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A Reverse Isoperimetric Inequality for Convex Plane Curves*

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Abstract. In this note we present a reverse isoperimetric inequality for closed convex curves, which states that if γ is a closed strictly convex plane curve with length L and enclosing an area A, then one gets

$$L^2 \leq 4\pi(A + |\tilde{A}|),$$

where $\tilde{A}$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality holds if and only if γ is a circle.

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1. Introduction

The *classical isoperimetric inequality* in the Euclidean plane $\mathbb{R}^2$ states that for a simple closed curve γ of length L, enclosing a region of area A, one gets

$$L^2 - 4\pi A \geq 0, \tag{1.1}$$

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and the equality holds if and only if γ is a circle. This fact was known to the ancient Greeks, the first complete mathematical proof was only given in the 19th century by Edler [5] (based on the arguments of Steiner [14]). There are various proofs, sharpened forms and generalizations of this inequality, see for instance [2], [3], [4], [8], [9], [11], [12], [13], [15], etc., and the literature therein.

In [6], there is a reverse isoperimetric inequality for the plane curves under some assumption on curvature. In the present note, we now establish a reverse isoperimetric inequality for convex curves, which states that if γ is a closed strictly convex curve in the plane $\mathbb{R}^2$ with length L and enclosing an area A, then we get

$$L^2 \leq 4\pi(A + |\tilde{A}|), \tag{1.2}$$

where $\tilde{A}$ denotes the oriented area of the domain enclosed by the locus of curvature centers of γ, and the equality holds if and only if γ is a circle. See also Theorem 4.2 below.

It should be pointed out that the above reverse isoperimetric inequality (1.2) is obtained by the integration of the radius of curvature, our curves must be strictly convex. We wonder if this sort of inequalities can be obtained for any (simple) closed plane curves. And furthermore, it would be interesting to generalize inequality (1.2) to higher dimensional spaces.

In the following, we first recall some facts about Minkowski's support function of closed convex plane curves, then give some properties of the locus of curvature centers of closed strictly convex plane curves, and finally present the above reverse isoperimetric inequality.

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2. Minkowski's support function for convex plane curves

From now on, without loss of generality, suppose that γ is a smooth regular positively oriented and closed strictly convex curve in the plane. Take a point O inside γ as the origin of our frame. Let p be the oriented perpendicular distance from O to the tangent line at a point on γ, and θ the oriented angle from the positive x_1-axis to this perpendicular ray. Clearly, p is a single-valued periodic function of θ with period 2π and γ can be parameterized in terms of θ and $p(\theta)$ as follows

$$\gamma(\theta) = \big(\gamma_1(\theta),\ \gamma_2(\theta)\big) = \big(p(\theta)\cos\theta - p'(\theta)\sin\theta,\ p(\theta)\sin\theta + p'(\theta)\cos\theta\big), \tag{2.1}$$

(see for instance [7]). The couple $(\theta, p(\theta))$ is usually called the *polar tangential coordinate* on γ, and $p(\theta)$ its *Minkowski's support function.*

Then, the curvature k of γ can be calculated according to $k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta)+p''(\theta)} > 0$, or equivalently, the radius of curvature ρ of γ is given by

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta). \tag{2.2}$$

Denote L and A the length of γ and the area it bounds, respectively. Then one can get

$$L = \int_\gamma ds = \int_0^{2\pi} \rho(\theta)d\theta = \int_0^{2\pi} p(\theta)d\theta, \tag{2.3}$$

and

$$A = \frac{1}{2}\int_\gamma p(\theta)ds = \frac{1}{2}\int_0^{2\pi} p(\theta)\Big[p(\theta)+p''(\theta)\Big]d\theta = \frac{1}{2}\int_0^{2\pi} \Big[p^2(\theta) - p'^2(\theta)\Big]d\theta. \tag{2.4}$$

(2.3) and (2.4) are known as *Cauchy's formula* and *Blaschke's formula*, respectively.

3. Some properties of the locus of curvature centers

We now turn to studying the properties of the locus of curvature centers of a closed strictly convex plane curve γ which is given by (2.1). Let β represent the locus of centers of curvature of γ. Then $\beta(\theta) = \big(\beta_1(\theta),\ \beta_2(\theta)\big)$ can be given by

$$\beta(\theta)=\gamma(\theta)+\rho(\theta)\mathbf{N}(\theta)=\big(-p'(\theta)\sin\theta-p''(\theta)\cos\theta,\ p'(\theta)\cos\theta-p''(\theta)\sin\theta\big), \tag{3.1}$$

where $\mathbf{N}(\theta) = (-\cos\theta,\ -\sin\theta)$ is the unit inward normal vector field along γ.

Proposition 3.1. *The oriented area of the domain enclosed by β is nonpositive. And moreover, if β is simple, then the orientation of β is the reverse direction against that of the original curve γ and the total curvature of β is equal to -2π.*

Proof. To get the claimed results, we calculate the oriented area, denoted by $\tilde{A}$, of β by Green's formula. From (3.1), we get

$$\beta_1 d\beta_2 - \beta_2 d\beta_1 = p'(\theta)\big(p'(\theta) + p'''(\theta)\big)d\theta,$$

and thus $\tilde{A}$ is given by

$$\tilde{A}=\frac{1}{2}\int_\gamma \beta_1 d\beta_2-\beta_2 d\beta_1=\frac{1}{2}\int_0^{2\pi} p'(\theta)\big(p'(\theta)+p'''\big)d\theta=\frac{1}{2}\int_0^{2\pi} \big(p'^2(\theta)-p''^2\big)d\theta. \tag{3.2}$$

Using the Wirtinger inequality for 2π-periodic C^2 real functions gives us $\tilde{A} \le 0$. If β is simple, then, from Green's formula and the fact that $\tilde{A} \le 0$, it follows that the orientation of β is the reverse direction against that of γ and the total curvature of β is equal to -2π. □

The following result is essential to the proof of the main result of this note.

Proposition 3.2. *Let γ be a C^2 closed and strictly convex curve in the plane, ρ the radius of curvature of γ, A the area enclosed by γ and $\tilde{A}$ the oriented area enclosed by β. Then we have*

$$\int_0^{2\pi} \rho^2 d\theta = 2(A + |\tilde{A}|). \tag{3.3}$$

Proof. From (2.2), we have $p'' = \rho - p$, and thus,

$$p''^2 = \rho^2 - 2p\rho + p^2 = \rho^2 - 2p(p + p'') + p^2 = \rho^2 - 2pp'' - p^2.$$

Now, according to (3.2), $|\tilde{A}|$ can be rewritten as

$$\begin{aligned}
|\tilde{A}| &= \frac{1}{2}\int_0^{2\pi} (\rho^2 - 2pp'' - p^2 - p'^2) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \rho^2 - \int_0^{2\pi} pp'' d\theta - \frac{1}{2}\int_0^{2\pi} (p^2 + p'^2) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \rho^2 d\theta - pp'|_0^{2\pi} + \int_0^{2\pi} p'^2 d\theta - \frac{1}{2}\int_0^{2\pi} (p^2 + p'^2) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \rho^2 d\theta + \frac{1}{2}\int_0^{2\pi} (p'^2 - p^2) d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \rho^2 d\theta - A,
\end{aligned}$$

which completes the proof. □

We remark that the equality (3.3) is new, and it would be interesting to find a similar formula for higher dimensional convex surfaces.

4. A reverse isoperimetric inequality

Lemma 4.1. *Let γ be a smooth closed and strictly convex curve in the plane, ρ be the radius of curvature of γ, and L be the length of γ. We have*

$$\int_\gamma \rho ds \geq \frac{L^2}{2\pi}. \tag{4.1}$$

Furthermore, the equality in (4.1) *holds if and only if γ is a circle.*

Proof. Note that $\int_\gamma \rho ds = \int_0^{2\pi} \rho^2(\theta) d\theta$. From the Cauchy-Schwartz inequality, we get

$$2\pi \int_0^{2\pi} \rho^2(\theta) d\theta \geq \Big(\int_0^{2\pi} \rho(\theta) d\theta\Big)^2 = \Big(\int_\gamma ds\Big)^2 = L^2. \tag{4.2}$$

And furthermore, the equality in (4.2) holds if and only if ρ is a constant which means that γ is a circle, because a simple closed plane curve with constant curvature must be a circle. □

Now, from the above lemma and Proposition 3.2, one can easily get our main result.

Theorem 4.2. (A Reverse Isoperimetric Inequality) *If γ is a closed strictly convex plane curve with length L and enclosing an area A, let $\tilde{A}$ denote the oriented area bounded by its locus of centers of curvature, then we get*

$$L^2 \leq 4\pi(A + |\tilde{A}|), \tag{4.3}$$

where the equality holds if and only if γ is a circle. □

The following corollary is a direct consequence of the classical isoperimetric inequality (1.1) and our reverse isoperimetric inequality (4.3). Also, it can be thought of as a direct consequence of (3.2) and the Wirtinger inequality.

Corollary 4.3. *Let β be the locus of curvature centers of a closed strictly convex plane curve γ. Then the oriented area $\tilde{A}$ of β is zero if and only if γ is a circle and thus β is a point which is the center of γ.* □

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