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Abelian Varieties, Surfaces of General Type and Integrable Systems

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1. Introduction

In recent years, there has been much effort given for finding integrable Hamiltonian systems. However, there is still no general method for testing the integrability of a given dynamical system. In this paper, we shall be concerned with finite dimensional algebraic completely integrable systems. A dynamical system is algebraic completely integrable (in the sense of Adler-van Moerbeke [1]) if it can be linearized on an abelian variety (i.e., a complex algebraic torus $\mathbb{C}^n/lattice$). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. However, besides the fact that many Hamiltonian completely integrable systems possess this structure, another motivation for its study which sounds more modern is: algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Indeed a theorem of Adler-Kostant-Symes [10] applied to Kac-Moody algebras provides such systems which, by a theorem of van Moerbeke-Mumford [17], are algebraic completely integrable. Therefore there are hidden symmetries which have a group theoretical foundation. Also some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are

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coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense. The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. In fact, the overwhelming majority of dynamical systems, Hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space. In the present paper, we discuss an interesting interaction between complex algebraic geometry and dynamical systems. We construct a new integrable system in five unknowns having three quartics invariants. This 5-dimensional system is algebraic completely integrable and it establishes some correspondences for old and new integrable systems. The paper is organized as follows:

In Section 2, we construct a new and interesting integrable system of differential equations (1) in five unknowns having three quartics invariants (2). We make a careful study of the algebraic geometric aspect of the complex affine variety $A(3)$ defined by putting these invariants equal to generic constants. We find via the Painlevé analysis the principal balances of the Hamiltonian field defined by the Hamiltonian. To be more precise, we show that the system (1) possesses Laurent series solutions in t, which depend on 4 free parameters: α, β, γ and θ. These meromorphic solutions restricted to the surface $A(3)$ are parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface $\mathcal{C}(5)$ of genus 7, that intersect in two points at which they are tangent to each other. The affine variety $A(3)$ is embedded into $\mathbb{P}^{15}$ and completes into an abelian variety $\widetilde{A}$ by adjoining a divisor $\mathcal{D} = \mathcal{C}_1 + \mathcal{C}_{-1}$. The latter has geometric genus 17 and is very ample. The flow (1) evolves on $\widetilde{A}$ and is tangent to each Riemann surface $\mathcal{C}_{\pm 1}$ at the points of tangency between them. Consequently, the system (1) is algebraic integrable. In Section 3, we show that the system (1) includes in particular, a system (8) in $\mathbb{C}^4$ which is intimately related to the potential obtained by Ramani, Dorizzi and Grammaticos [16,5]. When one examines all possible singularities of the system (8), one finds that it is possible for the variable q_1 to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing the variables $z_1, \dots, z_5$ (used in Section 2), which restores the Painlevé property to the system. We show that the system (8) admits Laurent solutions in $t^{1/2}$, depending on 3 free parameters: u, v and w. These pole solutions restricted to the invariant surface $B(9)$ are parameterized by two copies $-_{-1}$ and $-_1$ of the same Riemann surface $-(11)$ of genus 16. Applying the method explained in Piovan [15], we show that the invariant variety $B(9)$ can be completed as a cyclic double cover $\overline{B}$ of the abelian variety $\widetilde{A}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with invariants: Euler characteristic of $\widetilde{B} \equiv \mathcal{X}(\widetilde{B}) = 1$ and geometric genus of $\widetilde{B} \equiv p_g(\widetilde{B}) = 2$. Consequently, the system (8) is algebraic completely integrable in the generalized sense. The paper is supported by two appendices which contain some basis concepts concerning abelian varieties and Hamiltonian systems. The methods which will be used are primarily analytical but heavily inspired by algebraic geometrical methods. Abelian varieties and cyclic coverings of abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can then be translated into differential equations and their Laurent solutions. Among the results presented in this paper, there is an explicit calculation of invariants for Hamiltonian systems which cut out an open set in an abelian variety or cyclic coverings of abelian varieties, and various Riemann surfaces related to these systems are given explicitly. The integrable dynamical systems presented here are interesting problems, particular to experts of abelian varieties who may want to see explicit examples of a correspondence for varieties defined by different Riemann surfaces.

2. A five-dimensional algebraic completely integrable system and abelian surface

Consider the following system of five differential equations in the five unknowns $z_1, \ldots, z_5$:

$$\begin{aligned} \dot{z}_1 &= 2z_4, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= z_2(3z_1 + 8z_2^2), \\ \dot{z}_4 &= z_1^2 + 4z_1z_2^2 + z_5, \\ \dot{z}_5 &= 2z_1z_4 + 4z_2^2z_4 - 2z_1z_2z_3, \end{aligned} \tag{1}$$

where the dot indicates the differentiation with respect to time variable t. The following three quartics are constants of motion for this system

$$\begin{aligned} F_1 &= \frac{1}{2}z_5 - z_1z_2^2 + \frac{1}{2}z_3^2 - \frac{1}{4}z_1^2 - 2z_2^4, \\ F_2 &= z_5^2 - z_1^2z_5 + 4z_1z_2z_3z_4 - z_1^2z_3^2 + \frac{1}{4}z_1^4 - 4z_2^2z_4^2, \\ F_3 &= z_1z_5 + z_1^2z_2^2 - z_4^2. \end{aligned} \tag{2}$$

This new system is completely integrable and the Hamiltonian structure is defined by the Poisson bracket

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^{5} J_{kl}\frac{\partial F}{\partial z_k}\frac{\partial H}{\partial z_l},$$

where $\frac{\partial H}{\partial z} = (\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5})^\top$, and

$$J = \begin{bmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4z_1z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1z_2^2 \\ -4z_4 & 0 & 4z_1z_2 & -2z_5 + 8z_1z_2^2 & 0 \end{bmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (1) can be written as

$$\dot{z} = J\frac{\partial H}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

where $H = F_1$. The second flow commuting with the first is regulated by the equations

$$\dot{z} = J\frac{\partial F_2}{\partial z}, \quad z = (z_1, z_2, z_3, z_4, z_5)^\top,$$

and is written explicitly as

$$\begin{aligned}
\dot{z}_1 &= -16z_1z_2^2z_4 + 8z_1^2z_2z_3 + 8z_4z_5 - 4z_4z_1^2,\\
\dot{z}_2 &= 4z_1z_2z_4 - 2z_1^2z_3,\\
\dot{z}_3 &= 8z_2z_4^2 - 4z_4z_1z_3 + 4z_1z_2z_5 - 2z_1^3z_2,\\
\dot{z}_4 &= 2z_5z_1^2 - 8z_1z_2z_3z_4 + 4z_1^2z_3^2 - 2z_1^4 + 4z_5^2 + 8z_1z_2^2z_5 - 4z_1^3z_2^2,\\
\dot{z}_5 &= 8z_4z_5z_1 - 16z_2z_3z_4^2 + 8z_4z_1z_3^2 - 4z_4z_1^3 - 8z_1^2z_2^2z_4 + 4z_1^3z_2z_3\\
&\quad +16z_5z_2^2z_4 - 8z_2z_5z_1z_3 + 32z_1z_2^4z_4 - 16z_1^2z_2^3z_3.
\end{aligned}$$

These vector fields are in involution: $\{F_1, F_2\} = \langle \frac{\partial F_1}{\partial z}, J\frac{\partial F_2}{\partial z}\rangle = 0$, and the remaining one is casimir: $J\frac{\partial F_3}{\partial z} = 0$. The invariant variety A defined by

$$A = \bigcap_{k=1}^{2} \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \tag{3}$$

is a smooth affine surface for generic values of $(c_1, c_2, c_3) \in \mathbb{C}^3$. So, the question I address is how does one find the compactification of A into an abelian surface? The idea of the direct proof we shall give here is closely related to the geometric spirit of the (real) Arnold-Liouville theorem [1, 2, 11]. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to an n-dimensional complex torus $\mathbb{C}^n/\mathit{Lattice}$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the main problem will be to complete A (3) into a non-singular compact complex algebraic variety $\widetilde{A} = A \cup \mathcal{D}$ in such a way that the vector fields X_{F_1} and X_{F_2} generated respectively by F_1 and F_2, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{A}$ is an algebraic complex torus (an abelian variety) and the coordinates $z_1, \ldots, z_5$ restricted to A are abelian functions. A naive guess would be to take the natural compactification $\overline{A}$ of A by projectivizing the equations: $\overline{A} = \bigcap_{k=1}^{3}\{F_k(Z) = c_k Z_0^4\} \subset \mathbb{P}^5$. Indeed, this can never work for a general reason: an abelian variety $\widetilde{A}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n$ by n-dim $\widetilde{A}$ global polynomial homogeneous equations. In other words, if A is to be the affine part of an abelian surface, $\overline{A}$ must have a singularity somewhere along the locus at infinity $\overline{A} \cap \{Z_0 = 0\}$. In fact, we shall show that the existence of meromorphic solutions to the differential equations (1) depending on 4 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

Proposition 2.1. *The system* (1) *possesses Laurent series solutions which depend on* 4 *free parameters:* α, β, γ *and* θ. *These meromorphic solutions restricted to the surface* $A(3)$ *are parameterized by two copies* $\mathcal{C}_{-1}$ *and* $\mathcal{C}_1$ *of the same Riemann surface* (5) *of genus* 7.

Proof. The first fact to observe is that if the system is to have Laurent solutions depending on 4 free parameters, the Laurent decomposition of such asymptotic solutions must have the following form

$$\begin{aligned}
z_1 &= \frac{1}{t}(z_1^{(0)} + z_1^{(1)}t + z_1^{(2)}t^2 + z_1^{(3)}t^3 + z_1^{(4)}t^4 + \cdots),\\
z_2 &= \frac{1}{t}(z_2^{(0)} + z_2^{(1)}t + z_2^{(2)}t^2 + z_2^{(3)}t^3 + z_2^{(4)}t^4 + \cdots),\\
z_3 &= \frac{1}{t^2}(-z_2^{(0)} + z_2^{(2)}t^2 + 2z_2^{(3)}t^3 + 3z_2^{(4)}t^4 + \cdots),\\
z_4 &= \frac{1}{2t^2}(-z_1^{(0)} + z_1^{(2)}t^2 + 2z_1^{(3t)}t^3 + 3z_1^{(4)}t^4 + \cdots),\\
z_5 &= \frac{1}{t^3}(z_5^{(0)} + z_5^{(1)}t + z_5^{(2)}t^2 + z_5^{(3)}t^3 + z_5^{(4)}t^4 + \cdots).
\end{aligned}$$

Putting these expansions into

$$\begin{aligned}
\ddot{z}_1 &= 2z_5 + 2z_1^2 + 8z_1z_2^2,\\
\ddot{z}_2 &= 3z_1z_2 + 8z_2^3,\\
\dot{z}_5 &= z_1\dot{z}_1 + 2z_2^2\dot{z}_1 - 2z_1z_2\dot{z}_2,
\end{aligned}$$

deduced from (1), solving inductively for the $z_k^{(j)}$ $(k = 1, 2, 5)$, one finds at the 0^{th} step (resp. 2^{th} step) a free parameter α (resp. β) and the two remaining ones γ, θ at the 4^{th} step. More precisely, we have

$$\begin{aligned}
z_1 &= \frac{1}{t}\alpha - \frac{1}{2}\alpha^2 + \beta t - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t^2 + \gamma t^3 + \cdots,\\
z_2 &= \frac{1}{2t}\varepsilon - \frac{1}{4}\varepsilon\alpha + \frac{1}{8}\varepsilon\alpha^2 t - \frac{1}{32}\varepsilon\left(-\alpha^3 + 12\beta\right)t^2 + \theta t^3 + \cdots,\\
z_3 &= -\frac{1}{2t^2}\varepsilon + \frac{1}{8}\varepsilon\alpha^2 - \frac{1}{16}\varepsilon\left(-\alpha^3 + 12\beta\right)t + 3\theta t^2 + \cdots,\\
z_4 &= -\frac{1}{2t^2}\alpha + \frac{1}{2}\beta - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t + \frac{3}{2}\gamma t^2 + \cdots,\\
z_5 &= \frac{1}{2t^2}\alpha^2 - \frac{1}{4t}\left(\alpha^3 + 4\beta\right) + \frac{1}{4}\alpha\left(\alpha^3 + 2\beta\right) - \left(\alpha^2\beta - 2\gamma + 4\varepsilon\theta\alpha\right)t + \cdots,
\end{aligned} \tag{4}$$

with $\varepsilon = \pm 1$. Using the majorant method, we can show that the formal Laurent series solutions are convergent. Substituting the solutions (4) into $F_1 = c_1$, $F_2 = c_2$

and $F_3 = c_3$, and equating the t^0-terms yields

$$\begin{aligned}
F_1 &= \frac{7}{64}\alpha^4 - \frac{1}{8}\alpha\beta - \frac{5}{2}\varepsilon\theta = c_1, \\
F_2 &= \frac{1}{16}\left(4\beta - \alpha^3\right)\left(4\alpha^2\beta - \alpha^5 + 64\varepsilon\theta\alpha - 32\gamma\right) = c_2, \\
F_3 &= -\frac{1}{32}\alpha^6 - \beta^2 - \frac{1}{4}\alpha^3\beta - 3\varepsilon\theta\alpha^2 + 4\alpha\gamma = c_3.
\end{aligned}$$

Eliminating γ and θ from these equations, leads to an equation connecting the two remaining parameters α and β:

$$\mathcal{C} : \Delta(\alpha, \beta) = 0, \tag{5}$$

where

$$\begin{aligned}
\Delta(\alpha, \beta) \equiv{}& 64\beta^3 - 16\alpha^3\beta^2 - 4\left(\alpha^6 - 32\alpha^2 c_1 - 16c_3\right)\beta \\
&+\alpha\left(32c_2 - 32\alpha^4 c_1 + \alpha^8 - 16\alpha^2 c_3\right).
\end{aligned}$$

The Laurent solutions restricted to the surface $A(3)$ are thus parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface $\mathcal{C}(5)$. We now compute the genus of $\mathcal{C}$. We have

$$\begin{aligned}
\Delta(\alpha, \beta) &= 64\beta^3 - 16\alpha^3\beta^2 - 4\alpha^6\beta - \alpha^9 + \text{lower order terms}, \\
&= \prod_{j=1}^{3}(\beta + a_j\alpha^3) + \text{lower order terms}.
\end{aligned}$$

Consider Δ as a cover with regard to α. In a neighbourhood of $\alpha = \infty$, we have $\beta = -a_j\alpha^3 + \cdots$, and

$$(\alpha)_\infty = -P - Q - R, \qquad (\beta)_\infty = -3P - 3Q - 3R.$$

Put $t = \frac{1}{\alpha}$, then

$$\Delta(\alpha, \beta) = \frac{1}{t^9}(64t^9\beta^3 - 16t^6\beta\beta^2 - 4t^3\beta - 97) + \cdots,$$

which suggests the following change of charts $(\alpha, \beta) \longmapsto (w = t^3\beta, t = \frac{1}{\alpha})$. Note that the function

$$\frac{\partial\Delta}{\partial\beta} = 192\frac{w^2}{t^6} - 32\frac{w}{t^6} - \frac{4}{t^6} + \cdots,$$

is meromorphic on $\mathcal{C}$. Then #zeroes of $\frac{\partial\Delta}{\partial\beta}$ = #poles of $\frac{\partial\Delta}{\partial\beta}$, and

$$(\frac{\partial\Delta}{\partial\beta})_P = -6P, \quad (\frac{\partial\Delta}{\partial\beta})_Q = -6Q, \quad (\frac{\partial\Delta}{\partial\beta})_R = -6R,$$

$$(\frac{\partial\Delta}{\partial\beta})_\infty = -6(P + Q + R).$$

Therefore, the number of zeroes of $\frac{\partial\Delta}{\partial\beta}$ in the affine part $\mathcal{C}\backslash\{P,Q,R\}$ is 18. According to the Riemann-Hurwitz formula, the genus of the Riemann surface $\mathcal{C}$ is 7, which establishes the proposition.

In order to embed $\mathcal{C}$ into some projective space, one of the key underlying principles used is the Kodaira embedding theorem (see Appendix A), which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N$ with the set of functions having a pole of order k along positive divisor on the manifold, provided k is large enough; fortunately, for abelian varieties, k need not be larger than three according to Lefschetz. These functions are easily constructed from the Laurent solutions (4) by looking for polynomials in the phase variables which in the expansions have at most a k-fold pole. The nature of the expansions and some algebraic properties of abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{f_0,\dots,f_N\}$, of increasing degree in the original variables $z_1,\dots,z_5$ having the property that the embedding $\mathcal{D}$ of $\mathcal{C}_1+\mathcal{C}_{-1}$ into $\mathbb{P}^N$ via those functions satisfies the relation (see Appendix A (16)): geometric genus $(\mathcal{D})\equiv g(\mathcal{D})=N+2$. A this point, it may be not so clear why $\mathcal{D}$ must really live on an abelian surface. Let us say, for the moment, that the equations of the divisor $\mathcal{D}$ (i.e., the place where the solutions blow up), as a Riemann surface traced on the abelian surface $\widetilde{A}$ (to be constructed in Proposition 2.3), must be understood as relations connecting the free parameters as they appear firstly in the expansions (4). In the present situation, this means that (5) must be understood as relations connecting α and β. Let

$$L^{(r)}=\left\{\begin{array}{l}\text{polynomials}\quad f=f(z_,\dots,z_5)\\ \text{of degree}\leq r,\quad\text{such that}\\ f(z(t))=t^{-1}(z^{(0)}+\cdots),\\ \text{with}\quad z^{(0)}\neq 0\quad\text{on}\quad\mathcal{D}\\ \text{and with}\quad z(t)\quad\text{as in}\quad(4)\end{array}\right\}/[F_k=c_k,k=1,2,3],$$

and let $(f_0,f_1,\dots,f_{N_r})$ be a basis of $L^{(r)}$. We look for r such that :

$$g(\mathcal{D}^{(r)})=N_r+2,\quad\mathcal{D}^{(r)}\subset\mathbb{P}^{N_r}.$$

We shall show (Proposition 2.2) that it is unnecessary to go beyond $r=4$.

Lemma 2.1. *The spaces $L^{(r)}$, nested according to weighted degree, are generated as follows*

$$\begin{aligned}L^{(1)}&=\{f_0,f_1,f_2\},\\ L^{(2)}&=L^{(1)}\oplus\{f_3,f_4,f_5,f_6\},\\ L^{(3)}&=L^{(2)}\oplus\{f_7,f_8,f_9,f_{10}\},\\ L^{(4)}&=L^{(3)}\oplus\{f_{12},f_{13},f_{14},f_{15}\},\end{aligned}\qquad(6)$$

where

$$
\begin{aligned}
f_0 &= 1, \\
f_1 &= z_1 = \frac{1}{t}\alpha + \cdots, \\
f_2 &= z_2 = \frac{1}{2t}\varepsilon + \cdots, \\
f_3 &= 2z_5 - z_1^2 = -\frac{1}{2}\frac{4\beta - \alpha^3}{t} + \cdots, \\
f_4 &= z_3 + 2\varepsilon z_2^2 = -\frac{1}{2t}\varepsilon\alpha + \cdots, \\
f_5 &= z_4 + \varepsilon z_1 z_2 = -\frac{1}{2t}\alpha^2 + \cdots, \\
f_6 &= [f_1, f_2] = \frac{1}{4}\varepsilon\frac{4\beta - \alpha^3}{t} + \cdots, \\
f_7 &= f_1 A = \frac{1}{2t}\alpha^3 + \cdots, \\
f_8 &= f_2 A = \frac{1}{4t}\varepsilon\alpha^2 + \cdots, \\
f_9 &= z_4 B = \frac{1}{8t}\alpha^3\left(-\alpha^3 + 4\beta\right) + \cdots, \\
f_{10} &= z_5 B = -\frac{1}{8t}\alpha^4\left(-\alpha^3 + 4\beta\right) + \cdots, \\
f_{11} &= f_5 A = -\frac{1}{4t}\alpha^4 + \cdots, \\
f_{12} &= f_1 f_2 B = -\frac{1}{8t}\alpha^3\varepsilon\left(-\alpha^3 + 4\beta\right) + \cdots, \\
f_{13} &= f_4 f_5 + [f_1, f_4] = \frac{3}{8}\alpha\varepsilon\frac{4\beta - \alpha^3}{t} + \cdots, \\
f_{14} &= [f_1, f_3] + 2\varepsilon\,[f_1, f_6] = \frac{1}{2}\alpha^3\frac{4\beta - \alpha^3}{t} + \cdots, \\
f_{15} &= f_3 - 2z_5 + 4f_4^2 = -\frac{\alpha^3}{t} + \cdots,
\end{aligned}
$$

with $[s_j, s_k] = \dot{s}_j s_k - s_j \dot{s}_k$, *the wronskien of* s_k *and* s_j, $A = f_1 + 2\varepsilon f_4$ *and* $B = f_3 + 2\varepsilon f_6$.

Proof. The proof of this lemma is straightforward and can be done by inspection of the expansions (4).

Proposition 2.2. $L^{(4)}$ *provides an embedding of* $\mathcal{D}^{(4)}$ *into projective space* $\mathbb{P}^{15}$ *and* $\mathcal{D}^{(4)}$ *has genus* 17.

Proof. It turns out that neither $L^{(1)}$, nor $L^{(2)}$, nor $L^{(3)}$, yield a Riemann surface of the right genus; in fact $g(\mathcal{D}^{(r)}) \neq \dim L^{(r)} + 1, \quad r = 1, 2, 3$. For instance, the embedding into $\mathbb{P}^2$ via $L^{(1)}$ does not separate the sheets, so we proceed to $L^{(2)}$ and the corresponding embedding into $\mathbb{P}^6$ is unacceptable since $g(\mathcal{D}^{(2)}) - 2 > 6$ and $\mathcal{D}^{(2)} \subset \mathbb{P}^6 \neq \mathbb{P}^{g-2}$, which contradicts the fact that $N_r = g(\mathcal{D}^{(2)}) - 2$. So we proceed to $L^{(3)}$ and we consider the corresponding embedding into $\mathbb{P}^{10}$, according to the functions $(f_0, \ldots, f_{10})$. For finite values of α and β, dividing the vector $(f_0, \ldots, f_{10})$ by f_2 and taking the limit $t \to 0$, to yield

$$[0 : 2\varepsilon\alpha : 1 : -\varepsilon(4\beta - \alpha^3) : -\alpha : -\varepsilon\alpha^2 : \frac{1}{2}(4\beta - \alpha^3) : \varepsilon\alpha^3 : \frac{1}{2}\alpha^2 :$$
$$\frac{1}{4}\varepsilon\alpha^3(4\beta - \alpha^3) : -\frac{1}{4}\varepsilon\alpha^4(4\beta - \alpha^3)].$$

The point $\alpha = 0$ requires special attention. Indeed near $\alpha = 0$, the parameter β behaves as follows: $\beta \sim 0, i\sqrt{c_3}, -i\sqrt{c_3}$. Thus near $(\alpha, \beta) = (0, 0)$, the corresponding point is mapped into the point $[0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ in $\mathbb{P}^{10}$ which is independent of $\varepsilon = \pm 1$, whereas near the point $(\alpha, \beta) = (0, i\sqrt{c_3})$ (resp. $(\alpha, \beta) = (0, -i\sqrt{c_3})$) leads to two different points: $[0 : 0 : 1 : -4\varepsilon i\sqrt{c_3} : 0 : 0 : 2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0]$ (resp. $[0 : 0 : 1 : 4\varepsilon i\sqrt{c_3} : 0 : 0 : -2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0]$), according to the sign of ε. The Riemann surface (5) has three points covering $\alpha = \infty$, at which β behaves as follows: $\beta \sim -\frac{1279}{216}\alpha^3$, $\frac{1}{432}\alpha^3\left(1333 - 1295i\sqrt{3}\right)$, $\frac{1}{432}\alpha^3\left(1333 + 1295i\sqrt{3}\right)$. Then by dividing the vector $(f_0, \ldots, f_{10})$ by f_{10}, the corresponding point is mapped into the point $[0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1]$ in $\mathbb{P}^{10}$. Thus, $g(\mathcal{D}^{(3)}) - 2 > 10$ and $\mathcal{D}^{(2)} \subset \mathbb{P}^{10} \neq \mathbb{P}^{g-2}$, which contradicts the fact that $N_r = g(\mathcal{D}^{(3)}) - 2$. Consider now the embedding $\mathcal{D}^{(4)}$ into $\mathbb{P}^{15}$ using the 16 functions $f_0, \ldots, f_{15}$ of $L^{(4)}(6)$. It is easily seen that these functions separate all points of the Riemann surface (except perhaps for the points at $\alpha = \infty$ and $\alpha = \beta = 0$): The Riemann surfaces $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ are disjoint for finite values of α and β except for $\alpha = \beta = 0$; dividing the vector $(f_0, \ldots, f_{15})$ by f_2 and taking the limit $t \to 0$, to yield

$$[0 : 2\varepsilon\alpha : 1 : -\varepsilon(4\beta - \alpha^3) : -\alpha : -\varepsilon\alpha^2 : \frac{1}{2}(4\beta - \alpha^3) : \varepsilon\alpha^3 : \frac{1}{2}\alpha^2 :$$
$$\frac{1}{4}\varepsilon\alpha^3(4\beta - \alpha^3) : -\frac{1}{4}\varepsilon\alpha^4(4\beta - \alpha^3) : -\frac{1}{2}\varepsilon\alpha^4 : -\frac{1}{4}\alpha^3(4\beta - \alpha^3) :$$
$$\frac{3}{4}\alpha\left(4\beta - \alpha^3\right) : \varepsilon\alpha^3\left(4\beta - \alpha^3\right) : -2\varepsilon\alpha^3].$$

As before, the point $\alpha = 0$ requires special attention and the parameter β behaves as follows: $\beta \sim 0, i\sqrt{c_3}, -i\sqrt{c_3}$. Thus near $(\alpha, \beta) = (0, 0)$, the corresponding point is mapped into the point $[0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ in $\mathbb{P}^{15}$ which is independent of $\varepsilon = \pm 1$, whereas near the point $(\alpha, \beta) = (0, i\sqrt{c_3})$ (resp. $(\alpha, \beta) = (0, -i\sqrt{c_3})$) leads to two different points : $[0 : 0 : 1 : -4\varepsilon i\sqrt{c_3} : 0 : 0 : 2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ (resp. $[0 : 0 : 1 : 4\varepsilon i\sqrt{c_3} : 0 : 0 : -2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$), according to the sign of ε. About the point $\alpha = \infty$, it is appropriate to divide by f_{10}; then the corresponding point is mapped into the point $[0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0]$, in $\mathbb{P}^{15}$ which is independent of ε. Hence from formula (Appendix A,(19)), the divisor $\mathcal{D}^{(4)}$ obtained in this way has genus 17 and $\mathcal{D}^{(4)} \subset \mathbb{P}^{15} = \mathbb{P}^{g-2}$, as desired (i.e., satisfying the requirement Appendix A, (16)). This ends the proof of the proposition.

Let $\mathcal{L} = L^{(4)}$ and $\mathcal{D} = \mathcal{D}^{(4)}$. Next we wish to construct a surface strip around $\mathcal{D}$ which will support the commuting vector fields. In fact, $\mathcal{D}$ has a good chance to be very ample divisor on an abelian surface, still to be constructed.

Proposition 2.3. *The variety* $A(3)$ *generically is the affine part of an abelian surface* $\widetilde{A}$*. The reduced divisor at infinity* $\widetilde{A}\backslash A = \mathcal{C}_1 + \mathcal{C}_{-1}$*, consists of two copies* $\mathcal{C}_1$ *and* $\mathcal{C}_{-1}$ *of the same genus* 7 *Riemann surface* $\mathcal{C}(5)$*. The system of differential equations* (1) *is algebraic completely integrable and the corresponding flows evolve on* $\widetilde{A}$*.*

Proof. We need to attach the affine part of the intersection of the three invariants (2) so as to obtain a smooth compact connected surface in $\mathbb{P}^{15}$. To be precise, the orbits of the vector field (1) running through $\mathcal{D}$ form a smooth surface Σ near $\mathcal{D}$ such that $\Sigma\backslash A \subseteq \widetilde{A}$ and the variety $\widetilde{A} = A \cup \Sigma$ is smooth, compact and connected. Indeed, let $\psi(t,p) = \{z(t) = (z_1(t), \ldots, z_5(t)) : t \in \mathbb{C}, 0 < |t| < \varepsilon\}$, be the orbit of the vector field (1) going through the point $p \in A$. Let $\Sigma_p \subset \mathbb{P}^{15}$ be the surface element formed by the divisor $\mathcal{D}$ and the orbits going through p, and set $\Sigma \equiv \cup_{p\in\mathcal{D}}\Sigma_p$. Consider the Riemann surface $\mathcal{D}' = \mathcal{H} \cap \Sigma$ where $\mathcal{H} \subset \mathbb{P}^{15}$ is a hyperplane transversal to the direction of the flow. If $\mathcal{D}'$ is smooth, then using the implicit function theorem the surface Σ is smooth. But if $\mathcal{D}'$ is singular at 0, then Σ would be singular along the trajectory (t-axis) which goes immediately into the affine part A. Hence, A would be singular which is a contradiction because A is the fibre of a morphism from $\mathbb{C}^5$ to $\mathbb{C}^3$ and so smooth for almost all the three constants of the motion c_k. Next, let $\overline{A}$ be the projective closure of A into $\mathbb{P}^5$, let $Z = [Z_0 : Z_1 : \ldots : Z_5] \in \mathbb{P}^5$ and let $I = \overline{A} \cap \{Z_0 = 0\}$ be the locus at infinity. Consider the map $\overline{A} \subseteq \mathbb{P}^5 \to \mathbb{P}^{15}$, $Z \mapsto f(Z)$, where $f = (f_0, f_1, \ldots, f_{15}) \in \mathcal{L}(\mathcal{D})$ and let $\widetilde{A} = f(\overline{A})$. In a neighbourhood $V(p) \subseteq \mathbb{P}^{15}$ of p, we have $\Sigma_p = \widetilde{A}$ and $\Sigma_p\backslash\mathcal{D} \subseteq A$. Otherwise there would exist an element of surface $\Sigma'_p \subseteq \widetilde{A}$ such that $\Sigma_p \cap \Sigma'_p = (t\text{-}axis)$, orbit $\psi(t,p) = (t\text{-}axis)\backslash\, p \subseteq A$, and hence A would be singular along the t-axis which is impossible. Since the variety $\overline{A} \cap \{Z_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{A}$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{P}^5 \times \mathbb{P}^{15}$ of the map f, which is irreducible together with $\overline{A}$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{gen.} \times \mathbb{P}^{15}\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}\}$ is connected and therefore the projection map $proj_{\mathbb{P}^{15}}\{\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}\}\} = f(I) \equiv \mathcal{D}$, is connected. Hence, the variety $A \cup \Sigma = \widetilde{A}$ is compact, connected and embeds smoothly into $\mathbb{P}^{15}$ via f. We wish to show that $\widetilde{A}$ is an abelian surface equipped with two everywhere independent commuting vector fields. For doing that, let ϕ^{τ_1} and ϕ^{τ_2} be the flows corresponding to vector fields X_{F_1} and X_{F_2}. The latter are generated respectively by F_1 and F_2. For $p \in \mathcal{D}$ and for small $\varepsilon > 0$, $\phi^{\tau_1}(p), \forall\tau_1, 0 < |\tau_1| < \varepsilon$, is well defined and $\phi^{\tau_1}(p) \in \widetilde{A}\backslash A$. Then we may define ϕ^{τ_2} on $\widetilde{A}$ by $\phi^{\tau_2}(q) =$

$\phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q), q \in U(p) = \phi^{-\tau_1}(U(\phi^{\tau_1}(p)))$, where $U(p)$ is a neighbourhood of p. By commutativity one can see that ϕ^{τ_2} is independent of τ_1; $\phi^{-\tau_1-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1+\varepsilon_1}(q) = \phi^{-\tau_1}\phi^{-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1}\phi^{\varepsilon_1} = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q)$. We affirm that $\phi^{\tau_2}(q)$ is holomorphic away from $\mathcal{D}$. This because $\phi^{\tau_2}\phi^{\tau_1}(q)$ is holomorphic away from $\mathcal{D}$ and that ϕ^{τ_1} is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U(\phi^{\tau_1}(p))$. Now, since the flows ϕ^{τ_1} and ϕ^{τ_2} are holomorphic and independent on $\mathcal{D}$, we can show along the same lines as in the Arnold-Liouville theorem [1,2,11] that $\widetilde{A}$ is a complex torus $\mathbb{C}^2/lattice$ and so in particular $\widetilde{A}$ is a Kähler variety. And that will be done, by considering the local diffeomorphism $\mathbb{C}^2 \to \widetilde{A}, (\tau_1, \tau_2) \mapsto \phi^{\tau_1}\phi^{\tau_2}(p)$, for a fixed origin $p \in A$. The additive subgroup $\{(\tau_1, \tau_2) \in \mathbb{C}^2 : \phi^{\tau_1}\phi^{\tau_2}(p) = p\}$ is a lattice of $\mathbb{C}^2$, hence $\mathbb{C}^2/lattice \to \widetilde{A}$ is a biholomorphic diffeomorphism and $\widetilde{A}$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$. As mentioned in appendix A, a compact complex Kähler variety having the required number as (its dimension) of independent meromorphic functions is a projective variety. In fact, here we have $\widetilde{A} \subseteq \mathbb{P}^{15}$. Thus $\widetilde{A}$ is both a projective variety and a complex torus $\mathbb{C}^2/lattice$ and hence an abelian surface as a consequence of Chow theorem. This completes the proof of the proposition.

Remark 2.1. Note that the reflection σ on the affine variety A amounts to the flip $\sigma : (z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, -z_2, z_3, -z_4, z_5)$, changing the direction of the commuting vector fields. It can be extended to the $(-Id)$-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1, t_2) \mapsto (-t_1, -t_2)$ on $\widetilde{A}$, where t_1 and t_2 are the time coordinates of each of the flows X_{F_1} and X_{F_2}. The involution σ acts on the parameters of the Laurent solution (4) as follows $\sigma : (t, \alpha, \beta, \gamma, \theta) \longmapsto (-t, -\alpha, -\beta, -\gamma, \theta)$, interchanges the Riemann surfaces $\mathcal{C}_\varepsilon$ and the linear space $\mathcal{L}$ can be split into a direct sum of even and odd functions. Geometrically, this involution interchanges $\mathcal{C}_1$ and $\mathcal{C}_{-1}$, i.e., $\mathcal{C}_{-1} = \sigma\mathcal{C}_1$.

Remark 2.2. Consider on $\widetilde{A}$ the holomorphic 1-forms dt_1 and dt_2 defined by $dt_i(X_{F_j}) = \delta_{ij}$, where X_{F_1} and X_{F_2} are the vector fields generated respectively by F_1 and F_2. Taking the differentials of $\zeta = 1/z_1$ and $\xi = z_1/z_2$ viewed as functions of t_1 and t_2, using the vector fields and the Laurent series (4) and solving linearly for dt_1 and dt_2, we obtain the holomorphic differentials

$$\begin{aligned}
\omega_1 &= dt_1|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}(\frac{\partial \xi}{\partial t_2}d\zeta - \frac{\partial \zeta}{\partial t_2}d\xi)|_{\mathcal{C}_\varepsilon} = \frac{8}{\alpha\left(-4\beta + \alpha^3\right)}d\alpha, \\
\omega_2 &= dt_2|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}(\frac{-\partial \xi}{\partial t_1}d\zeta - \frac{\partial \zeta}{\partial t_1}d\xi)|_{\mathcal{C}_\varepsilon} = \frac{2}{\left(-4\beta + \alpha^3\right)^2}d\alpha,
\end{aligned}$$

with $\Delta \equiv \frac{\partial\zeta}{\partial t_1}\frac{\partial\xi}{\partial t_2} - \frac{\partial\zeta}{\partial t_2}\frac{\partial\xi}{\partial t_1}$. The zeroes of ω_2 provide the points of tangency of the vector field X_{F_1} to $\mathcal{C}_\varepsilon$. We have $\frac{\omega_1}{\omega_2} = \frac{4}{\alpha}\left(-4\beta + \alpha^3\right)$, and X_{F_1} is tangent to $\mathcal{H}_\varepsilon$ at the point covering $\alpha = \infty$.

3. A four-dimensional generalized algebraic complete integrable system and cyclic covering of abelian surface

Consider the case $F_3 = 0$, and the following change of variables

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = p_1 q_1, \quad z_5 = p_1^2 - q_1^2 q_2^2.$$

Substituting this into the constants of motion F_1, F_2, F_3 leads obviously to the relations

$$\begin{aligned} H_1 &= \frac{1}{2}p_1^2 - \frac{3}{2}q_1^2 q_2^2 + \frac{1}{2}p_2^2 - \frac{1}{4}q_1^4 - 2q_2^4, \\ H_2 &= p_1^4 - 6q_1^2 q_2^2 p_1^2 + q_1^4 q_2^4 - q_1^4 p_1^2 + q_1^6 q_2^2 + 4q_1^3 q_2 p_1 p_2 - q_1^4 p_2^2 + \frac{1}{4}q_1^8, \end{aligned} \tag{7}$$

whereas the last constant leads to an identity. Using the differential equations (1) combined with the transformation above leads to the system of differential equations

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= p_2, \\ \dot{p}_1 &= q_1 \left(q_1^2 + 3q_2^2\right), \\ \dot{p}_1 &= q_2 \left(3q_1^2 + 8q_2^2\right). \end{aligned} \tag{8}$$

The last equation (1) for z_5 leads to an identity. Thus, we obtain the potential constructed by Ramani, Dorozzi and Grammaticos [16,5]. Evidently, the functions H_1 and H_2 commute :

$$\{H_1, H_2\} = \sum_{k=1}^{2} \left(\frac{\partial H_1}{\partial p_k}\frac{\partial H_2}{\partial q_k} - \frac{\partial H_1}{\partial q_k}\frac{\partial H_2}{\partial p_k} \right) = 0.$$

The system (8) is weight-homogeneous[1] with q_1, q_2 having weight 1 and p_1, p_2 weight 2, so that H_1 and H_2 have weight 4 and 8 respectively. When one examines all possible singularities, one finds that it is possible for the variable q_1 to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the Painlevé test. However, these terms are trivially removed by introducing the variables $z_1, \ldots, z_5$ which restores the Painlevé property to the system. Let B be the affine variety defined by

$$B = \bigcap_{k=1}^{2} \{z \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{9}$$

where $(b_1, b_2) \in \mathbb{C}^2$.

Proposition 3.1. *The system* (8) *admits Laurent solutions in* $t^{1/2}$, *depending on* 3 *free parameters:* u, v *and* w. *These solutions restricted to the surface* $B(9)$ *are parameterized by two copies* $-_1$ *and* $-_{-1}$ *of the same Riemann surface of genus* 16.

[1] Recall that a system $\dot{z} = f(z)$ is weight-homogeneous with a weight ν_k going with each variable z_k if $f_k(\lambda^{\nu_i} z_1, \ldots, \lambda^{\nu_n} z_n) = \lambda^{\nu_k + 1} f_k(z_1, \ldots, z_n)$, for all $\lambda \in \mathbb{C}$.

Proof. The system (8) possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters u, v and w. There are precisely two such families, labeled by $\varepsilon = \pm 1$, and they are explicitly given as follows

$$\begin{aligned}
q_1 &= \frac{1}{\sqrt{t}}(u - \frac{1}{4}u^3 t + vt^2 - \frac{5}{128}u^7 t^3 + \frac{1}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots), \\
q_2 &= \frac{1}{t}(\frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon u^2 t + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{4}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + wt^4 + \cdots), \qquad (10) \\
p_1 &= \frac{1}{2t\sqrt{t}}(-u - \frac{1}{4}u^3 t + 3vt^2 - \frac{25}{128}t^3 u^7 + \\
&\qquad \frac{7}{8}u(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w)t^4 + \cdots), \\
p_2 &= \frac{1}{t^2}(-\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{2}\varepsilon u(\frac{1}{32}u^5 - 3v)t^3 + 3wt^4 + \cdots).
\end{aligned}$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter w linearly, leading to an equation connecting the two remaining parameters u and v :

$$\begin{aligned}
\Gamma : \quad & \frac{65}{4}uv^3 + \frac{93}{64}u^6 v^2 + \frac{3}{8192}\left(-9829u^8 + 26112H_1\right)u^3 v \qquad (11) \\
& -\frac{10299}{65536}u^{16} - \frac{123}{256}H_1 u^8 + H_2 + \frac{15362\,98731}{52} = 0.
\end{aligned}$$

According to Hurwitz' formula, this defines a Riemann surface Γ of genus 16. The Laurent solutions restricted to the surface $B(9)$ are thus parameterized by two copies Γ_{-1} and Γ_1 of the same Riemann surface Γ. This ends the proof of the proposition.

Remark 3.1. The asymptotic solution (10) can be read off from (4) and the change of variables: $q_1 = \sqrt{z_1}, q_2 = z_2, p_1 = z_4/q_1, p_2 = z_3$. The function z_1 has a simple pole along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$ and a double zero along a Riemann surface of genus 7 defining a double cover of $\widetilde{A}$ ramified along $\mathcal{C}_1 + \mathcal{C}_{-1}$.

Applying the method explained in Piovan [15], we have the

Proposition 3.2. *The invariant surface $B(9)$ can be completed as a cyclic double cover $\overline{B}$ of the abelian surface $\widetilde{A}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. The system (8) is algebraic complete integrable in the generalized sense. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{B}) = 1$ and $p_g(\widetilde{B}) = 2$.*

Proof. The morphism $\varphi : B \longrightarrow A, \quad (q_1, q_2, p_1, p_2) \longmapsto (z_1, z_2, z_3, z_4, z_5)$, maps the vector field (8) into an algebraic completely integrable system (1) in five unknowns and the affine variety $B(9)$ onto the affine part $A(3)$ of an abelian variety $\widetilde{A}$ with $\widetilde{A}\backslash A = \mathcal{C}_1 + \mathcal{C}_{-1}$. Observe that φ is an unramified cover. The Riemann surface Γ(11) plays an important role in the construction of a compactification $\overline{B}$ of B. Let us denote by G a cyclic group of two elements $\{-1, 1\}$ on $V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\}$, where $\tau = t^{1/2}$ and U_ε^j is an affine chart of Γ_ε for which the Laurent solutions (10) are defined. The action of G is defined by $(-1) \circ (u, v, \tau) = (-u, -v, -\tau)$ and is without fixed points in V_ε^j. So we can identify the quotient V_ε^j/G with the image of the smooth map $h_\varepsilon^j : V_\varepsilon^j \to B$ defined by the expansions (10). We have $(-1, 1).(u, v, \tau) = (-u, -v, \tau)$ and $(1, -1).(u, v, \tau) = (u, v, -\tau)$, i.e., $G \times G$ acts separately on each coordinate. Thus, identifying V_ε^j/G^2 with the image of $\varphi \circ h_\varepsilon^j$ in A. Note that $B_\varepsilon^j = V_\varepsilon^j/G$ is smooth (except for a finite number of points) and the coherence of the B_ε^j follows from the coherence of V_ε^j and the action of G. Now by taking B and by gluing on various varieties $B_\varepsilon^j\backslash\{\text{some points}\}$, we obtain a smooth complex manifold $\widehat{B}$ which is a double cover of the abelian variety $\widetilde{A}$ (constructed in Proposition 2.3) ramified along $\mathcal{C}_1 + \mathcal{C}_{-1}$, and therefore can be completed to an algebraic cyclic cover of $\widetilde{A}$. To see what happens to the missing points, we must investigate the image of $\Gamma \times \{0\}$ in $\cup B_\varepsilon^j$. The quotient $\Gamma \times \{0\}/G$ is birationally equivalent to the Riemann surface Υ of genus 7:

$$\Upsilon : \frac{65}{4}y^3 + \frac{93}{64}x^3y^2 + \frac{3}{8192}\left(-9829x^4 + 26112b_1\right)x^2y$$
$$+x\left(-\frac{10299}{65536}x^8 - \frac{123}{256}b_1x^4 + b_2 + \frac{15362\,98731}{52}\right) = 0,$$

where $y = uv, x = u^2$. The Riemann surface Υ is birationally equivalent to $\mathcal{C}$. The only points of Υ fixed under $(u, v) \mapsto (-u, -v)$ are the points at ∞, which correspond to the ramification points of the map $\Gamma \times \{0\} \overset{2-1}{\to} \Upsilon : (u, v) \mapsto (x, y)$ and coincide with the points at ∞ of the Riemann surface $\mathcal{C}$. Then the variety $\widehat{B}$ constructed above is birationally equivalent to the compactification $\overline{B}$ of the generic invariant surface B. So $\overline{B}$ is a cyclic double cover of the abelian surface $\widetilde{A}$ ramified along the divisor $\mathcal{C}_1+\mathcal{C}_{-1}$, where $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ have two points in common at which they are tangent to each other. It follows that the system (8) is algebraic completely integrable in the generalized sense. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type A_3) of $\mathcal{C}_1 + \mathcal{C}_{-1}$. In term of an appropriate local holomorphic coordinate system (X, Y, Z), the local analytic equation about this singularity is $X^4+Y^2+Z^2 = 0$. Now, let $\widetilde{B}$ be the resolution of singularities of $\overline{B}$, $\mathcal{X}(\widetilde{B})$ be the Euler characteristic of $\widetilde{B}$ and $p_g(\widetilde{B})$ the geometric genus of $\widetilde{B}$. Then $\widetilde{B}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{B}) = 1$ and $p_g(\widetilde{B}) = 2$. This concludes the proof of the proposition.

A. Appendix

In this appendix we recall some results about abelian surfaces which will be used in this paper (details can be found in [6,7]), as well as the basic techniques to study two-dimensional algebraic completely integrable systems (see Appendix B). Let

$M = \mathbb{C}/\Lambda$ be an n-dimensional abelian variety where Λ is the lattice generated by the $2n$ columns $\lambda_1, \ldots, \lambda_{2n}$ of the $n \times 2n$ period matrix Ω and let D be a divisor on M. Define $\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } M : (f) \geq -\mathcal{D}\}$, i.e., for $\mathcal{D} = \sum k_j \mathcal{D}_j$ a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a k_j-fold pole along $\mathcal{D}_j$. The divisor $\mathcal{D}$ is called ample when a basis $(f_0, \ldots, f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds M smoothly into $\mathbb{P}^N$ for some k, via the map $M \to \mathbb{P}^N$, $p \mapsto [1 : f_1(p) : \cdots : f_N(p)]$, then $k\mathcal{D}$ is called very ample. It is known that every positive divisor $\mathcal{D}$ on an irreducible abelian variety is ample and thus some multiple of $\mathcal{D}$ embeds M into $\mathbb{P}^N$. By a theorem of Lefschetz, any $k \geq 3$ will work. Moreover, there exists a complex basis of $\mathbb{C}^n$ such that the lattice expressed in that basis is generated by the columns of the $n \times 2n$ period matrix

$$\left(\begin{array}{ccc|c} \delta_1 & & 0 & \\ & \ddots & & Z \\ 0 & & \delta_n & \end{array}\right),$$

with $Z^\top = Z, \operatorname{Im} Z > 0, \delta_j \in \mathbb{N}^*$ and $\delta_j | \delta_{j+1}$. The integers δ_j which provide the so-called polarization of the abelian variety M are then related to the divisor as follows:

$$\dim \mathcal{L}(\mathcal{D}) = \delta_1 \ldots \delta_n. \tag{12}$$

In the case of a 2-dimensional abelian varieties (surfaces), even more can be stated: the geometric genus g of a positive divisor $\mathcal{D}$ (containing possibly one or several curves) on a surface M is given by the adjunction formula

$$g(\mathcal{D}) = \frac{K_M.\mathcal{D} + \mathcal{D}.\mathcal{D}}{2} + 1, \tag{13}$$

where K_M is the canonical divisor on M, i.e., the zero-locus of a holomorphic 2-form, $\mathcal{D}.\mathcal{D}$ denote the number of intersection points of $\mathcal{D}$ with $a + \mathcal{D}$ (where $a + \mathcal{D}$ is a small translation by a of $\mathcal{D}$ on M), whereas the Riemann-Roch theorem for line bundles on a surface tells you that

$$\chi(\mathcal{D}) = p_a(M) + 1 + \frac{1}{2}(\mathcal{D}.\mathcal{D} - \mathcal{D}K_M), \tag{14}$$

where $p_a(M)$ is the arithmetic genus of M and $\chi(\mathcal{D})$ the Euler characteristic of $\mathcal{D}$. To study abelian surfaces using Riemann surfaces on these surfaces, we recall that

$$\begin{aligned} \chi(\mathcal{D}) &= \dim H^0(M, \mathcal{O}_M(\mathcal{D})) - \dim H^1(M, \mathcal{O}_M(D)), \\ &= \dim \mathcal{L}(\mathcal{D}) - \dim H^1(M, \Omega^2(\mathcal{D} \otimes K_M^*)), \text{ (Kodaira-Serre duality)}, \\ &= \dim \mathcal{L}(\mathcal{D}), \text{ (Kodaira vanishing theorem)}, \end{aligned} \tag{15}$$

whenever $\mathcal{D} \otimes K_M^*$ defines a positive line bundle. However for abelian surfaces, K_M is trivial and $p_a(M) = -1$; therefore combining relations (12), (13), (14) and (15),

$$\chi(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}) = \frac{\mathcal{D}.\mathcal{D}}{2} = g(\mathcal{D}) - 1 = \delta_1 \delta_2. \tag{16}$$

A divisor $\mathcal{D}$ is called projectively normal, when the natural map $\mathcal{L}(\mathcal{D})^{\otimes k} \to \mathcal{L}(k\mathcal{D})$, is surjective, i.e., every function of $\mathcal{L}(k\mathcal{D})$ can be written as a linear combination of k-fold products of functions of $\mathcal{L}(\mathcal{D})$. Not every very ample divisor $\mathcal{D}$ is projectively normal but if $\mathcal{D}$ is linearly equivalent to $k\mathcal{D}_0$ for $k \geq 3$ for some divisor $\mathcal{D}_0$, then $\mathcal{D}$ is projectively normal [14,8].
Now consider the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\pi^*} \mathcal{O}_{\widetilde{C}} \longrightarrow X \longrightarrow 0,$$

where C is a singular connected Riemann surface, $\widetilde{C} = \sum C_j$ the corresponding set of smooth Riemann surfaces after desingularization and $\pi : \widetilde{C} \to C$ the projection. The exactness of the sheaf sequence shows that the Euler characteristic

$$\mathcal{X}(\mathcal{O}) = \dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}), \tag{17}$$

satisfies

$$\mathcal{X}(\mathcal{O}_C) - \mathcal{X}(\mathcal{O}_{\widetilde{C}}) + \mathcal{X}(X) = 0, \tag{18}$$

where $\mathcal{X}(X)$ only accounts for the singular points p of C; $\mathcal{X}(X_p)$ is the dimension of the set of holomorphic functions on the different branches around p taken separately, modulo the holomorphic functions on the Riemann surface C near that singular point. Consider the case of a planar singularity (in this paper, we will be concerned by a tacnode for which $\mathcal{X}(X) = 2$, as well), i.e., the tangents to the branches lie in a plane. If $f_j(x, y) = 0$ denotes the j^{th} branch of C running through p with local parameter s_j, then

$$\mathcal{X}(X_p) = \dim \Pi_j \mathbb{C}[[s_j]] / \frac{\mathbb{C}[[x, y]]}{\Pi_j f_j(x, y)}.$$

So using (15) and Serre duality, we obtain $\mathcal{X}(\mathcal{O}_C) = 1 - g(C)$ and $\mathcal{X}(\mathcal{O}_{\widetilde{C}}) = n - \sum_{j=1}^n g(C_j)$. Also, replacing in the formula (18), gives

$$g(C) = \sum_{j=1}^{n} g(C_j) + \mathcal{X}(X) + 1 - n. \tag{19}$$

Finally, recall that a Kähler variety is a variety with a Kähler metric, i.e., a hermitian metric whose associated differential 2-form of type $(1,1)$ is closed. The complex torus $\mathbb{C}^2/lattice$ with the euclidean metric $\sum dz_i \otimes d\bar{z}_i$ is a Kähler variety and any compact complex variety that can be embedded in projective space is also a Kähler variety. Now, a compact complex Kähler variety having as many independent meromorphic functions as its dimension is a projective variety [13].

B. Appendix

In this appendix we give some basic facts about integrable Hamiltonian systems. Let M be a $2n$-dimensional differentiable manifold and ω a closed non-degenerate differential 2-form. The pair (M, ω) is called a symplectic manifold. Let $H : M \to$

$\mathbb{R}$ be a smooth function. A Hamiltonian system on (M, ω) with Hamiltonian H can be written in the form

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \dots, \dot{q}_n = \frac{\partial H}{\partial p_n}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \dots, \dot{p}_n = -\frac{\partial H}{\partial q_n}, \tag{20}$$

where $(q_1, \dots, q_n, p_1, \dots, p_n)$ are coordinates in M. Thus the Hamiltonian vector field X_H is defined by $X_H = \sum_{k=1}^n \left(\frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} \right)$. If F is a smooth function on the manifold M, the Poisson bracket $\{F, H\}$ of F and H is defined by

$$X_H F = \sum_{k=1}^n \left(\frac{\partial H}{\partial p_k} \frac{\partial F}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial F}{\partial p_k} \right) = \{F, H\}. \tag{21}$$

A function F is an invariant (first integral) of the Hamiltonian system (20) if and only if the Lie derivative of F with respect X_H is identically zero. The functions F and H are said to be in involution or to commute, if $\{F, H\} = 0$. Note that equations (20) and (21) can be written in more compact form

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (q_1, \dots, q_n, p_1, \dots, p_n)^\top,$$

$$\{F, H\} = \left\langle \frac{\partial F}{\partial x}, J \frac{\partial H}{\partial x} \right\rangle = \sum_{k,l=1}^n J_{kl} \frac{\partial F}{\partial x_k} \frac{\partial H}{\partial x_l},$$

with $J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$, a skew-symmetric matrix where I is the $n \times n$ unit matrix and O the $n \times n$ zero matrix. A Hamiltonian system is completely integrable in the sense of Liouville if there exist n invariants $H_1 = H, H_2, \dots, H_n$ in involution (i.e., such that the associated Poisson brackets $\{H_k, H_l\}$ all vanish) with linearly independent gradients (i.e., $dH_1 \wedge \dots \wedge dH_n \neq 0$). For generic $(c_1, \dots, c_n)$ the level set $V = \{H_1 = c_1, \dots, H_n = c_n\}$ will be an n-manifold, and since $X_{H_k} H_l = \{H_k, H_l\} = 0$, the integral curves of each X_{H_k} will lie in V and the vector fields X_{H_k} span the tangent space of V. By a theorem of Arnold [2,11], if V is compact and connected, it is diffeomorphic to an n-dimensional torus $\mathbb{R}^n / \mathbb{Z}^n$ and each vector field will define a linear flow there. To be precise, in some open neighbourhood of the torus one can introduce regular symplectic coordinates $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$ in which ω takes the canonical form $\omega = \sum_{k=1}^n ds_k \wedge d\varphi_k$. Here the functions s_k (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the first integrals H_k. The functions φ_k (called angle-variables) give standard angular coordinates on the torus, and every vector field X_{H_k} can be written in the form $\dot{\varphi}_k = h_k(s_1, \dots, s_n)$, that is, its integral trajectories define a conditionally-periodic motion on the torus. Consequently, in a neighbourhood of the torus the Hamiltonian vector field X_{H_k} takes the following form $\dot{s_k} = 0, \dot{\varphi}_k = h_k(s_1, \dots, s_n)$, and can be solved by quadratures.

Consider now Hamiltonian problems of the form

$$X_H : \dot{x} = J \frac{\partial H}{\partial x}, \; x \in \mathbb{R}^m, \tag{22}$$

where H is the Hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket $\{H_i, H_j\} = \langle \frac{\partial H_i}{\partial x}, J\frac{\partial H_j}{\partial x}\rangle$, satisfies the Jacobi identities. The system (22) with polynomial right hand side will be called algebraic completely integrable (a.c.i.) in the sense of Adler-van Moerbeke [1] when:

i) The system possesses $n+k$ independent polynomial invariants $H_1, \ldots, H_{n+k}$ of which k lead to zero vector fields $J\frac{\partial H_{n+i}}{\partial x}(x) = 0$, $1 \leq i \leq k$, the n remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant varieties $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$ are assumed compact and connected. Then, according to the Arnold-Liouville theorem, there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\} \to \mathbb{R}^n/Lattice,$$

and the solutions of the system (22) are straight lines motions on these tori.

ii) The invariant varieties, thought of as affine varieties in $\mathbb{C}^m$ can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k} \{H_i = c_i, x \in \mathbb{C}^m\} \cup \mathcal{D} = \mathbb{C}^n/Lattice,$$

where $\mathbb{C}^n/Lattice$ is a complex algebraic torus (i.e., abelian variety) and $\mathcal{D}$ a divisor. Algebraic means that the torus can be defined as an intersection $\bigcap_{i=1}^{M} \{P_i(X_0, \ldots, X_N) = 0\}$ involving a large number of homogeneous polynomials P_i. In the natural coordinates $(t_1, \ldots, t_n)$ of $\mathbb{C}^n/Lattice$ coming from $\mathbb{C}^n$, the functions $x_i = x_i(t_1, \ldots, t_n)$ are meromorphic and (22) defines straight line motion on $\mathbb{C}^n/Lattice$. Condition i) means, in particular, there is an algebraic map $(x_1(t), \ldots, x_m(t)) \mapsto (\mu_1(t), \ldots, \mu_n(t))$ making the following sums linear in t:

$$\sum_{i=1}^{n} \int_{\mu_i(0)}^{\mu_i(t)} \omega_j = d_j t \ , \ 1 \leq j \leq n, \ d_j \in \mathbb{C},$$

where $\omega_1, \ldots, \omega_n$ denote holomorphic differentials on some algebraic curves.

Adler and van Moerbeke [1] have shown that the existence of a coherent set of Laurent solutions :

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} t^{j-k_i}, \quad k_i \in \mathbb{Z}, \qquad \text{some } k_i > 0,$$

depending on dim(phase space) $- 1 = m - 1$ free parameters is necessary and sufficient for a Hamiltonian system with the right number of constants of motion to be a.c.i. So, if the Hamiltonian flow (22) is a.c.i., it means that the variables x_i are meromorphic on the torus $\mathbb{C}^n/Lattice$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset \mathbb{C}^n/Lattice$. By the a.c.i. definition, the flow (22) is a straight line motion in $\mathbb{C}^n/Lattice$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus $\mathbb{C}^n/Lattice$, the total count is therefore $m-1 = dim(\text{phase space}) - 1$ parameters.

Next we assume that the divisor is very ample and in addition projectively normal. Consider a point $p \in \mathcal{D}$, a chart U_j around p on the torus and a function y_j in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at p. Then the vector $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$ provides a good system of coordinates in U_j. Then taking the derivative with regard to one of the flows

$$(\frac{y_i}{y_j})^{\cdot} = \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2}, \quad 1 \leq j \leq N,$$

are finite on U_j as well. Therefore, since y_j^2 has a double pole along $\mathcal{D}$, the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i \dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when $\mathcal{D}$ is projectively normal, we have that

$$(\frac{y_i}{y_j})^{\cdot} = \sum_{k,l} a_{k,l}(\frac{y_k}{y_j})(\frac{y_l}{y_j}),$$

i.e., the ratios y_i/y_j form a closed system of coordinates under differentiation. Using the majorant method [1], we can show that the formal Laurent series solution is convergent. At the bad points, the concept of projective normality plays an important role: this enables one to show that y_i/y_j is a bona fide Taylor series starting from every point in a neighbourhood of the point in question.

Some other integrable systems appear as coverings of algebraic completely integrable systems. The manifolds invariant by the complex flows are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense.

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