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On Reuleaux Triangles in Minkowski Planes

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Abstract. In this paper we prove some results on Reuleaux triangles in (Minkowski or) normed planes. For example, we reprove Wernicke's result (see [21]) that the unit disc and Reuleaux triangles in a normed plane are homothets if and only if the unit circle is either an affine regular hexagon or a parallelogram. Also we show that the ratio of the area of the unit ball of a Minkowski plane to that of a Reuleaux triangle of Minkowski width 1 lies between 4 and 6. The Minkowskian analogue of Barbier's theorem is obtained, and some inequalities on areas of Reuleaux triangles are given.

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1. Introduction

If a convex body K in $\mathbb{R}^d$, $d \geq 2$, has the same distance between any two parallel supporting hyperplanes, then K is called a *body of constant width.* There is a large variety of bodies of constant width in $\mathbb{R}^d$; see the surveys [4] and [8]. The most famous example of a non-circular planar figure of constant width is the *Reuleaux triangle* in the Euclidean plane. It is bounded by three circular arcs of radius

$r > 0$ which are centered at the vertices of an equilateral triangle with sides of length r.

It is natural to extend the notion of bodies of constant width and that of Reuleaux triangles to (Minkowski or) normed linear spaces; cf. the surveys [4] and [12].

In the literature one can find various results on the Minkowskian analogues of Reuleaux triangles. E.g., the Blaschke-Lebesgue theorem states that among all figures of constant width $r > 0$ in $\mathbb{R}^2$ the Reuleaux triangles of that width have minimum area. D. Ohmann and, independently, K. Günther showed in their dissertations (both in Marburg, 1948) that the analogous statement holds for normed planes; see also [15], [9], [3], and [6] for further approaches and modifications of that result. B. Wernicke [21] estimated the ratio of the areas of Reuleaux triangles and unit discs in Minkowski planes, clarifying also the equality cases. Constructions of curves of constant width and, in particular, of Reuleaux polygons in normed planes are discussed in Chapter 4 of [20], see also [11] and [10]. Further results on Minkowskian Reuleaux triangles can be found in [19] and [17], e.g. related to lattices.

The purpose of this paper is to investigate some properties of Reuleaux triangles of Minkowski width 1 which are generated by the 'equilateral' triangle of side-length 1 in Minkowski planes. We will give a new proof of the fact that the unit disc and a Reuleaux triangle in a Minkowski plane are homothetic if and only if the unit circle is either an affine regular hexagon or a parallelogram. Related to this, we will establish that the ratio of the area of the unit disc to the area of a Reuleaux triangle of Minkowski width 1 lies between 4 and 6. On the other hand, we will show that the product of the areas of Reuleaux triangles of Minkowski width 1 and of the dual unit disc lies between $\frac{3}{2}$ and $\frac{9}{4}$. The cases of equality will be completely clarified. Some of the results presented here were already obtained in [21], but we give a new and more unified approach to them. Also we will extend Barbier's theorem to Minkowski planes. Furthermore, we present various inequalities estimating the areas of Reuleaux triangles of Minkowski width 1 for different definitions of area, and also the perimeter of Minkowskian Reuleaux triangles is taken into consideration.

2. Preliminaries

Recall that a *convex body* $K \subset \mathbb{R}^d$ is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is centrally symmetric with respect to the origin 0 of $\mathbb{R}^d, d \geq 2$. As usual, we denote by S^{d-1} the standard Euclidean unit sphere of $\mathbb{R}^d$.

Let $(X, B) = \mathbb{M}^2$ be a two-dimensional normed linear space, i.e., a *normed or Minkowski plane* with unit disc B which is a centered planar convex body. Thus we consider X as $\mathbb{R}^2$ equipped with an arbitrary *norm* $\|\cdot\|$, and by $B^\circ \in X^*$ we denote the *dual* of B, where X^* is the dual space of X.

We will suppose that X possesses the standard Euclidean structure, and that λ is the Lebesgue measure induced by this structure. Since $d = 2$, we refer to this measure as *area* and denote it by $\lambda(\cdot)$. The area λ gives rise to a dual area λ^* defined for convex bodies in X^*. We may assume that $X = \mathbb{R}^2$.

If K_1 and K_2 are convex bodies in X, then the *Minkowski sum* and the *scalar multiplication* of these convex bodies are defined by

$$K_1 + K_2 = \{x : x = x_1 + x_2,\ x_1 \in K_1,\ x_2 \in K_2\},$$

$$\alpha K_1 = \{x : x = \alpha y,\ y \in K_1\}.$$

Let K be a convex body in X. The function h_K defined for $f \in \mathbb{R}^d$ by

$$h_K(f) := \sup\{\langle f, x\rangle : x \in K\}$$

is called the *support function* of K. When $X = \mathbb{R}^d$ and $u \in S^{d-1}$, then $h_K(u)$ is the distance from the origin to the supporting hyperplane of K with unit outer normal u.

Definition 1. *For each unit linear functional f in X^*, the Minkowski width of K in direction f, denoted by $\omega_B(K, f)$, is defined by*

$$\omega_B(K, f) = h_B(K, f) + h_B(K, -f),$$

where $h_B(K, f) = \sup\{\langle f, x\rangle : x \in K, ||f|| = 1\}$ is the Minkowskian support function; see [20]*, p.* 106*.*

One can show (see [2]) that if $u \in S^{d-1}$ and $X = \mathbb{R}^d$, then

$$\omega_B(K, u) = \frac{2\omega(K, u)}{\omega(B, u)}, \tag{1}$$

where $\omega(K, u)$ is the usual Euclidean width of K in direction u.

Definition 2. *A convex body K in (X, B) is said to be of constant Minkowski width $c \in \mathbb{R}^+$ if $\omega_B(K, f) = c$ for all unit linear functionals f in X^*.*

For the following known statement we refer, e.g., to [20], p. 107.

Theorem 3. *If K is a convex body of constant Minkowski width c in (X, B), then $K + (-K) = cB$.*

Recall that the Rogers-Shephard inequality (see [18]) states that for a convex body K in $\mathbb{R}^d$

$$\lambda(K + (-K)) \le \binom{2d}{d}\lambda(K) \tag{2}$$

holds, with equality if and only if K is a simplex. For our next theorem we refer to [5] and [16].

Theorem 4. *If a convex body K in $\mathbb{M}^2$ is of constant Minkowski width and, in addition, has an equichordal point (that is, all chords of K passing through this point have the same Minkowski length), then K is homothetic to the unit disc B of $\mathbb{M}^2$.*

For a two-dimensional Minkowski space (X, B) we can define the *mixed volume* of convex bodies K_1 and K_2 by

$$V(K_1,\ K_2) = 2^{-1} \int_{\partial K_1} h_{K_2} ds(x) = 2^{-1} \int_{\partial K_2} h_{K_1} ds(x), \tag{3}$$

where $ds(x)$ is the Euclidean element of arc length at x.

The Minkowski inequality for mixed volumes states that

$$V^2(K_1,\ K_2) \geq \lambda(K_1)\lambda(K_2)\,,$$

with equality if and only if K_1 and K_2 are homothetic.

Using formula (3), one can rewrite the Minkowski length of the boundary ∂K of K in terms of mixed volumes (see also [20], p. 120), namely by

$$\mu_B(\partial K) = \int_{\partial K} h_{I_B} ds(x) = 2V(K,\ I_B), \tag{4}$$

where the convex body I_B is the polar reciprocal of B (with respect to the Euclidean circle) rotated through $90°$.

Barbier's theorem states that for all convex bodies of constant width c in the Euclidean plane

$$l = \pi c$$

holds, where l is the respective perimeter.

We will extend this theorem to Minkowski planes.

Theorem 5. *If K is a convex body of constant Minkowski width $c \in \mathbb{R}^+$ in a normed plane (X, B), then*

$$\mu_B(\partial K) = \frac{c}{2}\mu_B(\partial B).$$

Proof. It follows from (1) that $\omega(K, u) = \frac{c}{2}\omega(B, u), u \in S^1$. This gives us

$$h_K(u) + h_K(-u) = \frac{c}{2}(h_B(u) + h_B(-u)).$$

Integrating both sides along ∂I_B, we obtain

$$\int_{\partial I_B} (h_K(u) + h_K(-u))ds(x) = \frac{c}{2} \int_{\partial I_B} (h_B(u) + h_B(-u))ds(x).$$

From (3) we have

$$\int_{\partial K} (h_{I_B}(u) + h_{I_B}(-u))ds(x) = \frac{c}{2} \int_{\partial B} (h_{I_B}(u) + h_{I_B}(-u))ds(x).$$

Since I_B is symmetric about the origin, we have $h_{I_B}(u) = h_{I_B}(-u)$. Hence the result follows from (4). $\square$

Corollary 6. *If K is a convex body of constant Minkowski width 1 in a normed plane (X, B), then*

$$3 \le \mu_B(\partial K) \le 4,$$

with equality on the left if and only if B is an affine regular hexagon, and equality on the right if and only if B is a parallelogram.

Proof. It is well known (see, e.g., [14]) that $6 \le \mu_B(\partial B) \le 8$ with equality on the left if and only if the unit ball B is an affine regular hexagon and equality on the right if and only if B is a parallelogram. Hence the result follows from Theorem 3. □

We remark that H. G. Eggleston showed in [5] that if the unit ball is a parallelogram, then every convex body of constant Minkowski width is homothetic to the unit ball.

3. Reuleaux triangles

Let $ABCDEF$ be an affine regular hexagon inscribed to the unit disc B of a Minkowski plane (X, B). We denote this hexagon by H.

One can construct Reuleaux triangles in Minkowski planes in the following way: we start with the triangle OAB which is equilateral in (X, B) and denote this triangle by T. Over the side OA of T we construct the arc $\widehat{EF}$ of ∂B, over the side OB of T the arc $\widehat{CD}$ of ∂B, and over the side AB of T the arc $\widehat{AB}$ of ∂B. The convex body bounded by these three arcs is a (Minkowskian) *Reuleaux triangle* of constant Minkowski width 1 and is said to be generated by T. We denote this Reuleaux triangle by T_R.

Example. Let the unit disc B be the hexagon $ABCDEF$ with vertex coordinates $\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$. One can see that the triangles OAB, OBC, OCD, ODE, OEF and OFA as well as $\frac{1}{2}B$ are Reuleaux triangles of constant Minkowski width 1. (There are further types of Reuleaux triangles, since this unit disc has, of course, also other inscribed affine regular hexagons.) From Theorem 5 it follows that all Reuleaux triangles described here have the Minkowskian perimeter 3.

From the above construction of Reuleaux triangles of Minkowski width 1 we get

Proposition 7. *If T_R is a Reuleaux triangle of Minkowski width* 1 *generated by an equilateral triangle T in (X, B), then $T = T_R$ if and only if B is an affine regular hexagon.*

The next proposition is due to G. D. Chakerian [3]; see also [14], p. 108.

Proposition 8. *If H is an affine regular hexagon inscribed to B and T_R is a Reuleaux triangle of constant Minkowski width* 1 *constructed from H as above, then*

$$\lambda(T_R) = \frac{\lambda(B)}{2} - \frac{\lambda(H)}{3}. \tag{5}$$

Also the proof of the following lemma can be found in Chakerian's paper [3], or in Thompson's book [20], p. 108.

Lemma 9. *If (X, B) is a Minkowski plane and H is an affine regular hexagon inscribed to the unit disc B, then*

$$\lambda(B) \leq \frac{4\lambda(H)}{3},$$

with equality if and only if B is an affine regular hexagon or a parallelogram.

In [3], Chakerian showed the minimal property of Reuleaux triangles regarding the isoperimetric problem for convex bodies of constant Minkowski width in normed planes: if K is a convex body of constant Minkowski width 2 in a normed plane (X, B), then $\lambda(K) \geq \lambda(\hat{T}_R)$, where $\hat{T}_B$ is a Reuleaux triangle of constant Minkowski width 2. Recall that the unit disc has the maximum area among all convex bodies of constant Minkowski width 2.

Theorem 10. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then T_R is homothetic to B if and only if B is an affine regular hexagon or a parallelogram.*

Proof. It follows from the above construction of a Reuleaux triangle that if B is an affine regular hexagon or a parallelogram, then there exists a Reuleaux triangle homothetic to B.

Let $\hat{T}_R$ be a Reuleaux triangle of width 2 that is homothetic to B. Then $\lambda(B) = \lambda(\hat{T}_R)$, and from (5) we have

$$\lambda(\hat{T}_R) = 2\lambda(B) - \frac{4\lambda(H)}{3}.$$

Therefore $\lambda(B) = \dfrac{4\lambda(H)}{3}$. It follows from Lemma 9 that this is the case when B is an affine regular hexagon or a parallelogram. □

In [3] (see also [17]) Chakerian proved that in Minkowski planes the ratio between the area of the unit disc and that of an equilateral triangle of side-length 1 lies between 6 and 8. We can prove the following for Reuleaux triangles of Minkowski width 1.

Theorem 11. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

$$4 \leq \frac{\lambda(B)}{\lambda(T_R)} \leq 6\,, \tag{6}$$

with equality on the left if and only if B is a parallelogram or an affine regular hexagon and T_R is homothetic to B, and on the right if and only if B is an affine regular hexagon and T_R is an equilateral triangle.

Proof. It follows from the Rogers-Shephard inequality (2) that $\lambda(T_R+(-T_R)) \leq 6\lambda(T_R)$, with equality if and only if T_R is an equilateral triangle. Hence the right inequality follows from Theorem 3 and Proposition 7.

To prove the left inequality we can write $\lambda(T_R+(-T_R))$ as

$$\lambda(T_R+(-T_R)) = \lambda(T_R) + 2V(T_R, -T_R) + \lambda(-T_R).$$

The Minkowski inequality for mixed volumes implies that $V(T_R, -T_R) \geq \lambda(T_R)$, with equality if and only if T_R is centrally symmetric. Therefore it follows from Theorem 4 that T_R and B are homothetic. Hence the result follows from Theorem 10. □

Corollary 12. *If K is a convex body of constant Minkowski width 2 in a normed plane (X, B), then*

$$\lambda(B) \leq \frac{3}{2}\lambda(K),$$

with equality if and only if B is an affine regular hexagon.

Proof. Theorem 11 implies that for all Reuleaux triangles $\hat{T}_R$ of Minkowski width 2 the inequality $\lambda(B) \leq \frac{3}{2}\lambda(\hat{T}_R)$ holds. We also know that there exists a Reuleaux triangle $\hat{T}_R$ such that $\lambda(K) \geq \lambda(\hat{T}_R)$. Hence the result follows. □

The measure of symmetry between a centrally symmetric convex body and its inscribed affine regular hexagon has been investigated by many authors (see, for example, [1], [2], and [7]). The following statements refer to the measure of symmetry between the dual of a centrally symmetric convex body and its inscribed affine regular hexagon.

Theorem 13. *If B is the unit disc of a normed plane $\mathbb{M}^2$ and H is an affine regular hexagon inscribed to B, then*

$$6 \leq \lambda(H)\lambda(B^\circ) \leq 9,$$

with equality on the left if and only if B is a parallelogram, and on the right if and only if B is an affine regular hexagon.

Proof. Since

$$\lambda(H) \leq \lambda(B) \leq \frac{4}{3}\lambda(H)$$

with equality on the left if and only if B is an affine regular hexagon, and on the right if and only if B is a parallelogram or an affine regular hexagon, we have

$$8 \leq \lambda(B)\lambda(B^\circ) \leq \frac{4}{3}\lambda(H)\lambda(B^\circ).$$

Thus, the left side of the inequality follows.

By $H \subseteq B$ we have $B^\circ \subseteq H^\circ$, and therefore

$$\lambda(H)\lambda(B^\circ) = \lambda(H)\lambda(I_B) \leq V^2(H,\ I_B) = \frac{1}{4}\mu_B^2(\partial H) = 9.$$

Equality holds when H and I_B are homothetic or H and B° are homothetic. Hence equality holds when B is an affine regular hexagon. $\square$

Corollary 14. *If B is the unit disc and T is an equilateral triangle of a normed plane $\mathbb{M}^2$, then*

$$1 \leq \lambda(T)\lambda(B^\circ) \leq \frac{3}{2},$$

with equality on the left if and only if B is a parallelogram, and on the right if and only if B is an affine regular hexagon.

4. Areas of Reuleaux triangles

Recall that in a Minkowski plane (X, B) the area μ is defined by $\mu(\cdot) = \sigma(X)\lambda(\cdot)$, where σ is a fixed constant. Choosing the correct constant σ is not as easy as it seems. Also, these two measures μ and λ must agree in the standard Euclidean plane.

Here are some well-known definitions of area:

i) The *Busemann definition of area* of a convex body K in a two-dimensional Minkowski space (X, B) is given by

$$\mu_B^{Bus}(K) := \frac{\pi}{\lambda(B)}\lambda(K).$$

ii) The *Holmes-Thompson definition of area* of K in a normed plane (X, B) is given by

$$\mu_B^{HT}(K) := \frac{\lambda(K)\lambda^*(B^\circ)}{\pi}.$$

iii) The *Benson definition of area* of K in a two-dimensional Minkowski space (X, B) is given by

$$\mu_B^{Ben}(K) := \frac{4}{\lambda(P)}\lambda(K),$$

where P is a minimal parallelogram circumscribed about B.

From Theorem 11 we can establish Wernicke's result; see [21].

Proposition 15. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

$$\frac{\pi}{6} \leq \mu_B^{Bus}(T_R) \leq \frac{\pi}{4},$$

with equality on the left if and only if B is a affine regular hexagon and T_R is an equilateral triangle, and on the right if and only if B is a parallelogram or affine regular hexagon and T_R is homothetic to B.

Proposition 16. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

$$\frac{1}{2} \leq \mu_B^{Ben}(T_R) \leq 1,$$

with equality on the left if and only if B is an affine regular hexagon, and on the right if and only if B is a parallelogram.

Proof. In [13] it was shown that if P is a minimal parallelogram circumscribed about B , then $\dfrac{\lambda(B)}{\lambda(P)} \geq \dfrac{3}{4}$ with equality if and only if B is an affine regular hexagon. Hence the left side of the inequality follows from Theorem 11. The right side of the inequality is obvious. □

Theorem 17. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

$$\frac{3}{2} \leq \lambda(T_R)\lambda(B^\circ) \leq \frac{9}{4},$$

with equality on the left side if and only if B is an affine regular hexagon and T_R is an equilateral triangle, and on the right side if and only if B is an affine regular hexagon and T_R is homothetic to B.

Proof. Since $(rB)^\circ = r^{-1}(B^\circ)$ for any $r > 0$, the quantity $\lambda(T_R)\lambda(B^\circ)$ is unchanged by dilation. Therefore we may assume that $\lambda(B^\circ) = 1$. Since $\lambda(T) \leq \lambda(T_R)$, the quantity $\lambda(T_R)\lambda(B^\circ)$ attains its minimum when $\lambda(T) = \lambda(T_R)$. By Proposition 7 this is the case when B is an affine regular hexagon. Hence the left side follows.

Since $\lambda(\hat{T}_R) \leq \lambda(B)$, where $\hat{T}_R$ is a Reuleaux triangle of Minkowski width 2, the quantity $\lambda(T_R)\lambda(B^\circ)$ attains its maximum when $4\lambda(T_R) = \lambda(B)$. By Theorem 10 this is the case when B is either an affine regular hexagon or a parallelogram. Simple calculation shows that the maximum is attained when B is an affine regular hexagon. Hence the right side follows. □

Corollary 18. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

$$\frac{3}{2\pi} \leq \mu_B^{HT}(T_R) \leq \frac{9}{4\pi},$$

with equality on the left side if and only if B is an affine regular hexagon and T_R is an equilateral triangle, and on the right side if and only if B is an affine regular hexagon and T_R is homothetic to B.

Also we can investigate the isoperimetric inequality for Reuleaux triangles in Minkowski planes.

Proposition 19. *If T_R is a Reuleaux triangle of Minkowski width 1 in a normed plane (X, B), then*

i) $$\frac{\mu_B^2(\partial T_R)}{\mu_B^{HT}(T_R)} \geq 4\pi$$

with equality if and only if B is an affine regular hexagon;

ii)
$$\frac{\mu_B^2(\partial T_R)}{\mu_B^{Bus}(T_R)} \geq \frac{36}{\pi}$$

with equality if and only if B is an affine regular hexagon;

iii)
$$\frac{\mu_B^2(\partial T_R)}{\mu_B^{Ben}(T_R)} \geq \frac{9}{4}.$$

Proof. i) Minkowski's inequality for mixed volumes implies that

$$\mu_B^2(\partial T_R) = 4V^2(T_R,\ I_B) \geq 4\lambda(T_R)\lambda(I_B),$$

with equality if and only if T_R and I_B are homothetic. Hence equality follows from Theorem 4 and the fact that $\lambda(I_B) = \lambda(B^\circ)$.

The inequalities ii) and iii) follow from Propositions 13 and 14, respectively, and from Corollary 6. □

Obviously, inequality iii) is not sharp. To find the sharp bound in this case should be interesting.

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