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Classification of Locally Projectively Homogeneous Torsion-less Affine Connections in the Plane Domains*

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Abstract. The aim of this paper is to classify (locally) all torsion-less locally projectively homogeneous affine connections on 2-dimensional manifolds. Especially, we express, in a simple and explicit form, all such connections which are not projectively flat (Theorem 1). We make also some conclusions from this classification (Theorem 2 and Theorem 3).

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1. Introduction and main results

Let (M, ∇), $(\overline{M}, \overline{\nabla})$ be two smooth manifolds with torsion-free affine connections. These manifolds are said to be *projectively equivalent* if there is a diffeomorphism of M onto $\overline{M}$ (called a *projective map*) transforming every geodesic (equipped with an affine parameter) into a geodesic (with an arbitrary parametrization). A manifold (M, ∇) is said to be *projectively flat* if there is, for each point $p \in M$, a neighborhood $\mathcal{U}_p$ which is projectively equivalent with an open domain of the Euclidean space $\mathbb{R}^n$, $n = \dim M$.

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If a manifold (M, ∇) is given (still with a torsion-free connection ∇), then a projective map $\phi : \mathcal{U} \to \mathcal{V}$ between two domains $\mathcal{U}, \mathcal{V} \subset (M, \nabla)$ is called a *local projective transformation.* Further, a *projective Killing vector field* on (M, ∇) is a vector field X generating a local group of local projective transformations. Finally, a manifold (M, ∇) will be said to be *locally projectively homogeneous* if, at each point $p \in M$, there are at least n independent projective Killing vector fields, $n = \dim M$. If M is connected, then an equivalent definition is that, for every two points $p, q \in M$, there is a local projective transformation of a neighborhood of p onto a neighborhood of q. All projective Killing vector fields on (M, ∇) form a Lie algebra. (So do the affine Killing vector fields, and, in the case of a Riemannian manifold, metric Killing vector fields).

Let us recall the following characterization of a projective Killing vector field (see [8], p. 45): A vector field X is a projective Killing vector field on (M, ∇) if it satisfies

$$(\mathcal{L}_X \nabla)(Y, Z) = \pi(Y)\, Z + \pi(Z)\, Y \tag{1}$$

for any vector fields Y and Z, π being a 1-form and $\mathcal{L}$ denoting the Lie derivative.

Now, we shall pass to the dimension $n = 2$. In this dimension, the explicit classification of all locally projectively homogeneous pseudo-Riemannian manifolds has been presented by A. V. Aminova [1]. (Recently, V. S Matveev pointed out that there might be a small gap in this classification). For the affine manifolds, such a classification was not known explicitly up to now. But it was well-known that, for every locally projectively homogeneous manifold (M, ∇), the full algebra of projective Killing vector fields is either 8-dimensional and isomorphic to $\mathfrak{sl}(3, \mathbb{R})$ (in such a case, (M, ∇) is projectively flat), or 3-dimensional and isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, or 2-dimensional and nonabelian (a private communication by V. S. Matveev).

In this paper we are going to classify all projectively homogeneous affine torsion-less connections from the group-theoretical point of view. This means that we always start with a specific transitive Lie algebra of vector fields from the list of P. J. Olver [6] and we are looking for all affine connections for which this algebra is an algebra of projective Killing vector fields.

This method was used earlier, by the present authors and B. Opozda, for an alternative classification of (affinely) locally homogeneous affine connections in dimension two (see [5]). The credit for the first explicit classification (in different form) belongs to B. Opozda [7] (see Section 7).

We try to organize our computation in (possibly) most systematic way so that the whole procedure is not excessively long. Also, because this topic is an ideal subject for a computer-aided research, we are using the software Maple 8, (c) Waterloo Maple Inc., throughout this work. But we put stress on the full transparency of this procedure.

According to [6], taking into account the comments in page 61, the classification of all transitive Lie algebras of vector fields in $\mathbb{R}^2$ is given by Table 1 and Table 6 ([6], pages 472 and 476, respectively). We present the Olver's tables with a slight modification, denoting them as Table 1 and Table 2.

	Generators	Dim	Structure
1.1	$\partial_v,\ v\partial_v - u\partial_u,\ v^2\partial_v - 2uv\partial_u$	3	$\mathfrak{sl}(2)$
1.2	$\partial_v,\ v\partial_v - u\partial_u,\ v^2\partial_v - (2uv+1)\partial_u$	3	$\mathfrak{sl}(2)$
1.3	$\partial_v,\ v\partial_v,\ u\partial_u,\ v^2\partial_v - uv\partial_u$	4	$\mathfrak{gl}(2)$
1.4	$\partial_v,\ v\partial_v,\ v^2\partial_v,\ \partial_u,\ u\partial_u,\ u^2\partial_u$	6	$\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$
1.5	$\partial_v,\ \eta_1(v)\partial_u,\ \ldots,\ \eta_k(v)\partial_u$	$k+1$	$\mathbb{R}\ltimes\mathbb{R}^k$
1.6	$\partial_v,\ u\partial_u,\ \eta_1(v)\partial_u,\ \ldots,\ \eta_k(v)\partial_u$	$k+2$	$\mathbb{R}^2\ltimes\mathbb{R}^k$
1.7	$\partial_v,\ v\partial_v + \alpha u\partial_u,\ \partial_u,\ v\partial_u,\ \ldots,\ v^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1)\ltimes\mathbb{R}^k$
1.8	$\partial_v,\ v\partial_v + (ku+v^k)\partial_u,\ \partial_u,\ v\partial_u,\ \ldots,\ v^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1)\ltimes\mathbb{R}^k$
1.9	$\partial_v,\ v\partial_v,\ u\partial_u,\ \partial_u,\ v\partial_u,\ \ldots,\ v^{k-1}\partial_u$	$k+3$	$\mathfrak{c}(1)\ltimes\mathbb{R}^k$
1.10	$\partial_v,\ 2v\partial_v + (k-1)u\partial_u,\ v^2\partial_v + (k-1)uv\partial_u,$ $\partial_u,\ v\partial_u,\ \ldots,\ v^{k-1}\partial_u$	$k+3$	$\mathfrak{sl}(2)\ltimes\mathbb{R}^k$
1.11	$\partial_v,\ v\partial_v,\ v^2\partial_v + (k-1)uv\partial_u,\ u\partial_u,$ $\partial_u,\ v\partial_u,\ \ldots,\ v^{k-1}\partial_u$	$k+4$	$\mathfrak{gl}(2)\ltimes\mathbb{R}^k$

Table 1. Transitive, imprimitive Lie algebras of vector fields in $\mathbb{R}^2$

Remarks. (from [6]): Here $\mathfrak{c}(1) = \mathfrak{a}(1)\oplus\mathbb{R}$.

In cases 1.5 and 1.6, the functions $\eta_1(v),\ldots,\eta_k(v)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u]=0$.

In cases 1.5–1.11 we require $k\geq 1$. Note, though, that if we set $k=0$ in case 1.10, and replace u by u^2, we obtain case 1.1. Similarly, if we set $k=0$ in case 1.11, we obtain case 1.3., cases 1.7 and 1.8 for $k=0$ are equivalent to the Lie algebra span$\{\partial_v,\, \mathrm{e}^v\partial_u\}$ of type 1.5. Case 1.9 for $k=0$ is equivalent to the Lie algebra span$\{\partial_v,\, \partial_u,\, u\partial_u\}$ of type 1.6.

	Generators	Dim	Structure
2.1	$\partial_v,\ \partial_u,\ \alpha(v\partial_v + u\partial_u) + u\partial_v - v\partial_u$	3	$\mathbb{R}\ltimes\mathbb{R}^2$
2.2	$\partial_v,\ v\partial_v + u\partial_u,\ (v^2-u^2)\partial_v + 2uv\partial_u$	3	$\mathfrak{sl}(2)$
2.3	$u\partial_v - v\partial_u,\ (1+v^2-u^2)\partial_v + 2uv\partial_u,$ $2uv\partial_v + (1-v^2+u^2)\partial_u$	3	$\mathfrak{so}(3)$
2.4	$\partial_v,\ \partial_u,\ v\partial_v + u\partial_u,\ u\partial_v - v\partial_u$	4	$\mathbb{R}^2\ltimes\mathbb{R}^2$
2.5	$\partial_v,\ \partial_u,\ v\partial_v - u\partial_u,\ u\partial_v,\ v\partial_u$	5	$\mathfrak{sa}(2)$
2.6	$\partial_v,\ \partial_u,\ v\partial_v,\ u\partial_v,\ v\partial_u,\ u\partial_u$	6	$\mathfrak{a}(2)$
2.7	$\partial_v,\ \partial_u,\ v\partial_v + u\partial_u,\ u\partial_v - v\partial_u,$ $(v^2-u^2)\partial_v + 2uv\partial_u,\ 2uv\partial_v + (u^2-v^2)\partial_u$	6	$\mathfrak{so}(3,1)$
2.8	$\partial_v,\ \partial_u,\ v\partial_v,\ u\partial_v,\ v\partial_u,\ u\partial_u,$ $v^2\partial_v + uv\partial_u,\ uv\partial_v + u^2\partial_u$	8	$\mathfrak{sl}(3)$

Table 2. Primitive Lie algebras of vector fields in $\mathbb{R}^2$

Next, let us recall the following criterion of projective flatness in dimension 2.

Due to [8], p. 15, we introduce a bilinear form $L(X,Y)$ on $\mathcal{X}(M)$ by putting

$$L(X,Y) = \frac{1}{3}\left[2\ \mathrm{Ric}(X,Y) + \mathrm{Ric}(Y,X)\right]. \tag{2}$$

(There is a sign misprint in [8], formula (4.10)). Then (M,∇) is projectively flat if and only if

$$(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) = 0 \quad \text{for all } X,Y,Z. \tag{3}$$

In any coordinate domain $\mathcal{U}(u,v) \subset M$ we express a vector field X in the form $X = a(u,v)\,\partial_u + b(u,v)\,\partial_v$. Then, for a torsion-less connection ∇ in $\mathcal{U}(u,v)$ we put

$$\begin{aligned}
\nabla_{\partial_u}\partial_u &= A(u,v)\,\partial_u + B(u,v)\,\partial_v,\\
\nabla_{\partial_u}\partial_v &= \nabla_{\partial_v}\partial_u = C(u,v)\,\partial_u + D(u,v)\,\partial_v,\\
\nabla_{\partial_v}\partial_v &= E(u,v)\,\partial_u + F(u,v)\,\partial_v.
\end{aligned} \tag{4}$$

Writing the formula (1) in local coordinates, we find that any projective Killing vector field X must satisfy six basic equations. We shall write these equations in the simplified notation:

$$\begin{array}{lll}
a_{uu} & +A\,a_u - B\,a_v + 2C\,b_u + A_u a + A_v b & = 2K,\\
b_{uu} & +2B\,a_u + (2D-A)\,b_u - B\,b_v + B_u a + B_v b & = 0,\\
a_{uv} & +(A-D)\,a_v + E\,b_u + C\,b_v + C_u a + C_v b & = L,\\
b_{uv} & +D\,a_u + B\,a_v - (C-F)\,b_u + D_u a + D_v b & = K,\\
a_{vv} & -E\,a_u + (2C-F)\,a_v + 2E\,b_v + E_u a + E_v b & = 0,\\
b_{vv} & +2D\,a_v - E\,b_u + F\,b_v + F_u a + F_v b & = 2L.
\end{array} \tag{5}$$

Here $K = K(u,v)$ and $L = L(u,v)$ are arbitrary functions depending always on the particular choice of the functions $a = a(u,v)$, $b = b(u,v)$. Recall that for $K = L = 0$ we obtain the equations for an affine Killing vector field.

Now, the equations (5) imply four homogeneous equations

$$\begin{array}{lll}
b_{uu} & +2Ba_u + (2D-A)b_u - Bb_v + B_u a + B_v b & = 0,\\
a_{vv} & -Ea_u + (2C-F)a_v + 2Eb_v + E_u a + E_v b & = 0,\\
a_{uu} - 2b_{uv} & +(A-2D)a_u - 3Ba_v + 2(2C-F)b_u & \\
 & +(A-2D)_u a + (A-2D)_v b & = 0,\\
b_{vv} - 2a_{uv} & -2(A-2D)a_v - 3Eb_u - (2C-F)b_v & \\
 & -(2C-F)_u a - (2C-F)_v b & = 0,
\end{array} \tag{6}$$

which are the only relevant ones for the proper projective case. Let us denote

$$\begin{aligned}
\widehat{A}(u,v) &= A(u,v) - 2\,D(u,v),\ \ \widehat{B}(u,v) = B(u,v),\\
\widehat{F}(u,v) &= F(u,v) - 2\,C(u,v),\ \ \widehat{E}(u,v) = E(u,v).
\end{aligned} \tag{7}$$

Then the equations (6) can be rewritten in the form

$$\begin{array}{llllllll}
b_{uu} & +2\,\widehat{B}\,a_u & & & -\widehat{A}\,b_u & -\widehat{B}\,b_v & +\widehat{B}_u a & +\widehat{B}_v b & = 0,\\
a_{vv} & -\widehat{E}\,a_u & -\widehat{F}\,a_v & & & +2\,\widehat{E}\,b_v & +\widehat{E}_u a & +\widehat{E}_v b & = 0,\\
a_{uu} - 2\,b_{uv} & +\widehat{A}\,a_u & -3\,\widehat{B}\,a_v & -2\,\widehat{F}\,b_u & & & +\widehat{A}_u a & +\widehat{A}_v b & = 0,\\
b_{vv} - 2\,a_{uv} & & -2\,\widehat{A}\,a_v & -3\,\widehat{E}\,b_u & & +\widehat{F}\,b_v & +\widehat{F}_u a & +\widehat{F}_v b & = 0.
\end{array} \tag{8}$$

We shall now formulate our main results.

Theorem 1. *For every locally projectively homogeneous torsion-less affine connection ∇ on a connected two-dimensional manifold M exactly one of the following cases occurs:*

1. *The full algebra of all projective Killing vector fields is 8-dimensional and isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. ∇ is projectively flat and, in a neighborhood of each point $p \in M$, there is a system (u,v) of local coordinates such that $A(u,v) = 2\,D(u,v)$, $B(u,v) = E(u,v) = 0$, $F(u,v) = 2\,C(u,v)$, and $C(u,v)$, $D(u,v)$ are arbitrary functions.*
2. *The full algebra of all projective Killing vector fields is 3-dimensional and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In a neighborhood of each point $p \in M$, there is a system (u,v) of local coordinates such that $A(u,v) = 2\,D(u,v) - 3/(2u)$, $B(u,v) = 0$, $E(u,v) = e\,u^3, e \neq 0$, $F(u,v) = 2\,C(u,v)$ and $C(u,v)$, $D(u,v)$ are arbitrary functions.*
3. *The full algebra of all projective Killing vector fields is 2-dimensional and nonabelian. In a neighborhood of each point $p \in M$, there is a system (u,v) of local coordinates such that $A(u,v) = 2\,D(u,v) + 3\,c_0\,u + c_1$, $B(u,v) = c_0$, $E(u,v) = c_0\,u^3 + c_1\,u^2 + c_2\,u + c_3$, $F(u,v) = 2\,C(u,v) + 3\,c_0\,u^2 + 2\,c_1\,u + c_2$, where the constants c_0, c_1 are not both equal to zero. Here $C(u,v)$, $D(u,v)$ are arbitrary functions.*

In the cases 2 and 3 the connections are not projectively flat. In each of the three cases, for a generic choice of the functions $C(u,v)$, $D(u,v)$, the connection ∇ does not admit any nonzero affine Killing vector field.

Remark. Here, 'generic choice' means that the corresponding functions $A(u,v)$, $\dots, F(u,v)$ do not satisfy any partial differential equation of the form (5) with $K = L = 0$ and such that $a(u,v)\,\partial_u + b(u,v)\,\partial_v$ is any of the vector fields belonging to the case 2.8, or 1.1, or 1.5 of the Olver's tables, respectively. (See more details in the Sections 4–7.)

The fact that Theorem 1 consists of three cases, as well as the last statement of Theorem 1, are well known. What is new, to the authors' knowledge, are the explicit formulas in cases 2 and 3.

Here we also obtain the following

Corollary 1. *A locally projectively homogeneous affine connection is projectively flat if and only if it admits a primitive algebra of projective Killing vector fields.*

The next result is a counter-part of the last statement of Theorem 1:

Theorem 2. *For every locally projectively homogeneous but projectively nonflat affine connection ∇ on M there is, in a neighborhood $\mathcal{U}$ of each point $p \in M$, an affine connection $\overline{\nabla}$ which is projectively equivalent to ∇ and such that every projective Killing vector field of $\overline{\nabla}$ on $\mathcal{U}$ is an affine Killing vector field of $\overline{\nabla}$.*

The next Theorem unifies cases 2 and 3 of Theorem 1.

Theorem 3. *For every locally projectively homogeneous but projectively nonflat affine connection ∇ on M there is, in a neighborhood $\mathcal{U}$ of each point $p \in M$, a local coordinate system (u, v) in which the components of ∇ satisfy*

$$A(u,v) = 2\,D(u,v) + \widehat{a}/u,\;\; B(u,v) = \widehat{b}/u,$$

$$F(u,v) = 2\,C(u,v) + \widehat{f}/u,\;\; E(u,v) = \widehat{e}/u.$$

Here $\widehat{a}, \widehat{b}, \widehat{e}, \widehat{f}$ are constants satisfying at least one of the inequalities

$$3\,(\widehat{a}-1)\,\widehat{e} - \widehat{f}^{\,2} \neq 0,\;\; (\widehat{a}+2)\,\widehat{f} - 9\,\widehat{b}\,\widehat{e} \neq 0$$

and $C(u,v)$, $D(u,v)$ are arbitrary functions.

2. The case of commuting projective Killing vector fields

We start with the canonical choice of two commuting local vector fields, namely $X = \partial_u, Y = \partial_v$, and we express the conditions saying that these vector fields are projective Killing vector fields, i.e., we put

$$a(u,v) = 1,\;\; b(u,v) = 0, \tag{9}$$
$$a(u,v) = 0,\;\; b(u,v) = 1, \tag{10}$$

and substitute in (8). We obtain

$$\widehat{A}_u = 0,\;\; \widehat{B}_u = 0,\;\; \widehat{E}_u = 0,\;\; \widehat{F}_u = 0, \tag{11}$$
$$\widehat{A}_v = 0,\;\; \widehat{B}_v = 0,\;\; \widehat{E}_v = 0,\;\; \widehat{F}_v = 0. \tag{12}$$

We conclude hence that, for every connection ∇ admitting projective Killing vector fields (9),(10), we must have

$$\widehat{A}(u,v) = A_0,\;\; \widehat{B}(u,v) = B_0,\;\; \widehat{E}(u,v) = E_0,\;\; \widehat{F}(u,v) = F_0, \tag{13}$$

where A_0, B_0, E_0, F_0 are arbitrary constants. Due to (7) we see that all connections with the above property depend on two arbitrary functions $C(u,v)$, $D(u,v)$.

In the following we assume that the constants above are all zero, and we shall look for all possible projective Killing vector fields belonging to a connection ∇ determined by $C(u,v), D(u,v)$. Substituting (13) into (8), we obtain the system of four equations for the unknown functions $a(u,v), b(u,v)$:

$$\begin{aligned} b_{uu} &= 0,\\ a_{vv} &= 0,\\ a_{uu} - 2\,b_{uv} &= 0,\\ b_{vv} - 2\,a_{uv} &= 0. \end{aligned} \tag{14}$$

By an easy computation we get the general solution in the form

$$\begin{aligned} a(u,v) &= c_3u^2 + c_1uv + c_5u + c_2v + c_6,\\ b(u,v) &= c_3uv + c_1v^2 + c_4u + c_7v + c_8, \end{aligned} \tag{15}$$

involving eight arbitrary parameters $c_1, \dots, c_8$. The corresponding projective Killing vector fields are those from the case 2.8 of Table 2.

Substitute now (15) into the third and the fourth equation of (5), and put here $A(u,v) = 2\,D(u,v)$, $B(u,v) = E(u,v) = 0$, $F(u,v) = 2\,C(u,v)$ and $K(u,v) = L(u,v) = 0$. The corresponding equations are

$$\begin{aligned}(c_3u^2 + c_1uv + c_5u + c_2v + c_6)\,C_u + (c_3uv + c_1v^2 + c_4u + c_7v + c_8)\,C_v \\ +(c_3u + 2\,c_1v + c_7)\,C + (c_1u + c_2)\,D + c_1 = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}(c_3u^2 + c_1uv + c_5u + c_2v + c_6)\,D_u + (c_3uv + c_1v^2 + c_4u + c_7v + c_8)\,D_v \\ +(c_3v + c_4)\,C + (2\,c_3u + c_1v + c_5)\,D + c_3 = 0.\end{aligned} \tag{17}$$

We see that our space with the affine connection ∇ admits an affine Killing vector field if, and only if, the system of two PDE (16) and (17) is satisfied for some particular choice of the parameters $c_1, \dots, c_8$. We conclude that, for a generic choice of $C(u,v)$ and $D(u,v)$, the space does not admit a nonzero affine Killing vector field. Yet, the algebra of projective Killing vector fields is 8-dimensional.

We shall now solve our problem in general, i.e., when A_0, B_0, E_0, F_0 are arbitrary constants. Recall that the corresponding Ricci form is given, due to (5) or (7), by

$$\begin{aligned}\mathrm{Ric}(\partial_u, \partial_u) &= B_v - D_u + D(A - D) + B(F - C),\\ \mathrm{Ric}(\partial_u, \partial_v) &= D_v - F_u + CD - BE,\\ \mathrm{Ric}(\partial_v, \partial_u) &= C_u - A_v + CD - BE,\\ \mathrm{Ric}(\partial_v, \partial_v) &= E_u - C_v + E(A - D) + C(F - C).\end{aligned} \tag{18}$$

Then substituting from (4), (2) and (18) for two choices $(X,Y,Z){=}(\partial_u, \partial_u, \partial_v)$, $(X,Y,Z){=}(\partial_u, \partial_v, \partial_v)$ into (3) we obtain, in the notation (7), two conditions which are equivalent to (3):

$$\widehat{A}_{vv} + 3\widehat{E}_{uu} + 2\widehat{F}_{uv} + 3\widehat{E}\widehat{A}_u - \widehat{F}\widehat{A}_v + 6\widehat{E}\widehat{B}_v + 3\widehat{A}\widehat{E}_u + 3\widehat{B}\widehat{E}_v - 2\widehat{F}\widehat{F}_u = 0, \tag{19}$$

$$2\widehat{A}_{uv} + 3\widehat{B}_{vv} + \widehat{F}_{uu} - 2\widehat{A}\widehat{A}_v + 3\widehat{E}\widehat{B}_u + 3\widehat{F}\widehat{B}_v + 6\widehat{B}\widehat{E}_u - \widehat{A}\widehat{F}_u + 3\widehat{B}\widehat{F}_v = 0. \tag{20}$$

Thus, the equations (19), (20) are equivalent with the projective flatness of (M, ∇).

In particular, we see that each connection ∇ given by (13) trivially satisfies the equations (19), (20) and hence it is projectively flat. We shall derive now the consequences from this fact using another way of reasoning.

3. Digression

The formulas (19), (20) (in different notation) stem from the different background in the articles [2] and [3]. The last papers present a continuation of the classical research concerning invariants of the action of the pseudogroup of all local diffeomorphisms in the plane on an ordinary differential equation of second order. This topic was started by R. Liouville, M. A. Tresse and E. Cartan (see [3] for the full references).

We explain shortly this relationship. Let us recall the standard equations of geodesics in our 2-dimensional case. We obtain easily, denoting the local coordinates as x, y for this moment but still preserving the notation (4), the following formulas:

$$\begin{aligned} \tfrac{d^2x}{dt^2} + A\left(\tfrac{dx}{dt}\right)^2 + 2\,C\,\tfrac{dx}{dt}\tfrac{dy}{dt} + E\left(\tfrac{dy}{dt}\right)^2 = 0,\\ \tfrac{d^2y}{dt^2} + B\left(\tfrac{dx}{dt}\right)^2 + 2\,D\,\tfrac{dx}{dt}\tfrac{dy}{dt} + F\left(\tfrac{dy}{dt}\right)^2 = 0. \end{aligned} \tag{21}$$

From (21) we can eliminate one single equation for $y' = dy/dx$ and $y'' = d^2y/dx^2$. We obtain easily

$$y'' = -B + (A - 2\,D)\,y' + (2\,C - F)\,(y')^2 + E\,(y')^3 \tag{22}$$

or, using (7),

$$y'' = -\widehat{B} + \widehat{A}\,y' - \widehat{F}\,(y')^2 + \widehat{E}\,(y')^3,$$

where the coefficients are still functions of x and $y = y(x)$. The equation (22) is an alternative characterization of all geodesics in the form where the independent variable x is not necessarily the affine parameter for any corresponding geodesic $(x, y\,(x))$. Conversely, from the formula (22) we can easily go back to the standard equations (21).

Now, let us rewrite (22) in the more general form

$$y'' = C_0(x,y) + C_1(x,y)\,y' + C_2(x,y)(y')^2 + C_3(x,y)(y')^3. \tag{23}$$

The equations (19) and (20) can be hence formally rewritten as

$$\begin{gathered} C_{1,vv} + 3\,C_{3,uu} - 2\,C_{2,uv} + 3\,C_3C_{1,u} + C_2C_{1,v} - 6\,C_3C_{0,v}\\ +3\,C_1C_{3,u} - 3\,C_0C_{3,v} - 2\,C_2C_{2,u} = 0, \end{gathered} \tag{24}$$

$$\begin{gathered} 2\,C_{1,uv} - 3\,C_{0,vv} - C_{2,uu} - 2\,C_1C_{1,v} - 3\,C_3C_{0,u} + 3\,C_2C_{0,v}\\ -6\,C_0C_{3,u} + C_1C_{2,u} + 3\,C_0C_{2,v} = 0. \end{gathered} \tag{25}$$

Now, consider a more general second order ODE than (22), namely

$$y'' = F(x,y,p),\ \ p = y', \tag{26}$$

where F is an arbitrary smooth function of three variables.

The fundamental result from [3] can be expressed (using also [2]) as follows:

Theorem 4. *The following four conditions are mutually equivalent:*

(a) *Equation* (26) *has an* 8*-dimensional symmetry algebra, which is isomorphic to* $\mathfrak{sl}(3,\mathbb{R})$.

(b) *Equation* (26) *is locally equivalent to a linear equation.*

(c) *Equation* (26) *is locally equivalent to the equation* $y'' = 0$.

(d) *Equation* (26) *is of the form* (23) *where the conditions* (24) *and* (25) *hold.*

We can conclude with the following

Corollary 2. *A connection ∇ on a 2-dimensional manifold M is projectively flat if and only if, in a neighborhood of each point $p \in M$, there is a local coordinate system (u, v) for which all functions $\widehat{A}(u, v), \widehat{B}(u, v), \widehat{E}(u, v), \widehat{F}(u, v)$ vanish. In such a system of local coordinates, the algebra of all projective Killing vector fields is of type* 2.8 *from Table* 2.

Proof. The first part follows from the equivalence of conditions (c) and (d) and from the meaning of formulas (24), (25). The second part follows from (15).

Remark. From the geometric meaning of the equation (22) we see that two connections $\nabla, \overline{\nabla}$ with the same functions $\widehat{A}, \widehat{B}, \widehat{E}, \widehat{F}$ in a fixed system of local coordinates $\mathcal{U}(u, v)$ are projectively equivalent in $\mathcal{U}$.

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4. The cases leading to projectively flat connections - continuation

Recall first that the conclusion (13) from Section 2 implies projective flatness. From the previous Corollary 2 we now get easily

Proposition 1. *Let (M, ∇) admit two commuting projective Killing vector fields X, Y. Then, around each point $p \in M$, there is a coordinate neighborhood $\mathcal{U}(u, v)$ in which $X = \partial_u$, $Y = \partial_v$, the connection ∇ is projectively flat, and the generators of the Lie algebra of all projective Killing vector fields are those from the case* 2.8 *in Table* 2.

Looking carefully at Table 1 and Table 2, we obtain at once

Corollary 3. *The cases* 1.7 *and* 1.9 *for $k = 1, 2$, the case* 1.8 *for $k = 1$, the cases* 1.10 *and* 1.11 *for $k = 2$, and the cases* 2.1, 2.4, 2.5, 2.6 *reduce to the case* 2.8*, i.e., the corresponding Lie algebra extends to the* 8*-dimensional algebra of projective Killing vector fields. In convenient local coordinates, the connection ∇ is described by the formulas* (13) *in which the arbitrary constants can be supposed to vanish and $C(u, v), D(u, v)$ are arbitrary functions. In each of these cases, there is generically no affine Killing vector field.*

Next, we are going to prove

Proposition 2. *The cases* 1.2, 1.3*, the case* 1.5 *for $k = 1$ with $\eta_1(v)$ linear and for $k = 2$, the case* 1.6 *for $k = 1, 2$, and the cases* 2.2, 2.3 *lead to projectively flat connections.*

Proof. For the sake of brevity, we shall always calculate the functions $\widehat{A}(u, v)$, $\widehat{B}(u, v)$, $\widehat{E}(u, v)$, $\widehat{F}(u, v)$ given by (7) and then we substitute the result into the equations (19), (20).

In all cases in question, except the case 2.3, the first of the wanted projective Killing vector fields is ∂_v. From formula (12) we get

$$\widehat{A}_v = \widehat{B}_v = \widehat{E}_v = \widehat{F}_v = 0 \tag{27}$$

and hence we can put

$$\widehat{A}(u,v) = d(u),\ \widehat{B}(u,v) = p(u),\ \widehat{E}(u,v) = e(u),\ \widehat{F}(u,v) = f(u), \tag{28}$$

where d, p, e, f are arbitrary functions of the variable u.

In the case 1.2, we have the second vector field $v\partial_v - u\partial_u$ and, substituting $a(u,v) = -u,\ b(u,v) = v$ and (28) into (8), we get a system of equations

$$\begin{array}{rll} -3\,p(u) & -u\,p'(u) & = 0, \\ 3\,e(u) & -u\,e'(u) & = 0, \\ d(u) & +u\,d'(u) & = 0, \\ f(u) & +u\,f'(u) & = 0. \end{array} \tag{29}$$

The general solution of (29) can be written in the form

$$p(u) = c_1/u^3,\ c(u) = c_2 u^3,\ d(u) = c_3/u,\ f(u) = c_4 u. \tag{30}$$

Finally, substitute the third vector field of case 1.2, namely $a(u,v) = -2uv - 1$, $b(u,v) = v^2$ together with (28) and (30) into (8). We obtain just four linear relations between the parameters and we conclude finally

$$\widehat{A}(u,v) = 0,\ \widehat{B}(u,v) = 0,\ \widehat{E}(u,v) = 4\,u^3,\ \widehat{F}(u,v) = 6\,u. \tag{31}$$

Substituting (31) into (19) and (20) we see that both equations are satisfied. Hence the case 1.2 leads just to projectively flat connections.

In the case 1.3, we have the second vector field $v\partial_v$ and substituting $a(u,v) = 0,\ b(u,v) = v$ and (28) into (8) we get

$$p(u) = e(u) = f(u) = 0. \tag{32}$$

Substituting the third vector field of the case 1.3, namely $a(u,v) = u,\ b(u,v) = 0$ together with (28) and (32) into (8), we get

$$u\,d'(u) + d(u) = 0 \Rightarrow d(u) = c_3/u. \tag{33}$$

Finally, substituting the fourth vector field of case 1.3, namely $a(u,v) = -uv$, $b(u,v) = v^2$ together with (28), (32) and (33) into (8), we get $c_3 = -2$ and the conclusion

$$\widehat{A}(u,v) = -2/u,\ \widehat{B}(u,v) = E(u,v) = F(u,v) = 0. \tag{34}$$

Substituting this into (19) and (20) we approve that the case 1.3 leads to projectively flat connections.

In the case 1.5 for $k = 1$ and linear $\eta_1(v)$, we can suppose, without loss of generality, that the second vector field is $v\partial_u$. Substituting $a(u,v) = v$, $b(u,v) = 0$ and (28) into (8) we get

$$v\,p'(u) = 0,\ v\,e'(u) = f(u),\ v\,d'(u) = 3\,p\,(u),\ v\,f'(u) = 2\,d(u)$$

and thus

$$d(u) = p(u) = f(u) = 0,\ e(u) = c_1. \tag{35}$$

We conclude

$$\widehat{A}(u,v) = \widehat{B}(u,v) = \widehat{F}(u,v) = 0,\ \widehat{E}(u,v) = c_1, \tag{36}$$

which is a special case of (13). Thus this case leads to projectively flat connections.

In the case 1.5 for $k = 2$ and nonlinear $\eta_1(v)=g(v)$, substituting $a(u,v) = g(v)$, $b(u,v) = 0$ and (28) into (8), we get

$$\begin{aligned}
&p'(u)\,g(v) &&= 0,\\
&g''(v) - g'(v)f(u) + g(v)\,e'(u) &&= 0,\\
&-3\,p(u)\,g'(v) + g(v)\,d'(u) &&= 0,\\
&-2\,d(u)\,g'(v) + g(v)\,f'(u) &&= 0.
\end{aligned} \tag{37}$$

According to the remarks below Table 1, $g(v)$ must satisfy a 2^{nd} order ODE with constant coefficients. Hence the coefficients in the equation (37_2) must be constant and we get

$$f(u) = c_0,\ e(u) = c_1 u + c_2. \tag{38}$$

From (37_4) we get $d(u) = 0$ and from (37_3) we then obtain $p(u) = 0$. Due to (28), we can summarize:

$$\widehat{A}(u,v) = \widehat{B}(u,v) = 0,\ \widehat{E}(u,v) = c_1 u + c_2,\ \widehat{F}(u,v) = c_0. \tag{39}$$

According to (19) and (20) we obtain projective flatness.

Notice that the equation (37_2) is uniquely determined by (38) or (39).

In the case 1.6, $k = 1$, we are first in the same situation as in the corresponding case 1.5. For $\eta_1(v)$ linear we derived conformal flatness. The remaining case of a nonabelian algebra $\{\partial_v,\ \eta_1(v)\partial_u\}$ is, without the loss of generality, the case $\eta_1(v) = \mathrm{e}^v$. Here substituting (28) and $a(u,v) = \mathrm{e}^v$, $b(u,v) = 0$ into (8) we get

$$p'(u) = 0,\ 1 - f(u) + e'(u) = 0,\ -3\,p(u) + d'(u) = 0,\ -2\,d(u) + f'(u) = 0. \tag{40}$$

Hence we obtain easily (using also (28))

$$\begin{aligned}
&\widehat{A}(u,v) = 3\,c_0 u + c_1,\ \widehat{B}(u,v) = c_0,\\
&\widehat{E}(u,v) = c_0 u^3 + c_1 u^2 + (c_2 - 1)u + c_3,\ \widehat{F}(u,v) = 3\,c_0 u^2 + 2\,c_1 u + c_2,
\end{aligned} \tag{41}$$

where c_0, c_1, c_2, c_3 are constants. (This formula will be used also in the next section.)

Now, we have additional vector field $u\partial_u$, i.e., $a(u,v) = u$, $b(u,v) = 0$. If we substitute this and (41) into (8), we obtain $c_0 = c_1 = c_3 = 0$. Hence

$$\widehat{A}(u,v) = \widehat{B}(u,v) = 0,\ \widehat{F}(u,v) = c_2,\ \widehat{E}(u,v) = (c_2 - 1)u, \tag{42}$$

which is a special case of (39). We obtain the projective flatness.

In the case 1.6, $k = 2$, we are first in the same situation as in the case 1.5 and we obtain the formula (39) and projective flatness. The new vector field $u\partial_u$ is, according to (8), also a projective Killing vector field if, and only if, we put $c_2 = 0$ in (39).

Next, consider the case 2.2. Here we have again (28). For the next vector field $a(u,v) = u$, $b(u,v) = v$ we obtain by (8), for all functions $p(u), d(u), c(u), f(u)$, the same differential equation $\varphi(u) + u\,\varphi'(u) = 0$. Hence

$$p(u) = c_1/u,\ e(u) = c_2/u,\ d(u) = c_3/u,\ f(u) = c_4/u. \tag{43}$$

Substituting (43) and the last vector field $a(u,v) = 2\,uv$, $b(u,v) = v^2 - u^2$ into (8), we obtain a system of four linear equations for the constants c_i and, finally, we get

$$\widehat{B}(u,v) = \widehat{F}(u,v) = 0,\ \widehat{A}(u,v) = \widehat{E}(u,v) = 1/u. \tag{44}$$

We again check easily the projective flatness.

Finally, we investigate the case 2.3. By a rather long computation, which is completely analogous to the case 6.3 in [5], pp. 96–97, we conclude with the following formula:

$$\widehat{A}(u,v) = \widehat{E}(u,v) = \partial g(u,v)/\partial u,\ \widehat{B}(u,v) = \widehat{F}(u,v) = \partial g(u,v)/\partial v \tag{45}$$

where $g(u,v) = \ln(u^2 + v^2 + 1)$. Now, the projective flatness follows either by the direct check of the formulas (19), (20), or, from the fact that this connection is the Levi-Civita connection of a Riemannian space of constant positive curvature (cf. [5], Theorem 6.4, part 4).

This concludes the proof of Proposition 2. □

5. The cases leading to projectively homogeneous but projectively nonflat connections

Proposition 3. *The case* 1.5 *for* $k = 1$ *where* $\eta_1(v) = \mathrm{e}^v$ *leads to projectively nonflat connections whose full algebra of projective Killing vector fields is* 2-*dimensional and nonabelian.*

Proof. The corresponding family of connections is given here by the formula (41). If either $c_0 \neq 0$ or $c_1 \neq 0$, we see by the direct check of (19) and (20) that the corresponding connections are not projectively flat. □

Proposition 4. *The case* 1.1 *leads to projectively nonflat connections whose full algebra of projective Killing vector fields is* 3-*dimensional and isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$.

Proof. Consider the case 1.1. The operator ∂_v is still here and the formulas (28) are valid. Take the next vector field $a(u,v) = -u$, $b(u,v) = v$. Substituting this and (27) into (8), we obtain a system of equations

$$3\,p(u)+up'(u) = 0,\ 3\,e(u)-ue'(u) = 0,\ d(u)+ud'(u) = 0,\ f(u)-uf'(u) = 0. \quad (46)$$

Hence

$$p(u) = c_1/u^3,\ e(u) = c_2/u^3,\ d(u) = c_3/u,\ f(u) = c_4 u. \quad (47)$$

Now, we express the condition that also the vector field $a(u,v) = -2\,uv$, $b(u,v) = v^2$ satisfies the equation (8) with the basic functions $\widehat{A}, \widehat{B}, \widehat{E}, \widehat{F}$ given by (47) and (28). We obtain a system of four linear equations for the corresponding constants and we get hence $c_1 = c_4 = 0$, $c_3 = -3/2$. We conclude with

$$\widehat{A}(u,v) = -3/(2u),\ \widehat{B}(u,v) = \widehat{F}(u,v) = 0,\ \widehat{E}(u,v) = c_2 u^3. \quad (48)$$

If $c_2 \neq 0$, we check easily from (19) and (20) that the corresponding connections are not projectively flat. □

6. The cases which do not determine any projectively homogeneous connections

Proposition 5. *The case* 1.4, *the cases* 1.5 *and* 1.6 *for* $k > 2$, *the cases* 1.7 *and* 1.8 *for* $k > 1$, *the cases* 1.10, 1.11 *for* $k \neq 2$ *and the case* 2.7 *do not produce any projectively homogeneous affine connection.*

Proof. In the case 1.4, the operators ∂_u, ∂_v are still present. From Section 2 we know that the formulas (13) hold. Now, assuming that the vector fields $u\partial_u$ and $v\partial_v$ are projective Killing vector fields, as well, we derive easily from (8) and formulas (13) that all constants in (13) are equal to zero. Hence $\widehat{A} = \widehat{B} = \widehat{E} = \widehat{F} = 0$.

Finally, assuming that the vector field $a(u,v) = 0$, $b(u,v) = v^2$ is projective Killing one, we obtain from (8_1) a contradiction.

In the case 1.5 for $k > 2$, we use the formulas (37) once again. Yet, we cannot use the same simple argument as in the case $k = 2$ and we must modify our procedure. So, consider a vector field $g(v)\,\partial_u$ in this algebra, where $g(v) \neq \text{const}$. First, from (37_1) and (37_3) we get

$$p(u) = c_0,\ d(u) = 3\,c_0\,u\,g'(v)/g(v) + c_1. \quad (49)$$

Further, we can write from (37_2)–(37_4)

$$\begin{aligned} &g''(v) - f(u)\,g'(v) + e'(u)\,g(v) = 0, \\ &-3\,c_0\,g'(v) + c_1\,g(v) = 0,\ -2\,c_1\,g'(v) + f'(u)\,g(v) = 0. \end{aligned} \quad (50)$$

Suppose first $c_0 \neq 0$. Then from (50_2) we see that $g'(v)/g(v) \neq 0$ is a well-defined constant. We get $g(v) = \mathrm{e}^{cv}$, $c = c_1/(3c_0)$. This is the only solution and $k \leq 1$, which is a contradiction.

Suppose now $c_0 = 0$. Then $p(u) = 0$, $d(u) = c_1 = 0$ follow from (49) and (50_2). From (50_3) we see $f'(u) = 0$ and $f(u) = c_2$. The equation (50_1) takes on the form

$$g''(v) - c_2\, g'(v) + e'(u)\, g(v) = 0. \tag{51}$$

Hence $e'(u)$ must be a constant and $e(u) = c_3\, u + c_4$. Consequently, the corresponding connections are expressed, up to the numeration of constants, by the formula (39). As we see again, the equation (51) is uniquely determined by the (modified) formula (39) and hence there are at most two additional vector fields $\eta_1(v)\,\partial_u$, $\eta_2(v)\,\partial_u$ which can be projective Killing. Hence $k \leq 2$, a contradiction. This completes the proof in our case.

In the case 1.6 for $k > 2$, the procedure is similar as above.

The proofs in cases 1.7 and 1.8 for $k > 1$ are similar to that of the case 1.4. So are the proofs for the cases 1.10 and 1.11 for $k \neq 2$ and for the case 2.7. $\square$

7. Proofs of the main theorems

Proof of Theorem 1. It follows from Corollary 2 and Propositions 1–5. Here the case 1 corresponds directly to Corollary 2 from Section 3, the case 2 corresponds to Proposition 4 and the case 3 corresponds to Proposition 3.

Proof of Theorem 2. It was proved in [5], that the connections whose Christoffel symbols are given by the formula ([5](10), p. 93), with C_1 and C_2 not both equal to zero, are locally affinely homogeneous and their full algebra of affine Killing vector fields is 2-dimensional and nonabelian. (In fact, it corresponds to the family from Proposition 3.) We reproduce here this formula once again:

$$\begin{aligned} &A(u,v) = C_1 u + C_2,\ B(u,v) = C_1,\ D(u,v) = -C_1 u + C_3,\\ &C(u,v) = -C_1 u^2 + (C_3 - C_2)u + C_4,\ F(u,v) = C_1 u^2 - 2\,C_3 u + C_5,\\ &E(u,v) = C_1 u^3 + (C_2 - 2\,C_3)u^2 + (C_5 - 2\,C_4 - 1)u + C_6. \end{aligned} \tag{52}$$

($C_1, \ldots, C_6$ are constants and $C_1 \neq 0$.)

We see that the corresponding functions $\widehat{A}(u,v)$, $\widehat{B}(u,v)$, $\widehat{E}(u,v)$, $\widehat{F}(u,v)$ are of the form (41), when c_0, c_1 are not both equal to zero. Hence, every connection ∇ given by the formula (41) is, for a fixed choice of the parameters c_0, c_1, c_2, c_3, $(c_0)^2 + (c_1)^2 > 0$, projectively equivalent to a connection $\overline{\nabla}$ of the form (52). Moreover, the full algebra of projective Killing vector fields of $\overline{\nabla}$ coincides with the full algebra of affine Killing vector fields and it is of the type 1.5, $k = 1$.

Further, it was proved in [5] that each connection whose Christoffel symbols are given by formula ([5](17), p. 95), namely

$$A(u) = -\frac{1}{2\,u},\ B(u) = 0,\ C(u) = c\,u,\ D(u) = \frac{1}{2u},\ E(u) = e\,u^3,\ F(u) = 2\,c\,u, \tag{53}$$

admits a 3-dimensional algebra of affine Killing vector fields of the type 1.1. The corresponding functions $\widehat{A}(u,v)$, $\widehat{B}(u,v)$, $\widehat{E}(u,v)$, $\widehat{F}(u,v)$ are of the form given by the formula ([5](17), where c_2 stands for e. For $c_2 \neq 0$, these connections are not projectively flat and they have the full algebra of projective Killing vector field of the type 1.1. Once again, for each choice of a connection ∇ given by (48) with $c_2 \neq 0$, there is a connection $\overline{\nabla}$ of the form (53) with the corresponding functions $\widehat{A}, \ldots, \widehat{F}$ given by (48) and thus projectively equivalent to ∇. The full projective Killing algebra and the full affine Killing algebra coincide, being both of type 1.1. □

The Corollary 1 of Theorem 1 follows at once looking at Table 1 and Table 2. □

Proof of Theorem 3. This proof is a simple application of a highly nontrivial result by Barbara Opozda from [7], namely of the following

Theorem 5. *Let ∇ be a torsion-less locally homogeneous affine connection on a* 2*-dimensional manifold $\mathcal{M}$. Then, either ∇ is a Levi-Civita connection of constant curvature or, in a neighborhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants a, b, c, d, e, f such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*
Type A*:*

$$\nabla_{\partial_u}\partial_u = a\partial_u + b\partial_v, \ \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v, \ \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v. \tag{54}$$

Type B*:*

$$\nabla_{\partial_u}\partial_u = \frac{a\partial_u + b\partial_v}{u}, \ \nabla_{\partial_u}\partial_v = \frac{c\partial_u + d\partial_v}{u}, \ \nabla_{\partial_v}\partial_v = \frac{e\partial_u + f\partial_v}{u}. \tag{55}$$

Assume now that ∇ is a locally projectively homogeneous torsion-less connection which is not projectively flat. Then, according to Theorem 2, ∇ is projectively equivalent to a connection $\overline{\nabla}$, which is locally affinely homogeneous. This connection cannot be a Levi-Civita connection of constant curvature, or a connection with constant Christoffel symbols because these connections are obviously projectively flat. Thus, it must be of type B from Theorem 5. The rest follows immediately. □

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