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Point-reflections in Metric Plane

Victor Pambuccian

Department of Integrative Studies, Arizona State University West
Phoenix, AZ 85069-7100, U.S.A.
e-mail: pamb@math.west.asu.edu

Abstract. We axiomatize the class of groups generated by the point-reflections of a metric plane with a non-Euclidean metric, the structure of which turns out to be very rich compared to the Euclidean metric case, and state an open problem.

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1. Introduction

There is a very large literature on characterizations of groups of motions in terms of line-reflections or hyperplane-reflections (see [1]), but relatively little about groups generated by point-reflections. This subject has received some attention much later, in [4], [5], [6] (and, in a different setting, with an added differential structure, in e. g. [2] or [3]). The purpose of this paper is to determine the theories of point-reflections that one obtains from the groups of isometries of Bachmann's metric planes. If the metric plane is elliptic, i. e. if there are three line-reflections whose product is the identity, then the point-reflections coincide with the line-reflections, so that the axiom system of the group generated by point-reflections is identical to the one expressed in terms of line-reflections. The interesting case is thus that of non-elliptic metric planes.

2. Non-elliptic metric planes

2.1. Axiom system in terms of line-reflections

We shall first present non-elliptic metric planes as they appear in [1]. Our language will be a one-sorted one, with variables to be interpreted as 'rigid motions',

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containing a unary predicate symbol G, with $G(x)$ to be interpreted as 'x is a line-reflection', a constant symbol 1, to be interpreted as 'the identity', and a binary operation $\circ$, with $\circ(a,b)$, which we shall write as $a \circ b$, to be interpreted as 'the composition of a with b'. To improve the readability of the axioms, we introduce the following abbreviations:

$$\begin{aligned} a^2 &:\Leftrightarrow a \circ a, \\ \iota(g) &:\Leftrightarrow g \neq 1 \wedge g^2 = 1, \\ a|b &:\Leftrightarrow G(a) \wedge G(b) \wedge \iota(a \circ b), \\ J(abc) &:\Leftrightarrow \iota((a \circ b) \circ c), \\ pq|a &:\Leftrightarrow p|q \wedge G(a) \wedge J(pqa). \end{aligned}$$

The axioms are (we omit universal quantifiers whenever the axioms are universal sentences):

B 1. $(a \circ b) \circ c = a \circ (b \circ c)$

B 2. $(\forall a)(\exists b)\, b \circ a = 1$

B 3. $1 \circ a = a$

B 4. $G(a) \rightarrow \iota(a)$

B 5. $G(a) \wedge G(b) \rightarrow G(a \circ (b \circ a))$

B 6. $(\forall abcd)(\exists g)\, a|b \wedge c|d \rightarrow G(g) \wedge J(abg) \wedge J(cdg)$

B 7. $ab|g \wedge cd|g \wedge ab|h \wedge cd|h \rightarrow (g = h \vee a \circ b = c \circ d)$

B 8. $\bigwedge_{i=1}^{3} pq|a_i \rightarrow G(a_1 \circ (a_2 \circ a_3))$

B 9. $\bigwedge_{i=1}^{3} g|a_i \rightarrow G(a_1 \circ (a_2 \circ a_3))$

B 10. $(\exists ghj)\, g|h \wedge G(j) \wedge \neg j|g \wedge \neg j|h \wedge \neg J(jgh)$

B 11. $(\forall x)(\exists ghj)\, G(g) \wedge G(h) \wedge G(j) \wedge (x = g \circ h \vee x = g \circ (h \circ j))$

B 12. $G(a) \wedge G(b) \wedge G(c) \rightarrow a \circ (b \circ c) \neq 1$

Since $a \circ b$ with $a|b$ represents a point-reflection, we may think of an unordered pair (a,b) with $a|b$ as a *point*, an element a with $G(a)$ as a *line*, two lines a and b for which $a|b$ as a pair of perpendicular lines, and say that a point (p,q) is *incident* with the line a if $pq|a$. With these figures of speech in mind, the above axioms make the following statements: B1, B2, and B3 are the group axioms for the operation $\circ$; B4 states that line-reflections are involutions; B5 states the invariance of the set of line-reflections, B6 states that any two points can be joined by a line, which is unique according to B7 (we shall denote the line joining the points (a,b) and (c,d) by $\langle (a,b),(c,d) \rangle$); B8 and B9 state that the composition of three reflections in lines that have a common point or a common perpendicular is a line-reflection; B10 states that there are three lines g, h, j such that g are h are perpendicular, but j is perpendicular to neither g nor h, nor does it go through the intersection point of g and h; B11 states that every motion is the composition of two or three line-reflections, and B12 states that the composition of three line-reflections is never the identity. The function of the last axiom, B12, is to exclude elliptic geometries, and thus to ensure that the perpendicular from a point not on a line to that line is unique. The theory of non-elliptic metric planes, axiomatized by {B1-B12} will be denoted by $\mathcal{B}$.

2.2. Axiom system in terms of ternary geometric operations

The same class of models can also be axiomatized in the following manner: the language $\mathcal{L}$ contains only one sort of individual variables, to be interpreted as 'points', three individual constants a_0, a_1, a_2, to be interpreted as three non-collinear points, and two operation symbols, F and π. $F(abc)$ is the foot of the perpendicular from c to the line ab, if $a \neq b$, and a itself if $a = b$, and $\pi(abc)$ is the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, and arbitrary otherwise. By 'fourth reflection point' we mean the following: if we designate by σ_x the mapping defined by $\sigma_x(y) = \sigma(xy)$, i. e. the reflection of y in the point x, then, if a, b, c are three collinear points, by [1, §3,9, Satz 24b], the composition (product) $\sigma_c\sigma_b\sigma_a$, is the reflection in a point, which lies on the same line as a, b, c. That point is designated by $\pi(abc)$.

In order to formulate the axioms in a more readable way, we shall use the following abbreviations:

$$\sigma(ab) := \pi(aba), \tag{1}$$

$$R(abc) := \sigma(F(abc)c), \tag{2}$$

$$L(abc) :\leftrightarrow F(abc) = c \vee a = b, \tag{3}$$

where σ has the same meaning as above, $R(abc)$ stands for the reflection of c in ab (a line if $a \neq b$, the point a if $a = b$), and $L(abc)$ stands for 'the points a, b, c are collinear (but not necessarily distinct)'. The axiom system consists of the following axioms:

C 1. $F(aab) = a$

C 2. $\sigma(aa) = a$

C 3. $\sigma(a\sigma(ab)) = b$

C 4. $L(aba)$

C 5. $L(abc) \to L(cba) \wedge L(bac)$

C 6. $L(ab\sigma(ab))$

C 7. $L(abF(abc))$

C 8. $\sigma(ax) = \sigma(bx) \to a = b$

C 9. $a \neq b \wedge F(abx) = F(aby) \to L(xyF(abx))$

C 10. $a \neq b \wedge c \neq d \wedge F(abc) = c \wedge F(abd) = d \rightarrow F(abx) = F(cdx)$

C 11. $\neg L(abx) \wedge F(xF(abx)y) = y \rightarrow F(abx) = F(aby)$

C 12. $a \neq b \wedge a \neq c \wedge F(abc) = a \rightarrow F(acb) = a$

C 13. $a \neq x \wedge x \neq y \wedge F(axy) = x \rightarrow F(a\sigma(ax)\sigma(ay)) = \sigma(ax)$

C 14. $\sigma(\sigma(xa)\sigma(xb)) = \sigma(x\sigma(ab))$

C 15. $u \neq v \wedge a \neq b \wedge F(abc) = a \rightarrow F(R(uva)R(uvb)R(uvc)) = R(uva)$

C 16. $\neg L(oab) \wedge \neg L(obc) \rightarrow \sigma(F(xR(ocR(obR(oax)))o)x) = R(ocR(obR(oax)))$

C 17. $\neg L(oab) \wedge \neg L(obc) \wedge \sigma(mx) = R(ocR(obR(oax)))$
$\wedge \sigma(ny) = R(ocR(obR(oay))) \rightarrow L(omn)$

C 18. $a \neq b \wedge b \neq c \wedge F(abc) = c \wedge a \neq a' \wedge b \neq b' \wedge c \neq c' \wedge F(aba') = a \wedge F(bab')$
$= b \wedge F(cbc') = c \rightarrow \sigma(F(xR(cc'R(bb'R(aa'x)))\pi(abc))x)$
$= R(cc'R(bb'R(aa'x))) \wedge F(\pi(abc)cF(xR(cc'R(bb'R(aa'x)))\pi(abc)))$
$= \pi(abc)$

C 19. $\neg L(a_0a_1a_2)$

The axioms make the following statements: C1 defines the value of $F(abc)$ when $a = b$ — it is an axiom with no geometric function (we could have opted to leave it undefined, but that would have lengthened the statements of the axioms C16 and C18); C2: the point a is a fixed point of the reflection σ_a, C3: reflections in points are involutory transformations (or the identity); C8: reflections of a point in two different points do not coincide; C4: a lies on the line determined by a and b; C5: collinearity of three points is a symmetric relation; C6: the reflection of b in a is collinear with a and b; C7: for $a \neq b$, the foot of the perpendicular from c to the line ab lies on that line; C9 states the uniqueness of the perpendicular to the line ab in the point $F(abx)$; C10: the foot of the perpendicular from x to the line ab does not depend on the particular choice of points a and b that determine the line ab; C11: if x is a point outside of the line ab, and y is a point on the perpendicular from x to ab, then the feet of the perpendiculars of x and y to the line ab coincide; C12 states that perpendicularity is a symmetric relation (if ca is perpendicular to ab, then ba is perpendicular to ac); C13: if yx is perpendicular to xa, then so are $\sigma_a(y)\sigma_a(x)$ and $\sigma_a(x)a$; C14: reflections in points preserve midpoints; C15: reflections in lines preserve the orthogonality relation; C16 and C17 together state the three reflections theorem for lines having a point in common; C18 is the three reflections theorem for lines having a common perpendicular; C19: a_0, a_1, a_2 are three non-collinear points. With $\Sigma = \{$C1-C19$\}$, we proved in [8] the following

Theorem 1. *Σ is an axiom system for non-elliptic metric planes. In every model of Σ, the operations F and π have the intended interpretations.*

3. Axiom system for metric planes with non-Euclidean metric in terms of point-reflections

We now turn to yet another axiomatization of non-elliptic metric planes with non-Euclidean metric (i. e. in which there exists no rectangle), in terms of motions which are products of point-reflections, the individual constant 1, and the binary operation $\circ$, with $a \circ b$ standing for the composition of the motions a and b. In case a is a point-reflection, we will refer to a as a 'point' as well. To improve the readability of the axioms we introduce the following abbreviations:

$$\begin{aligned}
P(a) &:\Leftrightarrow a \neq 1 \wedge a \circ a = 1\\
P(a_1,\dots,a_n) &:\Leftrightarrow \bigwedge_{i=1}^{n} P(a_i)\\
L(abc) &:\Leftrightarrow (a \circ b) \circ c = (c \circ b) \circ a\\
\sigma(ab) &:= (a \circ b) \circ a\\
\varphi(eabcd) &:\Leftrightarrow (\neg L(abc) \wedge L(abd) \wedge L(cde) \wedge \sigma(e\sigma(bc)) = \sigma(\sigma(db)c))\\
&\qquad \vee(a \neq b \wedge L(abc) \wedge d = c) \vee (a = b \wedge d = a)\\
\pi(abc) &:= c \circ (b \circ a)\\
\varrho(eabcd) &:\Leftrightarrow \varphi(eabcu) \wedge d = \sigma(uc).
\end{aligned}$$

Here $P(a)$ stands for 'a is a point-reflection', given that, in the group generated by point-reflections the only involutory elements are the point-reflections themselves. The subsequent abbreviations will be used only when all the variables that appear in them are point-reflections. $L(abc)$ stands for 'a, b, c are collinear'; $\sigma(ab)$ is the point obtained by reflecting b in a; $\varphi(eabcd)$ holds, in case $a \neq b$, if d is the foot of the perpendicular from c to ab (as shown, for a, b, c not collinear in [9, Prop. 1] (e is a point needed in this construction, see Figure 1)) and, in case $a = b$, if $d = a$; and $\varrho(eabcd)$ stands for 'd is the reflection of c in the line ab if $a \neq b$ or in point a if $a = b$.

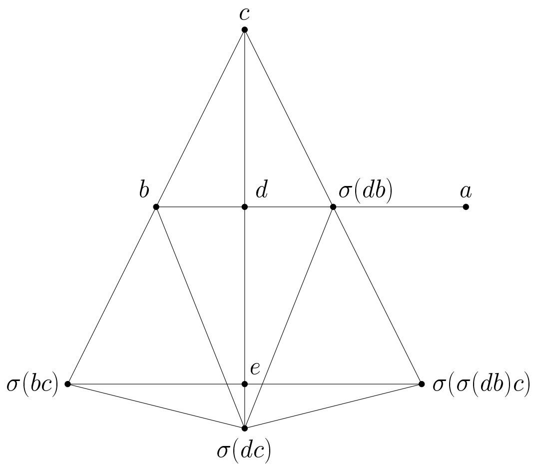

Figure 1. The definition of perpendicularity in terms of σ

The axioms for this axiom system are: B1, B2, B3, axiom P1, which ensures that, whenever a, b, c are collinear points, $\pi(abc)$ is a point as well, the axioms P2 and P3 stating the existence and uniqueness of the foot of the perpendicular from point c to line ab, whenever c does not lie on the line ab, as well as P4–P14, which are slightly changed variants of C8–C19.

P 1. $P(a,b,c) \rightarrow a \circ b \neq c$

P 2. $(\forall abc)(\exists de)\, P(a,b,c) \wedge \neg L(abc) \rightarrow P(e,d) \wedge \varphi(eabcd)$

P 3. $P(a,b,c,d,e,d',e') \wedge \neg L(abc) \wedge \varphi(eabcd) \wedge \varphi(e'abcd') \rightarrow d = d'$

P 4. $P(a,b,x) \wedge \sigma(ax) = \sigma(bx) \rightarrow a = b$

P 5. $P(a,b,d,e,f,x,y) \wedge a \neq b \wedge \varphi(eabxd) \wedge \varphi(fabyd) \rightarrow L(xyd)$

P 6. $P(a,b,c,d,e,f,u,v,x) \wedge a \neq b \wedge c \neq d \wedge L(abc) \wedge L(abd)$
$\wedge \varphi(eabxu) \wedge \varphi(fcdxv) \rightarrow u = v$

P 7. $P(a,b,d,e,f,u,v,x) \wedge \neg L(abx) \wedge \varphi(eabxu) \wedge L(xyu) \wedge \varphi(fabyv) \rightarrow u = v$

P 8. $P(a,b,c,x,y,u) \wedge \neg L(abc) \wedge \varphi(xabca) \wedge \varphi(yacbu) \rightarrow u = a$

P 9. $P(a,e,f,u,x,y) \wedge \neg L(axy) \wedge \varphi(eaxyx) \wedge \varphi(fa\sigma(ax)\sigma(ay)u) \rightarrow u = \sigma(ax)$

P 10. $P(a,b,c,a',b',c',e,m,n,p,q,u,v,x) \wedge \neg L(abc) \wedge \varphi(eabca) \wedge u \neq v$
$\wedge \varrho(nuvaa') \wedge \varrho(puvbb') \wedge \varrho(quvcc') \wedge \varphi(ma'b'c'x) \rightarrow x = a'$

P 11. $P(o,a,b,c,m,n,p,q,x,y,z,u,v) \wedge \neg L(oab) \wedge \neg L(obc) \wedge \varrho(moaxy) \wedge \varrho(nobyz)$
$\wedge \varrho(poczu) \wedge \varphi(qxuov) \rightarrow \sigma(vx) = u$

P 12. $P(o,a,b,c,m,n,p,m',n',p',x,y,z,u,x',y',z',u',t,t') \wedge \neg L(oab) \wedge \neg L(obc)$
$\wedge \varrho(moaxy) \wedge \varrho(nobyz) \wedge \varrho(poczu) \wedge \varrho(m'oax'y') \wedge \varrho(n'oby'z') \wedge \varrho(p'ocz'u')$
$\wedge \sigma(tx) = u \wedge \sigma(t'x') = u' \rightarrow L(ott')$

P 13. $P(a,b,c,a',b',c',m,n,p,t,u,v,w,x,y,z,w,o,g) \wedge L(abc) \wedge \neg L(aba') \wedge \neg$
$L(bab') \wedge \neg L(cbc') \wedge \varphi(maba'a) \wedge \varphi(nbab'b) \wedge \varphi(pcbc'c) \wedge \varrho(taa'xy) \wedge \varrho(ubb'yz)$
$\wedge \varrho(vcc'zw) \wedge \varphi(qxw\pi(abc)o) \wedge \varphi(r\pi(abc)cog) \rightarrow \sigma(ox) = w \wedge g = \pi(abc)$

P 14. $(\exists abc)\, P(a,b,c) \wedge \neg L(abc)$

Finally, we need an axiom that lets us know that every rigid motion is a product of point-reflections. We do not know whether every element of the subgroup generated by point-reflections of the motion group G of a metric plane can be written as a product of at most a certain fixed number of point-reflections (whereas we do know that every element of G can be written as a product of at most three line-reflections). Unless we establish that there is an upper bound on the number of point-reflections needed (or that such an upper bound does not exist), we cannot determine the first-order theory of the group generated by point-reflections (as it may be either (1) the theory axiomatized by the axioms {B1–B3,P1–P14} in case there are, for every natural number k, products of point-reflections that cannot be written as a product of at most k point-reflections, or (2) the theory axiomatized by those axioms and an axiom stating that every rigid motion is a product of at most k point-reflections, should k be the least number with this property).[1] What we can do is to determine the $L_{\omega_1\omega}$-theory of point-reflections, i. e. to state that every rigid motion is a product of an unspecified number of point-reflections as an infinite disjunction of first-order formulas. The axiom thus is

P 15. $(\forall a)\ \bigvee_{n=1}^{\infty}(\exists p_1 \dots p_n)\, P(p_1, \dots, p_n) \wedge a = p_1 \circ (\dots \circ p_n) \dots)$

Let $\Pi = \{$B1–B3,P1–P15$\}$. Given that we can define F in terms of φ, given that P2 and P3 ensure the existence and uniqueness of the value of $F(abc)$ for c not on ab, that C1–C7, C14 follow from our definitions of L, F and σ, and that the axioms C8–C19 follow from their translations P4–P14 into our language, we deduce that in every model of Π the individual variables x for which $P(x)$ holds can be interpreted as points, the defined notions F and π have the desired geometric interpretation, and thus, that the resulting structure is a non-elliptic metric plane with non-Euclidean metric (the metric cannot be Euclidean, given that, by P14 there are three non-collinear points a, b, c, and by P2, P3, there is precisely one point d on the line ab for which a point e with $\varphi(eabcd)$ exists; had the metric been Euclidean, then all points would have been "collinear" and for every point d on ab there would have been a point e with $\varphi(eabcd)$). Let $\mathfrak{M}$ be a model of Π. We now associate to every element $a \in \mathfrak{M}$ a mapping of the set S of points, i. e. of those members x of the universe of $\mathfrak{M}$ for which $P(x)$ holds, into itself, which we denote by $\tilde{a}$ and define by $\tilde{a}(x) := a \circ x \circ a^{-1}$. Note that $\widetilde{a \circ b} = \tilde{a} \cdot \tilde{b}$, where by $\cdot$ we have denoted the operation of composition of maps. If $a \in S$, then $\tilde{a}(x) = \sigma(ax)$, so the set $\{\tilde{a} : P(a)\}$ generates a group $\mathfrak{G}$ inside $Sym(S)$, which is precisely the group generated by the point-reflections of a non-elliptic metric plane with non-Euclidean metric (given that we know that σ has the desired interpretation). Given P15, the map $\tilde{}$ defines an isomorphism of $\mathfrak{M}$ onto $\mathfrak{G}$. We have shown that:

Theorem 2. Π *is an $L_{\omega_1\omega}$-axiom system for the group generated by the point-reflections of non-elliptic metric planes with non-Euclidean metric.*

We now turn to metric planes whose metric is Euclidean, also called 'metric-Euclidean planes'.

4. Point-reflections in metric-Euclidean planes

If axiom R ("There exists a rectangle" (see [1, §6,7])) holds in a metric plane, i. e.

$$(\exists abcd)\, a \neq b \wedge c \neq d \wedge a|c \wedge a|d \wedge b|c \wedge b|d,$$

is added to $\mathcal{B}$, then the group generated by point-reflections can be described very simply by means of B1–B3 and

[1] It was shown in [11] that, under additional assumptions, it is possible to write every product of point-reflections as a product of at most 4 point-reflections, but no such reduction is known in the general non-Euclidean metric case.

E 1. $(\exists ab)\, P(a,b) \wedge a \neq b$

E 2. $P(a,b) \wedge a \circ b = b \circ a \to a = b$

E 3. $(\forall x)(\exists ab)\, P(a,b) \wedge (x = a \vee x = a \circ b)$

E 4. $P(a,b,c) \to P(a \circ (b \circ c))$

Let $\mathfrak{M}$ be a model of B1–B3, E1–E4, let M be its universe, and let $P = \{m \in M : P(m)\}$ be the set of points in M. We define on $P \times P$ an equivalence relation $\sim$ by $(a,b) \sim (c,d)$ if and only if $a \circ c = b \circ d$, and denote by $[a,b]$ the equivalence class of (a,b). Let $G := P \times P/\sim$. We define on G an addition operation $+$ by $[a,b] + [c,d] := [a, b \circ c \circ d]$, which turns G into an Abelian group, as can be easily checked. We fix a point o in P, and consider all elements in G written as $[o,x]$ (notice that $[a,b] = [o, b \circ (a \circ o)]$). Writing $\mathbf{x}$ for $[o,x]$, we check that $\sigma(\mathbf{ab}) = 2\mathbf{a} - \mathbf{b}$ (i. e. that $[o,\sigma(ab)] + [o,b] = [o,a] + [o,a]$). By E2 we know that G must satisfy $2x = 0 \to x = 0$. Any metric-Euclidean plane can be embedded in a Gaussian plane associated with the pair of fields (K,L), where $K \subset L$, $[L:K] = 2$ (a generalization of the Gauss plane over ($\mathbb{C}$,$\mathbb{R}$), see [7]), the points being elements of L, and thus the algebraic representation of point-reflections $\tilde{\mathbf{x}}$ is given by $\tilde{\mathbf{x}}(\mathbf{y}) = 2\mathbf{x} - \mathbf{y}$ and $(\tilde{\mathbf{x}} \cdot \tilde{\mathbf{y}})(\mathbf{z}) = 2(\mathbf{x} - \mathbf{y}) + \mathbf{z}$. Given that the only operations involved in the description of point-reflections and their composition are $+$ and $-$, every first-order sentence that is true in all metric-Euclidean planes must hold over arbitrary Abelian groups which satisfy $2x = 0 \to x = 0$ as well. Thus

Theorem 3. {B1–B3, E1–E4} *is an axiom system for the group generated by the point-reflections of metric-Euclidean planes.*

Related results have been proved in [10].

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