is invariant under all permutations of its variables. A *form of degree* d over k is a map $\varphi : V \to k$ on a finite-dimensional vector space V over k such that $\varphi(av) = a^d\varphi(v)$ for all $a \in k$, $v \in V$ and where the map $\theta : V \times \cdots \times V \to k$ defined by

$$\theta(v_1, \ldots, v_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_l \leq d} (-1)^{d-l}\varphi(v_{i_1} + \cdots + v_{i_l})$$

($1 \leq l \leq d$) is a d-linear form over k. If we can write φ in the form $a_1x_1^d + \ldots + a_mx_m^d$ we use the notation $\varphi = \langle a_1, \ldots, a_n\rangle$ and call φ *diagonal.*

A d-linear space (V, θ) is called *non-degenerate* if $v = 0$ is the only vector such that $\theta(v, v_2, \ldots, v_d) = 0$ for all $v_i \in V$. The *orthogonal sum* $(V_1, \theta_1) \perp (V_2, \theta_2)$ of two d-linear spaces (V_i, θ_i), $i = 1, 2$, is the k-vector space $V_1 \oplus V_2$ together with the d-linear form $(\theta_1 \perp \theta_2)(u_1 + v_1, \ldots, u_d + v_d) = \theta_1(u_1, \ldots, u_d) + \theta_2(v_1, \ldots, v_d)$.

A d-linear space (V, θ) is called *decomposable* if $(V, \theta) \cong (V_1, \theta_1) \perp (V_2, \theta_2)$ for two non-zero d-linear spaces (V_i, θ_i), $i = 1, 2$. A non-zero d-linear space (V, θ) is called *indecomposable* if it is not decomposable. We distinguish between indecomposable ones and *absolutely indecomposable* ones; i.e., d-linear spaces which stay indecomposable under each algebraic field extension.

Let l/k be a finite field extension and $s : l \to k$ a non-zero k-linear map. If $\Gamma : V \times \cdots \times V \to l$ is a non-degenerate d-linear form over l then $s\Gamma : V \times \cdots \times V \to k$

is a non-degenerate d-linear form over k, with V viewed as a k-vector space. If the map s is the trace of the field extension l/k, we write $tr_{l/k}(\Gamma)$ or $tr_{l/k}(V, \Gamma)$ instead of $(V, tr_{l/k}\Gamma)$.

A form $\varphi : V \to k$ of degree d over k is called *isotropic* if there is a non-zero element $x \in V$ such that $\varphi(x) = 0$, otherwise it is called *anisotropic*. The form $\varphi : V \to k$ is called *weakly isotropic* if, for some integer m, the orthogonal sum of m copies $m \times \varphi$ of φ is isotropic. It is called *strongly anisotropic* if the orthogonal sum of m copies $m \times \varphi_d(x)$ is anisotropic for all integers m.

1.2.

Let R be a unital commutative ring and let the term "algebra" over R refer to a unital non-associative strictly power-associative R-algebra. We assume that R can be viewed as a subring of the algebra A via the map $R \to A$, $a \to a1$.

Let A denote either a non-commutative unital ring with $1 \neq 0$, or an R-algebra. Write A^d for the set of dth powers of elements in A and ΣA^d for the set of all non-trivial sums of dth powers of elements in A; i.e., for the set of all elements of the form $\sum_{i=1}^m a_i^d$ where each $a_i \in A$ and not all a_i are zero. For an element $a \in A$ the smallest number n such that $a = a_1^d + \cdots + a_n^d$ with all $a_i \in A$ is the *length* $l_d(a)$ of a. The smallest positive integer m such that -1 is a sum of dth powers in A is called the dth *level* (or *power Stufe* in [13]) of A, denoted $s_d(A)$. If there is no such integer, we set $s_d(A) = \infty$. In case d is odd, $s_d(k) = 1$. We write $v_d(A)$ for the smallest number m (if it exists) such that every element $a \in A$ which can be written as a sum or difference of m dth powers of elements in A; i.e., $a = e_1 a_1^d + \cdots + e_m a_m^d$ with all $a_i \in A$ and with $e_i \in \{1, -1\}$, and ∞ otherwise. (For a comprehensive survey on the results in the commutative case until 1970, see [5, p. 38].) If every element in A can be written as a sum or difference of dth powers of elements in A and $s_d(A) < \infty$, then $A = \sum A^d$. The *dth Pythagoras number* $p_d(A)$ of A is the smallest number q (if it exists) such that every sum of dth powers of elements in A can be written as a sum of q dth powers of elements in A, and ∞ otherwise. In other words, $p_d(A) = \sup\{l_d(a) | a \in \sum A^d\}$. Note that $p_d(A) = v_d(A)$ for odd integers d, so the invariant $v_d(A)$ only is interesting for even d. Obviously, $l_d(-1) = s_d(A) \leq p_d(A)$ by definition.

The case $d = 2$ is easily settled in the non-commutative case as well, since [5, (7.9), (7.10)] also hold in this more general setting:

Lemma 1. (i) *Let A be a non-commutative ring where $2 \in A^\times$. Then $v_2(A) \leq 2$.*

(ii) *Let A be a non-associative algebra over a ring R with $2 \in R^\times$ where $R \subset$ Center$(A) = \{c \in A | [c, A] = [c, A, A] = [A, c, A] = [A, A, c] = 0\}$. Then $v_2(A) \leq 2$. In particular, if $s_2(R) < \infty$ then $A = \sum A^2$ and $p_2(A) \leq 1 + s_2(R)$.*

(iii) *Let A be a commutative non-associative algebra over a unital commutative ring R, where $R \subset$ Center(A) (e.g. a Jordan algebra). Then $v_2(A) \leq 3$. In particular, if $s_2(R) < \infty$, then $A = \sum A^2$ and $p_2(A) \leq 2 + s_2(R)$.*

2. Sums of powers in commutative rings and central simple algebras

The Pythagoras number is a very delicate invariant, which is already difficult to get a hold on for $d = 2$. For $d = 2$ it is most interesting if $s_2(R) = \infty$, because otherwise it is bounded above by $s_2(R) + 2$ or even by $s_2(R) + 1$ if 2 is a unit in R. This situation is also true for $d \geq 2$ and A an R-algebra:

Proposition 1. *Let R be a unital commutative ring where d is an invertible element. Let R contain a primitive dth root of unity ω. Let A be an algebra over R, where $R \subset \mathrm{Nuc}(A) \cap \mathrm{Comm}(A) = \mathrm{Center}(A)$. If $\omega \in \Sigma R^d$, then*

$$A = \Sigma A^d.$$

More precisely,

$$s_d(A) \leq p_d(A) \leq d^{d-2}(1 + l_d(\omega) + \cdots + l_d(\omega^{d-1}))$$

gives an upper estimate for the dth Pythagoras number of A. In particular, if $p_d(R)$ is finite, then

$$p_d(A) \leq d^{d-2}(1 + (d-1)p_d(R)).$$

Proof. The proof is similar to the one given in [9, 1.1] for $d = 2$: let $l_m = l_d(\omega^m)$, then $\omega^m = \Sigma_{i=1}^{l_m} x_{i,m}^d$ in R, with $x_{i,m} \in R$ for each m, $1 \leq m \leq d-1$. Let $d = 3$. Then

$$\begin{aligned}(a+1)^3 &= a^3 + 3a^2 + 3a + 1,\\ (a+\omega)^3 &= a^3 + 3\omega a^2 + 3\omega^2 a + 1 \text{ and}\\ (a+\omega^2)^3 &= a^3 + 3\omega^2 a^2 + 3\omega a + 1.\end{aligned}$$

Therefore

$$(a+1)^3 + \omega(a+\omega)^3 + \omega^2(a+\omega^2)^3 = 9a.$$

This implies

$$a = \frac{1}{9}((a+1)^3 + \omega(a+\omega)^3 + \omega^2(a+\omega^2)^3).$$

For every $a \in A$, we compute more generally

$$a = d^{d-2}((\frac{a+1}{d})^d + \omega(\frac{a+\omega}{d})^d + \cdots + \omega^{d-1}(\frac{a+\omega^{d-1}}{d})^d)$$

and thus $a \in \Sigma A^d$. Thus a is a sum of S dth powers of elements of the algebra A, where $S = d^{d-2}(1 + l_d(\omega) + \cdots + l_d(\omega^{d-1}))$. $\square$

Since $l_d(\omega^m) \geq 1$ for all m, notice that S must be at least as large as d^{d-1}.

If ω cannot be written as a sum of dth powers in A, this upper bound does not exist and we do not know whether $p_d(A)$ is finite at all.

Lemma 2. *If k is a field containing a primitive dth root of unity ω for some $d > 2$ (hence $\mathrm{char}\, k$ does not divide d), then k is non-real.*

Proof. If 4 divides d, then k contains $\sqrt{-1}$. If p divides d, where p is an odd prime, then k contains a primitive pth root of unity ζ. Since $\zeta = \zeta^{p+1}$, every ζ^m is a square in k. Then k is non-real, since $-1 = \zeta + \zeta^2 + \cdots + \zeta^{p-1}$. □

Remark 1. (i) Let k be a field which contains a primitive dth root of unity ω such that $\omega \in \Sigma k^d$. The dth Pythagoras number $p_d(k)$ of k was shown to be finite already in [5, p. 104]. However, the bounds obtained there were given using a function $V(d)$ with $V(d) \leq 3(d-2)((\mu-1)!)^\mu$ where $\mu = (d-1)^{d-1}$ for $d \geq 3$ and not using the $l_d(\omega^m)$, $1 \leq m \leq d-1$.

(ii) Let $p \neq 2$ be a prime and let k be a field of characteristic 0 or greater than p which contains a primitive pth root of unity ω. The form $\langle 1, \omega, \ldots, \omega^{p-1} \rangle$ of degree p over k is universal [17, 9.3 (iii)]; i.e., each element of k occurs as a value of the form. If $\omega \in \Sigma k^p$, then each element of k is a sum of S p-th powers of elements of k, where now

$$S = (1 + l_p(\omega) + \cdots + l_p(\omega^{p-1}))$$

and $p_d(k) \leq S$.

(iii) Let R be a unital commutative ring where $d \in R^\times$ containing a primitive dth root of unity ω. Then $\omega \in \sum R^d$ if and only if $R = \sum R^d$ (Proposition 1).

(iv) Let k be an infinite field containing a primitive dth root of unity ω, such that $|k^\times / k^{\times d}|$ is finite. Then $\omega \in \sum k^d$ if and only if $k = \sum k^d$ if and only if $-1 \in \sum k^d$. (The last equivalence was proved in [5, (7.14)].)

Remark 2. (i) If R is a non-real field containing a primitive dth root of unity ω satisfying $\omega \in \sum R^d$, then $p_d(R)$ is finite and so is $p_d(A)$ for any algebra A over R as in Proposition 1. If R is a formally real field and d even, however, $p_d(R)$ may be infinite [5, (7.30)]. For a field R of characteristic zero, Tornheim [19] proved the upper bound

$$p_d(R) \leq (d+1) s_d(R) G(d) \leq (d+1) 2^d s_d(R)$$

where $G(d)$ is the Waring constant.

(ii) If A is a unital commutative algebra over a field k of characteristic 0 such that $s_d(A)$ is finite, then

$$p_d(A) \leq 2^{d-2}(1 + s_d(A))$$

[5, (7.29)]. So again the dth Pythagoras number of A seems to be most interesting when A is an algebra over a formally real field k (and when d is even), or when $s_d(A) = \infty$.

Corollary 1. *Let k be a field of characteristic 0 with $s_d(k) < \infty$ containing a primitive dth root of unity ω where $\omega \in \sum k^d$. Then*

$$p_d(A) \leq d^{d-2}(1 + (d-1) 2^{d-2}(1 + s_d(k))).$$

Proof. Since $r = p_d(k) \leq 2^{d-2}(1 + s_d(k))$, we get $p_d(A) \leq d^{d-2}(1 + (d-1)r) \leq d^{d-2}(1 + (d-1)2^{d-2}(1 + s_d(k)))$. $\square$

In particular, if k is a non-real field of characteristic 0 such that $|k^\times/k^{\times d}|$ is finite containing a primitive dth root of unity, then

$$p_d(A) \leq d^{d-2}(1 + (d-1)2^{d-2}(1 + s_d(k)))$$

for any algebra A as above.

Remark 3. Let D be a central simple division algebra over a field k. Given any integer $d \geq 2$ it is clear that $-1 \in \sum D^d$ implies $0 \in \sum D^d$. If k contains a primitive dth root of unity ω and $\omega \in \sum k^d$, then k is non-real and we also know $D = \sum D^d$ by Proposition 1. For $d = 2$, $-1 \in \sum D^d$ if and only if $0 \in \sum D^d$ if and only if $D = \sum D^d$ [9, Theorem D].

Let p be a prime number. For sums of dth powers, $d = p^r$, fields of characteristic p play a special role. Let k be a field of characteristic p and let A be an octonion algebra over k (indeed, even any algebra with a scalar involution), or a central simple associative algebra over k. Then

$$\sum A^{p^r} \subset \{x \in A | tr_A(x) \in k^{p^r}\},$$

because

$$tr_{A/k}(x)^{p^r} = tr_{A/k}(x^{p^r}).$$

For $d = 2$ and central simple associative algebras the above inclusion was proved to be an equality in [9, Theorem C]. This generalizes to sums of dth powers in A for $d = p^r$:

Theorem 1. *Let k be a field of prime characteristic p and A a central simple associative algebra over k. Then*

$$\sum A^{p^r} = \{a \in A \,|\, tr_{A/k}(a) \in k^{p^r}\}.$$

In particular, $A = \sum A^{p^r}$ if and only if k is perfect.

Proof. The proof that $\{a \in A \,|\, tr_A(a) \in k^{p^r}\} \subset \sum A^{p^r}$ is analogous to the one given in [9], we sketch it for the convenience of the reader: If $A = k$ this is trivial, so assume that A is different from k. $\sum A^{p^r}$ is an additive subgroup of A which is invariant.

Suppose first that A is not a division algebra. If $A \cong M_2(\mathbb{F}_2)$ we obtain the desired result by a tedious but straightforward computation. If $A \cong M_n(D)$ for a division algebra D over k, $n > 1$, and $A \not\cong M_2(\mathbb{F}_2)$ then $\ker tr_{A/k} \subset \sum A^{p^r}$ (Kasch's Theorem [9, (4.1)]).

If A is a division algebra over k, view A as an algebra over the field k^{p^r}. Then A is algebraic over k^{p^r} and $\sum A^{p^r}$ is an invariant k^{p^r}-subspace of A. Therefore $\ker(tr_{A/k}) \subset \sum A^{p^r}$ (Asano's Theorem [9, (4.2)]).

To see that $M = \{a \in A | tr_{A/k}(a) \in k^{p^r}\} \subset \sum A^{p^r}$ let $a \in M$, then $tr_{A/k}(a) = s^{p^r} \in k^{p^r}$. Since $tr_{A/k} : A \to k$ is surjective, there exists an element $b \in A$ such that $tr_{A/k}(b) = 1$, thus $tr_{A/k}(b^{p^r}) = tr_{A/k}(b)^{p^r} = 1$ and

$$tr_{A/k}(a + s^{p^r}(p-1)b^{p^r}) = tr_{A/k}(a) + s^{p^r}(p-1)tr_{A/k}(b^{p^r}) = s^{p^r} + (p-1)s^{p^r} = 0,$$

hence

$$a - (sb)^{p^r} = a - s^{p^r}(p-1)b^{p^r} \in \ker(tr_{A/k}) \subset \sum A^{p^r}$$

and therefore also $a \in \sum A^{p^r}$. □

3. Trace forms of higher degree

We fix the ensuing conventions: Let k be a field and let A be a unital, not necessarily associative, strictly power-associative k-algebra which is finite-dimensional as a k-vector space. Let

$$P_{A,a}(X) = X^n - s_1(a)X^{n-1} + s_2(a)X^{n-2} + \cdots + (-1)^n s_n(a)$$

be the generic minimal polynomial of $a \in A$. The coefficient $s_1(a) = tr_{A/k}(a)$ is called the *generic trace* of $a \in A$, n the *degree*. The generic trace induces a bilinear form $t_A : A \times A \to F$, $t_A(x, y) = tr_{A/k}(xy)$, the *bilinear trace form* of A. Its associated quadratic form is given by $x \to tr_{A/k}(x^2)$. If the bilinear trace form on A is symmetric, non-degenerate and *associative* (i.e., $tr_{A/k}(xy, z) = tr_{A/k}(x, yz)$), then A is separable. Conversely, if A is associative, alternative or a Jordan algebra, and if A is separable, then the bilinear trace form $tr_{A/k}$ on A is symmetric, non-degenerate and associative [6, (32.4) ff.].

Let $d \geq 2$ and let $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > d$. For any algebra A over k,

$$\varphi_d : A \to k, \varphi_d(a) = tr_{A/k}(a^d)$$

is a form of degree d over k, the *higher trace form* of degree d on A. If A has a non-degenerate associative symmetric bilinear trace form, φ_d is non-degenerate [16].

Lemma 3. *Let k be a field of characteristic* 0. *Let l be a finite Galois extension of k. The following are equivalent:*

(i) *l is not formally real.*

(ii) 0 *is a non-trivial sum of dth powers of elements in l for all positive integers $d \geq 2$.*

(iii) *The form $\varphi_d(x) = tr_{l/k}(x^d)$ of degree d is weakly isotropic for all positive integers $d \geq 2$.*

Proof. The equivalence of (i) and (ii) was proved in [5, p. 84].

The fact that (ii) implies (iii) is trivial.

It remains to show that (iii) implies (ii):

Let $n = [l : k]$. Let $\sigma_1, \dots, \sigma_n$ be the distinct embeddings of l in an algebraic closure of k. We have

$$tr_{l/k}(b) = \sum_{i=1}^{n} \sigma_i(b)$$

for each element $b \in l$. If the higher trace form $\varphi_d(x) = tr_{l/k}(x^d)$ of degree d is weakly isotropic then there are elements $a_i \in l$ which are not all zero such that

$$0 = \sum_{i=1}^{m} tr_{l/k}(a_i^d) = \sum_{i=1}^{m}(\sum_{j=1}^{n} \sigma_j(a_i)^d).$$

Hence 0 is a non-trivial sum of dth powers in l. $\square$

Remark 4. Let l be a finite Galois extension of k. If 0 is a non-trivial sum of dth powers in l (e.g. if l is non-real), then analogously as above, the form $\varphi(x) = tr_{l/k}(cx^d)$ of degree d, also denoted $tr_{l/k}(\langle c\rangle)$, is weakly isotropic for any $c \in l^\times$.

Conversely, suppose $c \in \sum l^d$. Let $\sigma_1, \dots, \sigma_n$ be the distinct embeddings of l in an algebraic closure of k, then

$$tr_{l/k}(b) = \sum_{i=1}^{n} \sigma_i(b)$$

for each element $b \in l$. If the form $tr_{l/k}(\langle c\rangle)$ of degree d is weakly isotropic, then there are elements $a_i \in l$ which are not all zero such that

$$0 = \sum_{i=1}^{m} tr_{l/k}(ca_i^d) = \sum_{i=1}^{m}(\sum_{j=1}^{n} \sigma_j(c)\sigma_j(a_i)^d).$$

Hence 0 is a non-trivial sum of dth powers in l since $c \in \sum l^d$ by assumption.

Lemma 4. *Let $d \geq 2$, and let A be an algebra over k in which 0 is a non-trivial sum of dth powers of elements in A. Then the higher trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is weakly isotropic.*

This was proved in [9, Lemma 2.1] for central simple associative algebras over k and in [15, 2.4] for non-associative algebras over k with scalar involution, both times for $d = 2$. Note that for d odd, the trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is always weakly isotropic for any algebra A over k. The proof of Lemma 4 is trivial. The more interesting implication is of course the remaining one.

Proposition 2. *Let A be any k-algebra with a scalar involution $^-$ (e.g. a composition algebra), and let $d \geq 2$. Then 0 is a non-trivial sum of dth powers in A if and only if the higher trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is weakly isotropic.*

Proof. If φ_d is weakly isotropic then there are $a_i \in A$ not all zero such that $0 = \sum_{i=1}^m tr_{A/k}(a_i^d) = a_1^d + \overline{a}_1^d + \cdots + a_m^d + \overline{a}_m^d$ and thus 0 is a non-trivial sum of dth powers in A. $\square$

For a central simple algebra A over a field k of characteristic not 2, 0 is a non-trivial sum of squares if and only if the quadratic trace form $\varphi_2(x) = tr_{A/k}(x^2)$ is weakly isotropic (Lewis [10, Theorem]). For $d \geq 2$ we obtain:

Proposition 3. *Let A be a central simple associative k-algebra, and let $d \geq 2$.*

(i) *Let A be a division algebra over k and let k be formally real. Then 0 is a non-trivial sum of dth powers in A if and only if the higher trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is weakly isotropic.*

(ii) *If k is not formally real, then 0 is a non-trivial sum of dth powers in A and the higher trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is weakly isotropic.*

Proof. (i) The proof closely follows the one of [10, Theorem].

(ii) If k is not formally real, then -1 is a sum of dth powers in k [5, p. 84], and thus 0 is a sum of dth powers already in k (and by Lemma 3, the higher trace form $\varphi_d(x) = tr_{A/k}(x^d)$ of degree d is weakly isotropic). $\square$

Remark 5. Let k be a formally real field and A a central simple algebra over k containing zero divisors. Then the higher trace form of A of degree d is isotropic, but for d even, we do not know whether 0 is a non-trivial sum of dth powers in A. However, Vaserstein [21] showed that for all sufficiently large n, every matrix in $\mathrm{Mat}_n(\mathbb{Z})$ is the sum of at most 10 dth powers. Hence 0 is a non-trivial sum of dth powers in $A = \mathrm{Mat}_n(D)$ for any division algebra D over k for all sufficiently large n.

For a unital non-commutative ring or an R-algebra A, clearly $\sum A^d \subset \sum A^e$ for each integer e dividing d. This implies that for a central simple division algebra D over k the fact that $0 \notin \sum D^2$ yields that $\sum D^2$ must be properly contained in D for any even integer d. With the help if this easy observation we rephrase some examples from [9]:

Example 1. (i) Let k be a formally real field (e.g. $k = \mathbb{Q}$). Put $K = k(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $D = (x_1, y_1)_K \otimes \cdots \otimes (x_n, y_n)_K$. Then D is a central simple algebra over K without zero divisors and $0 \notin \sum D^2$ [9, 2.5], thus $\sum D^d$ is a proper subset of D for any even integer d. Hence the absolutely indecomposable higher trace form $\varphi_d(x) = tr_{D/k}(x^d)$ of degree d is strongly anisotropic for even d. In particular, consider the function field of genus zero $K_0 = k(x,t)(\sqrt{at^2 + b})$ of the projective curve associated with a quaternion division algebra $(a,b)_K$ over $K = k(x)$. Put $D = (x,t)_{k(x,t)}$, then D is a quaternion division algebra over $k(x,t)$ which splits under the quadratic field extension K_0 of $k(x,t)$. Thus the absolutely indecomposable strongly anisotropic higher trace form φ_d of degree d on D becomes isotropic over K_0. (For a central simple algebra A over k containing zero divisors the higher trace form $\varphi_d(a) = tr_{A/k}(a^d)$ on A of degree d is isotropic for any $d \geq 2$.) It is an example of a strongly anisotropic absolutely indecomposable form of even degree, which becomes isotropic under a suitable quadratic field extension.

(ii) Let k be a formally real field, s an integer, and $E = UD(k, 2^s)$ the universal division algebra of degree 2^s over k. Then $0 \notin \sum E^2$ [9, 2.6], hence $\sum E^d$ is a proper subset of E for any even integer d and the absolutely indecomposable higher trace form $\varphi_d(x) = tr_{E/k}(x^d)$ of degree d is strongly anisotropic for every even integer d. For d even, the higher u-invariant $u(d, k) = \infty$ if k is formally real. For each integer m this gives an example of an anisotropic form of degree d and dimension $m2^{2s}$, which decomposes into absolutely indecomposable forms of dimension 2^{2s}.

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References

[1] Becker, E.: *Summen n-ter Potenzen in Körpern.* J. Reine Angew. Math. **307/308** (1979), 8–30. Zbl 0398.12012

[2] Cimpric, J.: *Higher product levels of noncommutative rings.* Commun. Alg. **29** (2001), 193–200. Zbl 0996.16028

[3] Dai, Z. D.; Lam, T.-Y.; Peng, C. K.: *Levels in algebra and topology.* Bull. Am. Math. Soc., New Ser. **3** (1980), 845–848. Zbl 0435.10016

[4] Harrison, D. K.: *A Grothendieck ring of higher degree forms.* J. Algebra **35** (1975), 123–138. Zbl 0309.15016

[5] Joly, J.-R.: *Sommes de puissances d-ièmes dans un anneau commutatif.* Acta Arith. **17** (1970), 37–114. Zbl 0206.34001

[6] Knus, M. A.; Merkurjev, A.; Rost, M.; Tignol, J.-P.: *The Book of Involutions.* AMS Coll. Publications, **44** (1998). Zbl 0955.16001

[7] Lam, T. Y.: *A First Course in Noncommutative Rings.* 2nd ed., Graduate Texts in Mathematics **131**, Springer-Verlag New-York 2001. Zbl 0980.16001

[8] Leep, D. B.: *Levels of Division Algebras.* Glasg. Math. J. **32** (1990), 365–370. Zbl 0727.12004

[9] Leep, D. B.; Shapiro, D. B.; Wadsworth, A. R.: *Sums of Squares in Division Algebras* Math. Z. **190** (1985), 151–162. Zbl 0548.16019

[10] Lewis, D.: *Sums of Squares in Central Simple Algebras.* Math. Z. **190** (1985), 497–498. Zbl 0563.16010

[11] Lewis, D.: *Levels of quaternion algebras.* Rocky Mt. J. Math. **19** (1989), 787–792. Zbl 0709.12002

[12] O'Ryan, M.; Shapiro, D. B.: *Centers of higher degree forms.* Linear Algebra Appl. **371** (2003), 301–314. Zbl 1047.11037

[13] Pall, S.; Rajwade, A. R.: *Power Stufe of Galois fields*. Bull. Soc. Math. Belg., Ser. A **35** (1983), 123–130. Zbl 0603.12009

[14] Pfister, A.: *Quadratic forms with applications to algebraic geometry and topology*. London Mathematical Society Lecture Notes Series **217**, Cambridge University Press, Cambridge 1995. Zbl 0847.11014

[15] Pumplün, S.: *Sums of squares in octonion algebras*. Proc. Am. Math. Soc. **133** (2005), 3143–3152. Zbl pre02188234

[16] Pumplün, S.: *Indecomposable forms of higher degree*. Math. Z. **253**(2) (2006), 347–360. Zbl 1090.11024

[17] Rupprecht, C.: *Cohomological invariants for higher degree forms*. PhD Thesis, Universität Regensburg, 2003.

[18] Szele, T.: *On ordered skewfields*. Proc. Amer. Math. Soc. **3** (1936), 81–92.

[19] Tornheim, L.: *Sums of n-th powers in fields of prime characteristics*. Duke Math. J. **4** (1938), 359–362. Zbl 0019.00302 and JFM 64.0093.02

[20] Wesolowski, A.: *Automorphism and similarity groups of forms determined by the characteristic polynomial*. Commun. Alg. **27**(7) (1999), 3109–3116. Zbl 0935.11014

[21] Vaserstein, L.: *On the sum of powers of matrices*. Linear Multilinear Algebra **21** (1987), 261–270. Zbl 0637.10041

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