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Uncountable Families of Partial Clones Containing Maximal Clones

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Abstract. Let A be a non singleton finite set. We show that every maximal clone determined by a prime affine or h-universal relation on A is contained in a family of partial clones on A of continuum cardinality.

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1. Introduction

Let $k \geq 2$ and $\mathbf{k}$ be a k-element set. Denote by $\mathrm{Par}(\mathbf{k})$ the set of all partial functions on $\mathbf{k}$ and let $\mathrm{Op}(\mathbf{k})$ be the set of all everywhere defined functions on $\mathbf{k}$. A *partial clone* on $\mathbf{k}$ is a subset of $\mathrm{Par}(\mathbf{k})$ closed under composition and containing all the projections on $\mathbf{k}$. A partial clone contained in $\mathrm{Op}(\mathbf{k})$ is called a *clone* on $\mathbf{k}$. The partial clones on $\mathbf{k}$, ordered by inclusion, form an algebraic dually atomic lattice $\mathcal{P}_k$ (see e.g., [2, 7]). The set of all clones on $\mathbf{k}$, ordered by inclusion, forms a dually atomic sublattice $\mathcal{O}_k$ of $\mathcal{P}_k$ (see [16], p. 80). In 1941 E. L. Post fully described the lattice $\mathcal{O}_2$ ([17]), which is countably infinite and quite exceptional among the lattices $\mathcal{O}_k$; indeed $\mathcal{O}_k$ is of continuum cardinality whenever $k \geq 3$ ([12]). The study of partial clones on a 2-element set was initiated by Freivald in 1966 who described all 8 maximal elements of $\mathcal{P}_2$ and showed that this lattice is

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of continuum cardinality ([4]). The lattices $\mathcal{O}_k$ ($k \geq 3$) and $\mathcal{P}_k (k \geq 2)$ are quite unknown, and so a significant effort was concentrated on special parts of them, mainly the upper and lower parts (for lists of references see [22] for the total case and [3, 9, 14, 23] for the partial case). A remarkable result in Universal Algebra is the classification of all maximal elements of $\mathcal{O}_k$ due to Ivo G. Rosenberg for arbitrary $k \geq 3$. His result will be discussed and used in part in this paper.

The *total component* of a partial clone C is the clone $C \cap \mathrm{Op}(\mathbf{k})$. A natural problem arises here: given a total clone M, describe the set $\widetilde{M}$ of all partial clones whose total component is M. This problem was first considered by Alekseev and Voronenko in [1], followed by Strauch in [24, 25] for some maximal clones over $\{0, 1\}$. Implicit results in this direction can be found in [8, 10, 19]. The same problem has been studied in depth in the paper [11] for maximal clones in the general case. It is well known that the maximal clones on $\mathbf{k}$ ($k \geq 3$), as classified by Rosenberg, are grouped into six different families (see Theorem 1 below or, e.g., [21, 22]). Maximal clones from four of these families are considered in [11][1]. The interval $\widetilde{M}$ is completely described if M is a maximal clone determined by either a central or equivalence relation on $\mathbf{k}$. In both cases the interval $\widetilde{M}$ is finite. Now if the maximal clone M is determined by a bounded order, then a finite subinterval of $\widetilde{M}$ contained in the strong closure of M (see section 2 for the definition) is described in [11]. We point out here that describing the interval $\widetilde{M}$ for a maximal clone M determined by a bounded order may turn out to be a very difficult task. Finally it is shown in [11] that $\widetilde{M}$ is finite if M is a maximal clone determined by a fixed-point-free permutation consisting of cycles of same length p, where p is a prime divisor of k. A complete description of $\widetilde{M}$ is given for the two cases $p = 2, 3$.

In this paper we consider the two families of maximal clones not studied in [11], namely the families of maximal clones determined by prime affine relations and by h-universal relations on $\mathbf{k}$. We show that if M is a maximal clone in either family, then the strong closure of M is contained in uncountably many partial clones. Thus the interval of partial clones $\widetilde{M}$ is of continuum cardinality. We point out here that our result for the prime affine case generalizes one of the main results established in [1], namely that if L denotes the maximal clone of all linear functions on $\{0, 1\}$, then the interval of partial clones $\widetilde{L}$ is of continuum cardinality over $\{0, 1\}$.

2. Basic definitions and notations

Let $k \geq 2$ be an integer and $\mathbf{k} := \{0, 1, \ldots, k-1\}$. For a positive integer n, an n-ary partial function on $\mathbf{k}$ is a map $f : \mathrm{dom}\ (f) \to \mathbf{k}$ where $\mathrm{dom}\ (f) \subseteq \mathbf{k}^n$ is called the *domain* of f. Let $\mathrm{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial functions on $\mathbf{k}$ and let $\mathrm{Par}(\mathbf{k}) := \bigcup_{n\geq 1} \mathrm{Par}^{(n)}(\mathbf{k})$. Moreover set $\mathrm{Op}^{(n)}(\mathbf{k}) := \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) \mid \mathrm{dom}\ (f) =$

[1]The results from [11] are explained also in [15].

$\mathbf{k}^n\}$ and let $\mathrm{Op}(\mathbf{k}) := \bigcup_{n\geq 1} \mathrm{Op}^{(n)}(\mathbf{k})$, i.e., $\mathrm{Op}(\mathbf{k})$ is the set of all total functions on $\mathbf{k}$. In the sequel we will say "function" for "total function".

A partial function $g \in \mathrm{Par}^{(n)}(\mathbf{k})$ is a *subfunction* of $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ (in symbols $g \leq f$ or $g = f|_{\mathrm{dom}\,(g)}$) if $\mathrm{dom}\,(g) \subseteq \mathrm{dom}\,(f)$ and $g(\underline{a}) = f(\underline{a})$ for all $\underline{a} \in \mathrm{dom}\,(g)$.

For every positive integer n, and every $1 \leq i \leq n$, we denote by e_i^n the n-ary function *i-th projection* defined by $e_i^n(x_1, \ldots, x_n) := x_i$ for all $(x_1, \ldots, x_n) \in \mathbf{k}^n$. Furthermore let

$$J_{\mathbf{k}} := \{e_i^n \mid 1 \leq i \leq n < \infty\}$$

be the set of all projections on $\mathbf{k}$.

For $n, m \geq 1$, $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ and $g_1, \ldots, g_n \in \mathrm{Par}^{(m)}(\mathbf{k})$, the *composition* of f and $g_1, \ldots, g_n$, denoted $f[g_1, \ldots, g_n]$ is the m-ary partial function on $\mathbf{k}$ defined by

$$\mathrm{dom}\,(f[g_1, \ldots, g_n]) := \{\underline{v} \in \mathbf{k}^m \mid \underline{v} \in \bigcap_{i=1}^{n} \mathrm{dom}\,(g_i) \text{ and } (g_1(\underline{v}), \ldots, g_n(\underline{v})) \in \mathrm{dom}\,(f)\};$$

and

$$f[g_1, \ldots, g_n](\underline{v}) := f(g_1(\underline{v}), \ldots, g_n(\underline{v}))$$

for all $\underline{v} \in \mathrm{dom}\,(f[g_1, \ldots, g_n])$.

Definitions.

1. A *partial clone* on $\mathbf{k}$ is a composition closed subset of $\mathrm{Par}(\mathbf{k})$ containing $J_{\mathbf{k}}$. A partial clone contained in $\mathrm{Op}(\mathbf{k})$ is called a *clone* on $\mathbf{k}$.

As mentioned earlier, the set of partial clones on $\mathbf{k}$, ordered by inclusion, form an algebraic dually atomic lattice $\mathcal{P}_k$ in which arbitrary infimum is the set-theoretical intersection. For $F \subseteq \mathrm{Par}(\mathbf{k})$, we denote by $\langle F \rangle$ the partial clone *generated* by F, i.e., $\langle F \rangle$ is the intersection of all partial clones containing the set F.

2. A partial clone C is *strong* if it contains all subfunctions of its functions. Furthermore, if C is a clone on $\mathbf{k}$, then we denote by $\mathrm{Str}(C)$ the *strong closure* of C, i.e.,

$$\mathrm{Str}(C) := \{g \in \mathrm{Par}(\mathbf{k}) \mid g \leq f \text{ for some } f \in C\}.$$

It is easy to see that for every clone C the strong closure $\mathrm{Str}(C)$ of C is a strong partial clone on $\mathbf{k}$ containing C (see e.g., [3, 16, 18, 19]).

3. We introduce the concept of partial polymorphisms of a relation. We use the same notation as in [10]. Let $h \geq 1$ and ϱ be an h-ary relation on $\mathbf{k}$, (i.e., $\varrho \subseteq \mathbf{k}^h$), and let f be an n-ary partial function on $\mathbf{k}$. Denote by $\mathcal{M}(\varrho, \mathrm{dom}(f))$ ($\varrho \neq \emptyset$) the set of all $h \times n$ matrices M whose columns $M_{*j} \in \varrho$, for $j = 1, \ldots, n$ and whose rows $M_{i*} \in \mathrm{dom}\,(f)$ for $i = 1, \ldots, h$. We say that f *preserves* ϱ if for every $M \in \mathcal{M}(\varrho, \mathrm{dom}(f))$, the h-tuple $f(M) := (f(M_{1*}), \ldots, f(M_{h*})) \in \varrho$. Set $\mathrm{pPol}\,\varrho := \{f \in \mathrm{Par}(\mathbf{k}) \mid f \text{ preserves } \varrho\}$ and $\mathrm{Pol}\,\varrho = \mathrm{pPol}\,\varrho \cap \mathrm{Op}(\mathbf{k})$ (i.e., $\mathrm{Pol}\,\varrho$ is the set of all (total) functions that preserve the relation ϱ). It is well-known that for every relation ϱ, $\mathrm{Pol}\,\varrho$ is a clone (see e.g. [16]), while $\mathrm{pPol}\,\varrho$ is a strong partial clone called the (partial) clone *determined* by ϱ (see e.g. [18, 19, 14, 3]), (by the results of [18] and [19]), we know even more: a partial clone is strong if and only if it is of the form $\mathrm{pPol}\,Q$ for some set Q of finitary relations).

Notice that partial clones determined by relations are defined in a different but equivalent way in [11].

4. The partial clones on $\mathbf{k}$, ordered by inclusion, form an algebraic lattice ([19]) in which every meet is the set-theoretical intersection. A partial clone C *covers* a partial clone D if $D \subset C$ and the strict inclusions $D \subset C' \subset C$ hold for no partial clone C' on $\mathbf{k}$. Notice that this holds if and only if $\langle D \cup \{g\}\rangle = C$ for each $g \in C \setminus D$. Furthermore a partial clone (a clone) M is a *maximal partial clone* (a *maximal clone*) if M is covered by $\mathrm{Par}(\mathbf{k})$ (is covered by $\mathrm{Op}(\mathbf{k})$).

The main goal of this paper is to study families of partial clones containing some maximal clones on $\mathbf{k}$. We introduce some family of relations on $\mathbf{k}$ for the purpose recalling the Rosenberg classification of all maximal clones over $\mathbf{k}$. For $1 \le h \le k$ set

$$\iota_k^h := \{(a_1, \dots, a_h) \in \mathbf{k}^h \mid a_i = a_j \text{ for some } 1 \le i < j \le n\}.$$

Let $h \ge 1$, ϱ be an h-ary relation on $\mathbf{k}$ and denote by S_h the set of all permutations on $\{1, \dots, h\}$. For $\pi \in S_h$ set

$$\varrho^{(\pi)} := \{(x_{\pi(1)}, \dots, x_{\pi(h)}) \mid (x_1, \dots, x_h) \in \varrho\}.$$

The h-ary relation ϱ is said to be

1) *totally symmetric* (in case $h = 2$ *symmetric*) if $\rho^{(\pi)} = \rho$ for every $\pi \in S_h$,
2) *totally reflexive* (in case $h = 2$ *reflexive*) if $\iota_k^h \subseteq \varrho$,
3) *prime affine* if $h = 4$, $\mathbf{k} = \mathbf{p}^m$ where p is a prime number, $m \ge 1$, $\mathbf{p} := \{0, \dots, p-1\}$ and we can define an elementary Abelian p-group $< \mathbf{k}, + >$ on $\mathbf{k}$ so that
$$\rho := \{(\underline{a}, \underline{b}, \underline{c}, \underline{d}) \in \mathbf{k}^4 \mid \underline{a} + \underline{b} = \underline{c} + \underline{d}\}.$$
4) *central*, if $\varrho \neq \mathbf{k}^h$, ϱ is totally symmetric, totally reflexive and $\{c\} \times \mathbf{k}^{h-1} \subseteq \varrho$ for some $c \in \mathbf{k}$. Notice that for $h = 1$ each $\emptyset \neq \varrho \subset \mathbf{k}$ is central and for $h \ge 2$ such c is called a *central element* of ϱ,
5) *elementary*, if $k = h^m$, $h \ge 3$, $m \ge 1$ and
$$(a_1, a_2, \dots, a_h) \in \rho \iff (\forall i \in \{0, \dots, m-1\}\ (a_1^{[i]}, a_2^{[i]}, \dots, a_h^{[i]}) \in \iota_h^h),$$
where $a^{[i]}$ ($a \in \{0, 1, \dots, h^{m-1}\}$) denotes the i-th digit in the h-adic expansion
$$a = a^{[m-1]} \cdot h^{m-1} + a^{[m-2]} \cdot h^{m-2} + \dots + a^{[1]} \cdot h + a^{[0]},$$
6) a *homomorphic inverse image of an h-ary relation ϱ' on* $\mathbf{k}'$, if there exists a surjective mapping $q : \mathbf{k} \longrightarrow \mathbf{k}'$ with
$$(a_1, \dots, a_h) \in \varrho \iff (q(a_1), \dots, q(a_h)) \in \varrho'$$
for all $a_1, \dots, a_h \in \mathbf{k}$,
7) *h-universal*, if ϱ is a homomorphic inverse image of an h-ary elementary relation.

Denote by

$\mathcal{C}_k$ the set of all central relations on $\mathbf{k}$;

$\mathcal{C}_k^h$ the set of all h-ary central relations on $\mathbf{k}$;

$\mathcal{U}_k$ the set of all non-trivial equivalence relations on $\mathbf{k}$;

$P_{k,p}$ the set of all fixed point-free permutations on $\mathbf{k}$ consisting of cycles of the same prime length p ;

$\mathcal{S}_{k,p} := \{s^0 \mid s \in P_{k,p}\}$, where $s^0 := \{(x, s(x)) \mid x \in \mathbf{k}\}$ is the *graph* of s;

$\mathcal{S}_k := \bigcup\{\mathcal{S}_{k,p} \mid p \text{ is a prime divisor of } k\}$;

$\mathcal{M}_k$ the set of all order relations on $\mathbf{k}$ with a least and a greatest element;

$\mathcal{M}_k^\star$ the set of all lattice orders on $\mathbf{k}$;

$\mathcal{L}_k$ the set of all prime affine relations on $\mathbf{k}$;

$\mathcal{B}_k$ the set of all h-universal relations, $3 \leq h \leq k-1$.

The Rosenberg classification of all maximal clones on $\mathbf{k}$ is based on the above relations. We have:

Theorem 1. ([21]) *Let $k \geq 2$. Every proper clone on $\mathbf{k}$ is contained in a maximal one. Moreover a clone M is a maximal clone over $\mathbf{k}$ if and only if $M = \mathrm{Pol}\,\rho$ for some relation $\rho \in \mathcal{C}_k \cup \mathcal{M}_k \cup \mathcal{S}_k \cup \mathcal{U}_k \cup \mathcal{L}_k \cup \mathcal{B}_k$.*

We say that a partial clone C over $\mathbf{k}$ is of type $\mathcal{X} \in \{\mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{U}, \mathcal{L}, \mathcal{B}\}$ if $C \cap Op(\mathbf{k}) = \mathrm{Pol}\,\varrho$ for some $\varrho \in \mathcal{X}_k$. As mentioned earlier, the two authors together with I. G. Rosenberg studied partial clones of type $\mathcal{C}, \mathcal{M}, \mathcal{S}, \mathcal{U}$ in [11] and the present paper is devoted to the study of partial clones of type $\mathcal{B}, \mathcal{L}$. Our goal is to show the following:

Theorem 2. *Let $k \geq 3$ and M be a maximal clone determined by either an h-universal or prime affine relation or on $\mathbf{k}$. Then the set of partial clones containing M has the cardinality of continuum on $\mathbf{k}$.*

It is shown in [9] that every maximal clone is contained in exactly one maximal partial clone over $\mathbf{k}$. Moreover maximal partial clones containing maximal clones are all described in [9]. In particular it is shown that:

Proposition 3. ([9]) *Let $k \geq 2$. Every maximal clone is contained in exactly one maximal partial clone over $\mathbf{k}$. Let $M = \mathrm{Pol}\,\rho$ be a maximal clone over $\mathbf{k}$, then*

(i) *if ρ is an h-universal relation, then $\mathrm{pPol}\,\rho$ is the unique maximal partial clone over $\mathbf{k}$ that contains M.*

(ii) *if ρ is prime affine then $\mathbf{k} = \mathbf{p}^m$ where p is prime, $m \geq 1$ and*

$$\rho = \{(\underline{a}, \underline{b}, \underline{c}, \underline{d}) \in (\mathbf{p}^m)^4 \mid \underline{a} + \underline{b} = \underline{c} + \underline{d}\}.$$

Let λ_p be the p-ary relation on $\mathbf{p}^m$ defined by

$$\lambda_p := \{(\underline{a}, \underline{a} + \underline{b}, \underline{a} + 2 \cdot \underline{b}, \ldots, \underline{a} + (p-1) \cdot \underline{b}) \mid \underline{a}, \underline{b} \in \mathbf{p}^m\},$$

where $+$ and $\cdot$ are the operations of the vector space $\mathbf{p}^m$ on the field $\mathbf{p}$. Then $\mathrm{pPol}\,\lambda_p$ *is the maximal partial clone on $\mathbf{k}$ that properly contains the partial clone* $\mathrm{pPol}\,\varrho$ *(and consequently contains the maximal clone* $\mathrm{Pol}\,\varrho$*).* $\square$

3. Intervals of partial clones of type $\mathcal{B}$

Let $h \geq 3$, $m \geq 1$ and k be such that $3 \leq h^m \leq k$. Let $\mathbf{m} := \{0, \ldots, m-1\}$ and $\mathbf{h^m} := \{0, 1, \ldots, h^m - 1\}$. In the sequel ζ_m denotes an h-ary elementary relation on $\mathbf{h^m}$, i.e.,

$$(a_0, a_1, \ldots, a_{h-1}) \in \zeta_m \iff \forall i \in \mathbf{m} : \ (a_0^{[i]}, a_1^{[i]}, \ldots, a_{h-1}^{[i]}) \in \iota_h^h$$

for all $a_0, \ldots, a_{h-1} \in \mathbf{h^m}$.

Furthermore let $\rho \in \mathcal{B}_k$ be an h-ary universal relation that is a homomorphic inverse image of ζ_m, i.e., there is a surjective mapping $q : \mathbf{k} \longrightarrow \mathbf{h^m}$ with

$$(a_1, \ldots, a_h) \in \varrho \iff (q(a_1), \ldots, q(a_h)) \in \zeta_m$$

for all $a_1, \ldots, a_h \in \mathbf{k}$.

We need the following characterization of functions preserving ζ_m and ρ given by the second author in [13] (see also [15], Theorem 5.2.6.1).

Lemma 4. 1) *Let $f \in \mathrm{Op}^{(n)}(\mathbf{h^m})$ and $f_0, \ldots, f_{m-1}$ be the n-ary functions in $\mathrm{Op}(\mathbf{h^m})$ defined by*

$$f_i(x_1, \ldots, x_n) := (f(x_1, \ldots, x_n))^{[i]}$$

for all $i = 0, 1, \ldots, m-1$, i.e.,

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{m-1} f_i(x_1, \ldots, x_n) \cdot h^i$$

holds for all $x_1, \ldots, x_n \in \mathbf{h^m}$. Then

$$\begin{aligned} f \in \mathrm{Pol}\, \zeta_m \iff & \forall i \in \{0, 1, \ldots, m-1\} : \\ & \textit{either } |\mathrm{im}(f_i)| \leq h-1 \\ & \textit{or there are } j \in \{1, \ldots, n\},\ v \in \mathbf{m}, \textit{a permutation } s \textit{ on } \mathbf{h} \\ & \textit{such that } f_i(x_1, \ldots, x_n) = s((x_j)^{[v]}). \end{aligned}$$

2) *Let $f \in \mathrm{Op}^{(n)}(\mathbf{k})$ and $f_0, \ldots, f_{m-1}$ be the n-ary functions in $\mathrm{Op}(\mathbf{k})$ defined by*

$$f_i(x_1, \ldots, x_n) := (q(f(x_1, \ldots, x_n)))^{[i]}$$

for all $i = 0, 1, \ldots, m-1$. Then

$$\begin{aligned} f \in \mathrm{Pol}\, \rho \iff & \forall i \in \{0, 1, \ldots, m-1\} : \\ & \textit{either } |\mathrm{im}(f_i)| \leq h-1 \\ & \textit{or there are } j \in \{1, \ldots, n\},\ v \in \mathbf{m},\ \textit{a permutation } s \textit{ on } \mathbf{h} \\ & \textit{such that } f_i(x_1, \ldots, x_n) = s((q(x_j))^{[v]}) \end{aligned} \tag{1}$$

□

We illustrate this with the following

Examples. Let $h = 3$, $m = 2$, $k = 11$, $q : \mathbf{11} \longrightarrow \mathbf{9}$ be defined by $q(x) := x + 1\ (mod\, 9)$ for $x \in \mathbf{9}$, $q(9) = 4$ and $q(10) = 1$. Furthermore let the two permutations s_1 and s_2 be defined by

x	$s_1(x)$	$s_2(x)$
0	1	2
1	0	0
2	2	1

For $x = x^{[1]} \cdot 3 + x^{[0]} \in \mathbf{9}$, let $g(x) := s_1(x^{[0]})$ and $g'(x) := s_2(x^{[1]})$, i.e.,

x	$x^{[1]}$	$x^{[0]}$	$g(x)$	$g'(x)$
0	0	0	1	2
1	0	1	0	2
2	0	2	2	2
3	1	0	1	0
4	1	1	0	0
5	1	2	2	0
6	2	0	1	1
7	2	1	0	1
8	2	2	2	1

Then the ternary functions $f, h \in \mathrm{Op}(\mathbf{9})$ defined by

$$f(x_1, x_2, x_3) := g(x_1) \cdot 3 + g'(x_3)$$

(here $f_0(x_1, x_2, x_3) = g'(x_3) = s_2(x_3^{[1]})$ and $f_1(x_1, x_2, x_3) = g(x_1) = s_1(x_1^{[0]})$) and

$$h(x_1, x_2, x_3) := g(x_2) \cdot 3 + f'(x_1, x_2, x_3),$$

where $\mathrm{im}(f') \subset \{0, 1, 2\}$ and $|\mathrm{im}(f)| \leq 2$, both belong to Pol ζ_2.

The following is an example of a unary function $f \in \mathrm{Op}(\mathbf{11})$ (see last column below) that preserves the relation ρ :

x	$q(x)$	$(q(x))^{[1]}$	$(q(x))^{[0]}$	$s_1((q(x))^{[1]})$	$r(x)$	$q(f(x))$	$f(x)$
0	1	0	1	1	1	4	3
1	2	0	2	1	1	4	9
2	3	1	0	0	1	1	10
3	4	1	1	0	1	1	0
4	5	1	2	0	0	0	8
5	6	2	0	2	0	6	5
6	7	2	1	2	0	6	5
7	8	2	2	2	1	7	6
8	0	0	0	1	1	4	3
9	4	1	1	0	1	1	0
10	1	0	1	1	1	4	9

since

$$q(f(x)) = s_1((q(x))^{[1]}) \cdot 3 + r(x).$$

Now let ζ_m, ρ and q be as discussed in the beginning of this section. As the mapping $q : \mathbf{k} \to \mathbf{h}^{\mathbf{m}}$ is surjective and $m \geq 1$, we have $|\mathrm{im}(q)| \geq h$ and so there are $i_0, \ldots, i_{h-1} \in \mathbf{k}$ such that $\{q(i_0), \ldots, q(i_{h-1})\} = \mathbf{h}$. For notational ease we may assume that

$$\forall i \in \{0, 1, \ldots, h-1\} : \ q(i) = i, \tag{2}$$

and as $(0, 1, \ldots, h-1) \notin \zeta_m$ we have $(0, 1, \ldots, h-1) \notin \rho$. For $n \geq 2$ set

$$\begin{aligned}
\iota_{2n+h} &:= \{(a_1, \ldots, a_{2n+h}) \in \mathbf{h}^{2n+h} \mid |\{a_1, \ldots, a_{2n+h}\}| \leq h-1\},\\
\chi_{2n+h} &:= \{(a_1, \ldots, a_{2n+h}) \in \mathbf{h}^{2n+h} \mid |\{a_1, \ldots, a_{2n+h}\}| = h \text{ with}\\
&\qquad \text{1) } h-2 \text{ symbols occurring each once and}\\
&\qquad \text{2) one symbol occurring twice and}\\
&\qquad \text{3) one symbol occurring } 2n \text{ times in } a_1, \ldots, a_{2n+h}\ \}
\end{aligned}$$

and

$$\sigma_{2n+h} := \iota_{2n+h} \cup \chi_{2n+h}.$$

The relations σ_{2n+h} have been defined in Theorem 11 of [10], and were combined with an infinite family of partial functions to exhibit a family of partial clones of continuum cardinality. We will use some of the results related to these relations and established in [10] later on in Lemma 8.

Now using the mapping q and the relations σ_{2n+h} we define the family of relations $\sigma^{\star}_{2n+h}$ as follows :

$$\begin{aligned}
(a_1, \ldots, a_h) \in \sigma^{\star}_{2n+h} \ :\Longleftrightarrow\ & \forall i \in \{0, 1, \ldots, m-1\} :\\
& ((q(a_1))^{[i]}, (q(a_2))^{[i]}, \ldots, (q(a_h))^{[i]}) \in \sigma_{2n+h}.
\end{aligned}$$

The relations $\sigma^{\star}_{2n+h}$ and σ_{2n+h} are closely related. First we show that they have same restrictions on the set $\mathbf{h}$:

Lemma 5.

$$\sigma^{\star}_{2n+h} \cap \mathbf{h}^{2n+h} = \sigma_{2n+h} \cap \mathbf{h}^{2n+h}. \tag{3}$$

Proof. Obviously, by (2), we have

$$\begin{aligned}
&(a_1, \dots, a_{2n+h}) \in \sigma^{\star}_{2n+h} \cap \mathbf{h}^{2n+h} \implies \\
&(\ (a_1^{[0]}, \dots, a_{2n+h}^{[0]}) = (a_1, \dots, a_{2n+h}) \in \sigma_{2n+h}\ ) \quad \wedge \\
&(\forall i \in \{1, 2, \dots, m-1\} : a_1^{[i]} = \dots = a_{2n+h}^{[i]} = 0).
\end{aligned}$$

On the other hand, if $(a_1, \dots, a_{2n+h}) \in \sigma_{2n+h} \cap \mathbf{h}^{2n+h}$, then $(a_1^{[i]}, \dots, a_{2n+h}^{[i]}) \in \sigma_{2n+h}$ for all $i \in \mathbf{m}$ and, by the definition of $\sigma^{\star}_{2n+h}$, $(a_1, \dots, a_{2n+h}) \in \sigma^{\star}_{2n+h} \cap \mathbf{h}^{2n+h}$ holds. □

Our next result shows that pPol $\sigma^{\star}_{2n+h}$ is a partial clone of type $\mathcal{B}$ for all $h \geq 3$ and all $n \geq 2$.

Lemma 6. *Let* $n \geq 2$. *Then* Pol $\rho \subseteq$ pPol $\sigma^{\star}_{2n+h} \subseteq$ pPol ρ.

Proof. First we show that Pol $\rho \subseteq$ pPol $\sigma^{\star}_{2n+h}$. Let $f \in$ Pol ρ be n-ary and as in Lemma 4 let $f_0, \dots, f_{m-1}$ be the n-ary functions defined by

$$f_i(x_1, \dots, x_n) := (q(f(x_1, \dots, x_n)))^{[i]}$$

for all $i = 0, 1, \dots, m-1$. Thus

$$q(f(x_1, \dots, x_n)) = \sum_{i=0}^{m-1} f_i(x_1, \dots, x_n) \cdot h^i$$

for all $(x_1, \dots, x_n) \in \mathbf{k}^n$. As $f \in$ Pol ρ, the functions $f_0, \dots, f_{m-1}$ satisfy (4) (Lemma 4). We show that $f \in$ Pol $\sigma^{\star}_{2n+h}$. Let $(r_{t,i})$ be a $(2n+h) \times n$ matrix with all columns $(r_{1i}, r_{2i}, \dots, r_{2n+h,i}) \in \sigma^{\star}_{2n+h}$, $i = 1, \dots, n$. We show that

$$(f(r_{11}, r_{12}, \dots, r_{1n}), \dots, f(r_{2n+h,1}, r_{2n+h,2}, \dots, r_{2n+h,n})) \in \sigma^{\star}_{2n+h}$$

and this holds if and only if

$$(q(f(r_{11}, r_{12}, \dots, r_{1n}))^{[i]}, \dots, q(f(r_{2n+h,1}, r_{2n+h,2}, \dots, r_{2n+h,n}))^{[i]}) \in \sigma_{2n+h}$$

for all $i = 0, 1, \dots, m-1$. But

$$\begin{aligned}
&(q(f(r_{11}, r_{12}, \dots, r_{1n}))^{[i]}, \dots, q(f(r_{2n+h,1}, r_{2n+h,2}, \dots, r_{2n+h,n}))^{[i]}) \\
&= (f_i(r_{11}, \dots, r_{1n}), \dots, f_i(r_{2n+h,1}, \dots, r_{2n+h,n})) \\
&\in \begin{cases} \iota_{2n+h} & \text{if} \quad |\mathrm{im}(f_i)| \leq h-1, \\ \sigma_{2n+h} & \text{if} \quad f_i(x_1, \dots, x_n) = s((q(x_j))^{[v]}), \end{cases}
\end{aligned}$$

for all $i = 0, 1, \dots, m-1$. Thus f preserves the relation $\sigma^{\star}_{2n+h}$.

We use Proposition 3 to prove that pPol $\sigma^{\star}_{2n+h} \subseteq$ pPol ρ. Since the lattice $\mathcal{P}_k$ is dually atomic, each of the partial clones pPol $\sigma^{\star}_{2n+h}$ is contained in at least one maximal partial clone. Now by Proposition 3 the maximal clone Pol ϱ is contained in a unique maximal partial clone over $\mathbf{k}$, namely pPol ρ. If the inclusion pPol $\sigma^{\star}_{2n+h} \subseteq$ pPol ρ does not hold for some $n \geq 2$ and some $h \geq 3$, then pPol $\sigma^{\star}_{2n+h}$ would be contained in a maximal partial clone distinct from pPol ϱ, and so Pol ρ would be contained in two distinct maximal partial clones, contradicting Proposition 3. □

We need an infinite family of partial functions φ_{2n+h} defined in [10]. Let

$$\begin{aligned} v_0 &= (x_0, x_1, \dots, x_{2n+h-1}) \\ &:= (0, 1, 2, \dots, h-3, h-2, h-2, \underbrace{h-1, h-1, \dots, h-1}_{2n \text{ times}}) \end{aligned}$$

and, for $j = 0, \dots, 2n+h-1$, let

$$v_j := (x_j, x_{1+j \ (mod\ 2n+h)}, x_{2+j \ (mod\ 2n+h)}, \dots, x_{2n+h-1+j \ (mod\ 2n+h)}).$$

For $n \geq 2$ let φ_{2n+h} be the $(2n+h)$-ary function defined by

$$\text{dom } (\varphi_{2n+h}) := \{v_0, v_1, \dots, v_{2n+h-1}\}$$

and

$$\varphi_{2n+h}(x_1, \dots, x_{2n+h}) := \begin{cases} h-1 & \text{if } (x_1, \dots, x_{2n+h}) = v_{h-1}, \\ x_1 & \text{if } (x_1, \dots, x_{2n+h}) \\ & \quad \in \{v_1, \dots, v_{h-2}, v_h, \dots, v_{2n+h-1}\}. \end{cases}$$

Lemma 7. *Let $n, m \geq 2$. Then*

$$\varphi_{2n+h} \in \text{pPol } \ \sigma^{\star}_{2m+h} \iff \varphi_{2n+h} \in \text{pPol } \ \sigma_{2m+h}.$$

Proof. This follows from (3) and

$$\begin{aligned} &\forall (r_{11}, \dots, r_{2m+h,1}), \dots, (r_{1,2n+h}, \dots, r_{2m+h,2n+h}) \in (\sigma^{\star}_{2m+h} \cup \sigma_{2m+h}) \backslash \mathbf{h^{2m+h}} : \\ &\{(r_{11}, \dots, r_{1,2n+h}), \dots, (r_{2m+h,1}, \dots, r_{2m+h,2n+h})\} \not\subseteq \text{dom } (\varphi_{2n+h}). \end{aligned}$$ □

The following result comes from the proof of Theorem 11 in [10]:

Lemma 8. *Let $n,\ m \geq 2$. Then*

$$\varphi_{2m+h} \in \text{pPol } \sigma_{2n+h} \iff n \neq m.$$ □

We combine Lemmas 7, 8 to obtain

Lemma 9. *Let $n,\ m \geq 2$. Then*

$$\varphi_{2m+h} \in \text{pPol } \sigma^{\star}_{2n+h} \iff n \neq m.$$ □

Let $\mathcal{P}(N_{\geq 2})$ be the power set of $N_{\geq 2} := \{2, 3, \dots\}$. From Lemma 9, the correspondence

$$\chi : \mathcal{P}(N_{\geq 2}) \longrightarrow [\text{Str (Pol } \rho), \text{pPol } \ \rho]$$

defined by

$$\chi(X) := \bigcap_{n \in N_{\geq 2} \backslash X} \text{pPol } \ \sigma^{\star}_{2n+h}$$

$(X \in \mathcal{P}(N_{\geq 2}))$ is a one-to-one mapping. We have shown

Theorem 10. *Let $k \geq 3$ and $\rho \in \mathcal{B}_k$. Then the interval of partial clones* $[\text{Str(Pol } \ \rho), \text{pPol } \ \rho]$ *is of continuum cardinality on* $\mathbf{k}$. □

4. Intervals of partial clones of type $\mathcal{L}$

In this section we consider a maximal clone $L = \mathrm{Pol}\,\varrho$ where $\rho \in \mathcal{L}_k$ is a prime affine relation on $\mathbf{k}$. Thus $k = p^\ell$ for some $\ell \geq 1$, p is a prime number and $\varrho = \{(x,y,z,t) \in \mathbf{k}^4 \mid x+y = z+t\}$, where $\langle \mathbf{k}, + \rangle$ is an elementary Abelian p-group. Choose the notation so that $\langle \mathbf{k}, + \rangle = \langle \mathbf{p}^\ell, \oplus \rangle = \underbrace{\langle \mathbf{p}, + \rangle \times \ldots \times \langle \mathbf{p}, + \rangle}_{\ell}$ where $\langle \mathbf{p}, + \rangle$ is the cyclic group *mod* p on $\mathbf{p} := \{0, \ldots, p-1\}$. We will use the description given in [21] of the maximal clone L (see also [9]). Let $\mathbf{p}^{\ell \times \ell}$ be the set of all square matrices of size ℓ with entries from $\mathbf{p}$.

Proposition 11. [21] *Let* $\mathbf{k} = \mathbf{p}^\ell$, $\rho \in \mathcal{L}_k$ *and* $L = \mathrm{Pol}\,\rho$ *be as defined above. Then*

$$L = \bigcup_{n \geq 1} \{ f \in \mathrm{Op}^{(n)}(\mathbf{p}^\ell) \mid \exists\, \underline{a} \in \mathbf{p}^\ell,\ \exists\, A_1, \ldots, A_n \in \mathbf{p}^{\ell \times \ell} \text{ such that } \forall\, x_1, \ldots, x_n \in \mathbf{p}^\ell \ (\ f(x_1, \ldots, x_n) = \underline{a} \oplus \sum_{i=1}^{n} x_i \otimes A_i\ )\},$$

where $\oplus$ *and* $\otimes$ *are the usual matrix operations over the finite field* $(\mathbf{p};\ +,\ \cdot)$. $\square$

In the sequel we write E for $\mathbf{p}^\ell$.

Remark. The binary sum of the elementary Abelian p-group E is denoted by $\oplus$. For every $a \in E$ denote by $c_a \in \mathrm{Op}^{(1)}(E)$ the unary constant function defined by $c_a(x) := a$ for all $x \in E$. Moreover for every square matrix $A \in \mathbf{p}^{\ell \times \ell}$ let $\otimes_A \in \mathrm{Op}^{(1)}(E)$ denote the unary function defined by $\otimes_A(x) := x \otimes A$ for all $x \in E$. Put

$$L' := \{\oplus\} \cup \{c_a \mid a \in E\} \cup \{\otimes_A \mid A \in \mathbf{p}^{\ell \times \ell}\},$$

then using Proposition 11 one can easily verify that L' is a *generating set* for the maximal clone L, i.e., $L = \langle L' \rangle$. This fact will be used in the proof of Lemma 12. We will also use the Definability Lemma shown in [18] and used in [5, 7, 10]. It gives necessary and sufficient conditions under which $\mathrm{pPol}\,\lambda_1$ is contained in $\mathrm{pPol}\,\lambda_2$ for two relations λ_1 and λ_2.

We need to introduce some notations that will be used later on. For $x := (x_1, \ldots, x_\ell) \in E$ and $y \in E$, let $\ominus x := (p - x_1\ (mod\,p), \ldots, p - x_\ell\ (mod\,p))$ and $x \ominus y := x \oplus (\ominus y)$. Furthermore let $-1 := p-1$, and

$$\mathbf{0} := (0,0,\ldots,0),\ \mathbf{1} := (1,1,\ldots,1),\ -\mathbf{1} := (-1,-1,\ldots,-1) \in E.$$

If M is a nonempty set, then by $M^{r \times s}$ we denote the set of all $r \times s$ matrices with entries from M.

If $a := (a_1, a_2, \ldots, a_\ell) \in E$, then we denote by $a[i]$ the i-th coordinate of a, i.e., $a[i] := a_i$ for all $i \in \{1, \ldots, \ell\}$. For $r \geq 2p$ let

$$\lambda_r := \{(x_1, x_2, \ldots, x_r) \in E^r \mid x_1 \oplus x_2 \oplus \ldots \oplus x_r = \mathbf{0}\}.$$

For $x := (x_1, \ldots, x_n) \in E^n$ where $x_i := (x_{i1}, \ldots, x_{i\ell}) \in E$, for all $1 \le i \le n$, let

$$\widehat{x} := (x_{11}, \ldots, x_{1\ell}, x_{21}, \ldots, x_{2\ell}, \ldots, x_{n1}, \ldots, x_{n\ell}) \in \mathbf{p}^{n\ell}.$$

For $1 \le s \le n$ let

$$T_{n;s} := \{(x_1, \ldots, x_n) \in E^n \mid (\widehat{x_1, \ldots, x_n}) \in \{0,1\}^{n\ell} \text{ and } \sum_{i=1}^{n} \sum_{j=1}^{\ell} x_{ij} \le s\,\ell\},$$

thus $(x_1, \ldots, x_n) \in T_{n;s}$ iff $(\widehat{x_1, \ldots, x_n}) \in \{0,1\}^{n\ell}$ and the number of 1's in $(\widehat{x_1, \ldots, x_n})$ is at most $s\ell$.
For $2p \le r$ and $1 \le s \le pr - 1$ let $\tau_{r,s} \in \mathrm{Par}(E)$ denote the partial function with the arity

$$n(r,s) := (pr-1)s + 1$$

and defined by

$$\mathrm{dom}\ (\tau_{r,s}) := T_{n(r,s);s} \cup \{(-\mathbf{1}, \ldots, -\mathbf{1})\}$$

and

$$\tau_{r,s}(x) := \begin{cases} \mathbf{0} & \text{for} \quad x \in T_{n(r,s);s}, \\ \mathbf{1} & \text{for} \quad x_1 = \cdots = x_{n(r,s)} = -\mathbf{1}. \end{cases}$$

Lemma 12. *Let* $2p \le r$ *and* $1 \le s \le p\,r - 1$. *Then*
(a) $\tau_{r,s} \in \mathrm{pPol}\ \ \lambda_{pr}$,
(b) $\tau_{r,s} \notin \mathrm{pPol}\ \ \lambda_{p(r+1)}$,
(c) $\mathrm{pPol}\ \ \lambda_{p(r+1)} \subset \mathrm{pPol}\ \ \lambda_{pr}$,
(d) $\mathrm{Str}\ (L) \subseteq \mathrm{pPol}\ \ \lambda_{pr}$,
(e) $\mathrm{Op}(E) \cap \mathrm{pPol}\ \lambda_{pr} = L$.

Proof. To simplify the notation we write n instead of $n(r,s)$ (i.e., $n = (p \cdot r - 1) \cdot s + 1$), τ for $\tau_{r,s}$, λ for λ_{pr} and m for $p \cdot r$.
(a) We proceed by contradiction. Assume that $\tau \notin \mathrm{pPol}\ \lambda$. Then there is a matrix $A := (a_{ij}) \in E^{m \times n}$ such that

$$\forall i \in \{1, 2, \ldots, m\} : \ r_i := (a_{i1}, a_{i2}, \ldots, a_{in}) \in \mathrm{dom}\ (\tau), \tag{4}$$

$$\forall j \in \{1, 2, \ldots, n\} : \ (a_{1j}, a_{2j}, \ldots, a_{mj}) \in \lambda, \tag{5}$$

and

$$(\tau(r_1), \tau(r_2), \ldots, \tau(r_m)) \in E^m \setminus \lambda. \tag{6}$$

Clearly there is a row in A of the form $(-\mathbf{1}, -\mathbf{1}, \ldots, -\mathbf{1})$ since otherwise $r_i \in T_{n(r,s);s}$ for all $i = 1, \ldots, m$ and thus $(\tau(r_1), \tau(r_2), \ldots, \tau(r_m)) = (\mathbf{0}, \ldots, \mathbf{0}) \in \lambda$. W.l.o.g. we can assume that

$$r_1 = \cdots . = r_t = (-\mathbf{1}, -\mathbf{1}, \ldots, -\mathbf{1}) \tag{7}$$

and

$$\{\widehat{r_{t+1}}, \dots, \widehat{r_m}\} \subseteq \{0,1\}^{n\ell}. \tag{8}$$

By (6) and (7) we have $t \neq 0 \ (mod\, p)$.
Then by (5) and (7)

$$\forall j \in \{1, \dots, n\}\ \forall q \in \{1, \dots, \ell\} : \sum_{i=t+1}^{m} a_{ij}[q] \geq 1,$$

i.e.,

$$\sum_{j=1}^{n} \sum_{q=1}^{l} \sum_{i=t+1}^{m} a_{ij}[q] \geq n\ell = ((pr-1)s+1)\ell. \tag{9}$$

Furthermore, it follows from (4) and (8)

$$\forall i \in \{t+1, \dots, m\} : \sum_{j=1}^{n} \sum_{q=1}^{\ell} a_{ij}[q] \leq s\ell,$$

thus

$$\sum_{i=t+1}^{m} \sum_{j=1}^{n} \sum_{q=1}^{\ell} a_{ij}[q] \leq (m-t)sl \leq (pr-1)s\ell,$$

contradicting (9) and thus proving (a).
(b) Consider the matrix with $p(r+1)$ rows $b_1, \dots, b_{p(r+1)}$ and $(pr-1)s+1$ columns

$$B = \begin{pmatrix}
\underbrace{\mathbf{1\,1\,\dots\,1}}_{s} & \mathbf{0\,0\,\dots\,0} & \dots\dots & \mathbf{0\,0\,\dots\,0} & \mathbf{0} \\
\mathbf{0\,0\,\dots\,0} & \underbrace{\mathbf{1\,1\,\dots\,1}}_{s} & \dots\dots & \mathbf{0\,0\,\dots\,0} & \mathbf{0} \\
\vdots & & & & \vdots \\
\mathbf{0\,0\,\dots\,0} & \mathbf{0\,0\,\dots\,0} & \dots\dots & \underbrace{\mathbf{1\,1\,\dots\,1}}_{s} & \mathbf{0} \\
\mathbf{0\,0\,\dots\,0} & \mathbf{0\,0\,\dots\,0} & \dots\dots & \mathbf{0\,0\,\dots\,0} & \mathbf{1} \\
\mathbf{0\,0\,\dots\,0} & \mathbf{0\,0\,\dots\,0} & \dots\dots & \mathbf{0\,0\,\dots\,0} & \mathbf{0} \\
\vdots & & & & \vdots \\
\mathbf{0\,0\,\dots\,0} & \mathbf{0\,0\,\dots\,0} & \dots\dots & \mathbf{0\,0\,\dots\,0} & \mathbf{0} \\
\mathbf{-1}\ \ \dots & \mathbf{-1}\ \ \dots & \dots\dots & \mathbf{-1}\ \dots \mathbf{-1} & \mathbf{-1}
\end{pmatrix}$$

Clearly all columns of B belong to $\lambda_{p(r+1)}$. However

$$(\tau(b_1), \dots, \tau(b_{p(r+1)})) = (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{1}) \in E^{p(r+1)} \setminus \lambda_{p(r+1)},$$

completing the proof of (b).
(c) Since

$$\lambda_{pr} = \{(x_1, \dots, x_{pr}) \in E^{pr} \mid (x_1, x_2, \dots, x_{pr}, \underbrace{x_{pr}, x_{pr}, \dots, x_{pr}}_{p}) \in \lambda_{p(r+1)}\}$$

we have, by the general theory (see e.g., the Definability Lemma in [18]) that

$$\text{pPol } \lambda_{p(r+1)} \subseteq \text{pPol } \lambda_{pr}.$$

As $\tau_{r,s} \in$ pPol $\lambda_{pr} \setminus$ pPol $\lambda_{p(r+1)}$, (c) follows.

(d) As mentioned earlier $L =< L' >$, where $L' := \{\oplus\} \cup \{c_a \mid a \in E\} \cup \{\otimes_A \mid A \in \mathbf{p}^{\ell\times\ell}\}$.

It is easy to see that all functions in L' preserve the relation λ_m, i.e, $L' \subseteq$ pPol λ_m. Thus $L \subseteq$ pPol λ_m and as pPol λ_m is a strong partial clone, Str $(L) \subseteq$ pPol λ_m, proving (d).

(e) From (d) we have $L \subseteq \text{Op}(E) \cap$ pPol $\lambda_{pr} \subset \text{Op}(E)$. Now (e) follows from the maximality of the clone L. □

We need the concept of affine spaces for the next result. For $n \geq 1$ let $(\{0,1\}^n; +, \cdot)$ be the n-dimensional vector space over the field $(\{0,1\}; +, \cdot)$ (with the two usual binary operations *mod* 2). A subset $T \subseteq \{0,1\}^n$ is an *affine space of the dimension* t (in symbols $t :=$ dim T), if

$$T = b + U \ (mod\ 2) := \{b + u \mid u \in U\}$$

where $b \in \{0,1\}^n$ and U is a subspace of $\{0,1\}^n$ of dimension t. The next three results will be used in the proof of Lemma 16. They are essentially useful for the case where $|E|$ is a power of 2. For $1 \leq s \leq n$ let $R_{n,s}$ be the set of all 0-1 n-vectors containing at most s 1's, that is $R_{n,s} := \{(a_1, \ldots, a_n) \in \{0,1\}^n \mid \sum_{i=1}^{n} a_i \leq s\}$. We have:

Lemma 13. *Let $1 \leq s \leq n$ and let $A \subseteq \{0,1\}^n$ be an affine space. Then*
 (a) $A \subseteq R_{n;s} \implies \dim A \leq s$;
 (b) $A \subseteq \{0,1\}^n \setminus R_{n;s} \implies \dim A \leq n - s - 1$.

Proof. The statement in (a) is shown by V. B. Alekseev and L. L. Voronenko in [1].

(b) Let $A \subseteq \{0,1\}^n \setminus R_{n;s}$. Then $A' := (1,1,\ldots,1) + A$ (*mod* 2) is an affine space of same dimension as A and since vectors in A have at least $s+1$ entries equal 1 and since $1 + 1 = 0$, vectors in A' have at most $n - s - 1$ entries equal 1, i.e., $A' \subseteq T_{n;n-s-1}$. Thus by (a) dim $A =$ dim $A' \leq n - s - 1$. □

From Lemma 12 we have that $\tau_{r,s} \in$ pPol λ_{pr} and Str $(L) \subseteq$ pPol λ_{pr}. We now show that if $|E|$ is a power of 2 then there are subfunctions of $\tau_{r,s}$ that belong to Str (L).

Lemma 14. *Let $p = 2$, $E = \{0,1\}^\ell$, $r \geq 2$, $n := (2r-1)s+1$ and $A \subseteq$ dom $(\tau_{r,s})$ be such that $\widehat{A} := \{\widehat{x} \mid x \in A\} \subseteq \{0,1\}^{n\ell}$ is an affine space. Then $\tau_{r,s_{|A}} \in$ Str (L).*

Proof. If $|A| = 1$ or $A \subseteq T_{n;s}$ then by definition $\tau_{r,s_{|A}}$ is a constant function and so it belongs to Str (L). Assume that $|A| \geq 2$ and $A \not\subseteq T_{n;s}$, thus $a := (\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) \in A$ (notice that as $p = 2$ we have here $-1 = 1$). First we deal with the case $|A| = 2$. Let $A = \{a, b\}$ with $b := (b_1, \ldots, b_n) \in (\{0,1\}^\ell)^n \setminus \{a\}$. As $b \neq a$ there is an $1 \leq i \leq n$ with $b_i \neq \mathbf{1}$; say $b_1 \neq \mathbf{1}$. Then there is a matrix $D \in \{0,1\}^{\ell\times\ell}$ with

$b_1 \otimes D = \mathbf{0}\ (mod\ 2)$ and $\mathbf{1} \otimes D = \mathbf{1}\ (mod\ 2)$, i.e., $\tau_{r,s_{|A}}(b_1, \ldots, b_n) = b_1 \otimes D$ and $\tau_{r,s_{|A}}(\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1} \otimes D$. Thus $\tau_{r,s_{|A}} \in \mathrm{Str}\ (L^{(1)})$ follows from Proposition 11.

Next we show that $|A| \geq 3$ is impossible. Indeed if $|A| \geq 3$, then there are two vectors $b, c \in A \cap T_{n,s}$ with $b \neq c$. Therefore $\widehat{b} \oplus \widehat{c} \in \widehat{T_{n;2s}} \setminus \{(\underbrace{0, 0, \ldots, 0}_{n\ell})\}$ and $\widehat{a} \oplus \widehat{b} \oplus \widehat{c} \in \widehat{T_{n;n-1}} \setminus \widehat{T_{n;n-2s}}$. Furthermore, since $\widehat{A}$ is an affine space, we have $\widehat{d} := (d_1, \ldots, d_n) := \widehat{a} \oplus \widehat{b} \oplus \widehat{c} \in \widehat{A}$ and satisfies $\widehat{d} \notin \widehat{T_{n;n-2s}}$. Since $n - 2s = (2r-3)s+1 \geq s+1$, we obtain $\sum_{i=1}^{n} \sum_{j=1}^{\ell} d_{ij} \geq (s+1)\ell$, contradicting $d \in A \cap T_{n,s}$. $\square$

Put

$$\begin{aligned} &s_1 := 1, \\ &s_j := (p \cdot j - 1) \cdot s_{j-1} + 1 \ \text{ for } \ j \geq 2, \\ &\alpha_j := \tau_{j+1,s_j} \ \text{ for } \ j \geq 2, \end{aligned}$$

i.e., the function α_j has the arity $N := n(j+1, s_j) = (p \cdot (j+1) - 1) \cdot s_j + 1$ and

x_1	$\cdots$	$x_i := (x_{i1}, \ldots, x_{i\ell})$	$\cdots$	x_N	$\alpha_j(x_1, \ldots, x_N)$
$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$
a_1	$\cdots$	$a_i := (a_{i1}, \ldots, a_{i\ell})$ $a_i \in \{0,1\}^\ell$ $\sum_{i=1}^{N} \sum_{t=1}^{\ell} a_{it} \leq s_j \cdot \ell$	$\cdots$	a_N	$\mathbf{0}$
$\mathbf{-1}$	$\cdots$	$\mathbf{-1}$	$\cdots$	$\mathbf{-1}$	$\mathbf{-1}$
otherwise					not defined

We remark that α_j was already in [1] defined for $p = 2$ and $\ell = 1$.

Lemma 15. *Let $p \geq 3$, $i < j$, $n := (p(j+1)-1)s_j + 1$, $m := (p(i+1)-1)s_i + 1$, $b \in \mathbf{p}^{m\ell}$ and let $A \in \mathbf{p}^{n\ell \times m\ell}$ be a matrix which is not the zero matrix. Furthermore for $(\gamma, q) \in \{(n, j), (m, i)\}$ let*

$$D_{\gamma,q} := \{(\underbrace{-1, -1, \ldots, -1}_{\gamma\ell})\} \cup \{(x_1, \ldots, x_{\gamma\ell}) \in \{0,1\}^{\gamma\ell} \mid \sum_{t=1}^{\gamma\ell} x_t \leq s_q \ell\}.$$

Then

$$\exists\, x \in D_{n,j} :\ b + x \cdot A\ (mod\ p)\ \notin D_{m,i}.$$

Proof. In the proof below $+$ and $\cdot$ denote the addition and multiplication modulo p.

Let $A := (a_{uv})$. For $1 \leq u \leq n\ell$ and $1 \leq v \leq m\ell$ let
$r_u := (a_{u1}, a_{u2}, \ldots, a_{u,m\ell})$ and $c_v := (a_{1v}, a_{2v}, \ldots, a_{n\ell,v})$
be the u-th row and v-th column of A respectively. Furthermore for $t \geq 2$ let

$$(a)_t := (\underbrace{a, a, \ldots, a}_{t})$$

where $a \in E$, and for $1 \leq u < v \leq t$ let
$e_{t;u} := (\underbrace{0, 0, \ldots, 0}_{u-1}, 1, \underbrace{0, 0, \ldots, 0}_{t-u})$ and $e_{t;u,v} := (\underbrace{0, 0, \ldots, 0}_{u-1}, 1, \underbrace{0, 0, \ldots, 0}_{v-u-1}, 1, \underbrace{0, 0, \ldots, 0}_{t-v})$
and finally let $e_{t;v,u} := e_{t;u,v}$. Thus $e_{t;u,v}$ is the t-vector consisting of 1's at the u and v positions and 0's elsewhere. We proceed by contradiction. Assume that

$$\forall\, x \in D_{n,j} : \; b + x \cdot A \in D_{m,i}. \tag{10}$$

As $(0)_{n\ell} \in D_{n,j}$ we have $b \in D_{m,i}$ and so one of the following three cases occurs: (1) b is a zero vector, (2) b is nonzero 0-1 vector or (3) all entries of b are -1.

Case 1: $b = (0)_{m\ell}$.
Since $e_{n\ell;t} \in D_{n,j}$ and $e_{n\ell;t} \cdot A = r_t$, we deduce from (10)

$$\forall\, t \in \{1, 2, \ldots, n\ell\} : \; r_t \in D_{m,i}, \tag{11}$$

and so one of the following 2 cases is possible:

Case 1.1: $\exists\, q \in \{1, 2, \ldots, n\ell\} : \; r_q = (-1)_{m\ell}$.
If $r_t = (0)_{m\ell}$ for all $t \in \{1, 2, \ldots, n\ell\} \backslash \{q\}$ then $(-1)_{n\ell} \cdot A = (1)_{m\ell} \notin D_{m,i}$. On the other hand if there is a $t \in \{1, 2, \ldots, n\} \setminus \{q\}$ with $r_t \in D_{m,i} \setminus \{(0)_{m\ell}\}$, then $e_{n\ell;q,t} \cdot A \notin D_{m,i}$.

Since the Case 1.1 leads to a contradiction we have:

Case 1.2: $\forall\, q \in \{1, 2, \ldots, n\ell\} : \; r_q \in \{0, 1\}^{m\ell} \setminus \{(-1)_{m\ell}\}$. We distinguish three subcases here:

Case 1.2.1: $\exists\, t \in \{1, 2, \ldots, m\ell\}\; \exists\, u \neq v \in \{1, 2, \ldots, n\ell\} : \; a_{ut} = a_{vt} = 1$. Thus the t-th column of A has the form $(\ldots, c_{u-1,t}, 1, c_{u+1,t}, \ldots, c_{u-1,t}, 1, c_{u+1,t}, \ldots)$ and so

$$e_{n\ell;u,v} \cdot A = r_u + r_v = (\ldots, a_{u,t-1} + a_{v,t-1}, 2, a_{u,t+1} + a_{v,t+1}, \ldots).$$

Now if $p \geq 5$ then $2 \neq -1 \ (mod\, p)$ and thus $e_{n\ell;u,v} \cdot A \notin D_{m,i}$. On the other hand if $p = 3$, then $e_{n\ell;u,v} \cdot A$ belongs to $D_{m,i}$ only if $r_u = r_v = (1)_{m\ell}$, but then $r_u \notin D_{m,i}$, contradicting (11).

Case 1.2.2: Every column in A contains exactly one nonzero entry equal to 1, i.e., $\{c_1, c_2, \ldots, c_{m\ell}\} \subseteq \{e_{m\ell;1}, e_{m\ell;2}, \ldots, e_{m\ell;m\ell}\}$. Since $s_j = (p(j+1) - 1)s_{j-1} + 1$ (notice that the addition and multiplication are over the integers here), and since $i < j$ we have:

$$s_j \geq (p(i+1) - 1) \cdot s_i + 1 = m.$$

Therefore there is an $x \in D_{n,j}$ with $x \cdot A = (1)_{m\ell} \notin D_{m,i}$, a contradiction.

Case 1.2.3: A has a zero column and every column in A has at most one nonzero entry equal to 1. Then $(-1)_{n\ell} \cdot A \notin D_{m,i}$. This contradiction completes the proof for the Case 1.

Case 2: $b \neq (0)_{m\ell}$ is a 0-1 vector. Then w.l.o.g we may assume that all 1's in b are consecutive and occur to the left of the 0's, i.e., $b = (\underbrace{1, 1, \ldots, 1}_{t}, 0, \ldots, 0)$ and as $b \in D_{m,i}$ we have $1 \leq t \leq s_i\ell$.

Since $e_{n\ell;q} \in D_{n,j}$ and $e_{n\ell;q} \cdot A = r_q$ we have by (10) that $\forall q \in \{1, 2, \dots, n\ell\}$:

either $\quad r_q = (\underbrace{-2, -2, \dots, -2}_{t}, \underbrace{-1, -1, \dots, -1}_{m\ell - t})$

or $\quad (a_{q1}, \dots, a_{qt}) \in \{0, -1\}^t$ and $(a_{q,t+1}, \dots, a_{q,m\ell}) \in \{0, 1\}^{m\ell - t}$ and (12)
the number of 0's in $(a_{q1}, \dots, a_{qt})$ plus the number of 1's in $(a_{q,t+1}, \dots, a_{q,m\ell})$ is less or equal to $s_i \ell$.

Then we have four possible cases :

Case 2.1: $\exists q \in \{1, \dots, n\ell\}\ \forall u \in \{1, \dots, n\ell\} \setminus \{q\} :\ r_u = (0)_{m\ell}$.
If A has a zero column, then, since A is not the zero matrix, it is easy to check that $b + x \cdot A \notin D_{m,i}$ for certain $x \in D_{m,j}$. Consequently, we can assume that A does not have any zero column.

First we show that

$$m\ell - t > s_i \ell. \tag{13}$$

Indeed

$$\begin{array}{rcl} m\ell - t \geq \ell(m - s_i) & = & \ell((p(i+1) - 1)s_i + 1 - s_i) \\ & = & \ell((p(i+1) - 2)s_i + 1) \\ & \geq & \ell((3 \times 2 - 2)s_i + 1) \\ & > & \ell s_i. \end{array}$$

Combining this with the fact that A has no zero columns we obtain

$$r_q = (\underbrace{-2, -2, \dots, -2}_{t}, \underbrace{-1, -1, \dots, -1}_{m\ell - t}).$$

But this is a contradiction with (10), since

$$\begin{array}{rcl} b + (-1)_{n\ell} \cdot A & = & (\underbrace{1, 1, \dots, 1}_{t}, \underbrace{0, 0, \dots, 0}_{m\ell - t}) + (\underbrace{2, 2, \dots, 2}_{t}, \underbrace{1, 1, \dots, 1}_{m\ell - t}) \\ & = & (\underbrace{3, 3, \dots, 3}_{t}, \underbrace{1, 1, \dots, 1}_{m\ell - t > s_i \ell}) \\ & \notin & D_{m,i}. \end{array}$$

Case 2.2: $\exists u \neq v \in \{1, 2, \dots, n\ell\}$: $r_u = r_v = (\underbrace{-2, -2, \dots, -2}_{t}, \underbrace{-1, -1, \dots, -1}_{m\ell - t})$.

Here

$$b + e_{n\ell;u,v} \cdot A = (\underbrace{-3, -3, \dots, -3}_{t}, \underbrace{-2, -2, \dots, -2}_{m\ell - t > s_i \ell}) \notin D_{m,i}$$

a contradiction.

Case 2.3: $\exists u \neq v \in \{1, 2, \dots, n\ell\}\ \exists w \in \{1, \dots, t\} :\ a_{uw} = a_{vw} = -1$.
By (12) and (13) we have

$$(a_{u,t+1}, \dots, a_{u,m\ell}) \neq (1)_{m\ell - t} \neq (a_{v,t+1}, \dots, a_{v,m\ell}).$$

Therefore

$$b + e_{n\ell;u,v} \cdot A = (\underbrace{1, 1, \dots, 1}_{t}, \underbrace{0, 0, \dots, 0}_{m\ell - t}) + (\underbrace{\dots}_{w-1}, -2, \underbrace{\dots}_{t-w}, \underbrace{\dots}_{\neq (2,\dots,2)}) =$$
$$(\underbrace{\dots}_{w-1}, -1, \underbrace{\dots}_{t-w}, \underbrace{\dots}_{\neq (2,\dots,2)}) \notin D_{m,i}.$$

Case 2.4: $\forall u, v \in \{1, 2, \ldots, n\ell\}\ \forall w \in \{1, 2, \ldots, t\}$:
$$u \neq v \implies (a_{uw}, a_{vw}) \in \{(0,0), (0,-1), (-1,0)\}.$$
Here we distinguish two subcases:

Case 2.4.1: $\exists u \neq v \in \{1, 2, \ldots, n\ell\}\ \exists q \in \{t+1, \ldots, m\ell\} : a_{uq} = a_{vq} = 1$.
Then this leads to the contradiction

$$b + \underbrace{e_{n\ell;u,v} \cdot A}_{(\ldots,-1,\ldots,2,\ldots)} = (\ldots, 0, \ldots, 2, \ldots) \notin D_{m,i}.$$

Case 2.4.2: $\forall u, v \in \{1, 2, \ldots, n\ell\}\ \forall q \in \{t+1, \ldots, m\ell\}$:
$$u \neq v \implies (a_{uq}, a_{vq}) \in \{(0,0), (0,1), (1,0)\}.$$
Obviously, in this case we have
$$(0)_{n\ell} \neq \{-c_1, \ldots, -c_t, c_{t+1}, \ldots, c_{m\ell}\} \subseteq \{(0)_{nl}, e_{n\ell;1}, e_{n\ell;2}, \ldots, e_{n\ell;n\ell}\}.$$
Hence, there is an $n\ell$-vector $y \in T_{n\ell;s_j}$ with $b + y \cdot A \notin D_{m,i}$, contradicting (10).

Case 3: $b = (-1)_{m\ell}$.

Since $b + r_q \in D_{m,i}$ for all $q \in \{1, 2, \ldots, n\ell\}$, we have $\forall q \in \{1, 2, \ldots, n\ell\}$:

$$r_q \neq (0)_{m\ell} \Longrightarrow r_q \in \{1,2\}^{m\ell} \text{ and the number of 2's in } r_q \text{ is not greater than } s_i\ell. \tag{14}$$

Here one of the following two cases is possible:

Case 3.1: $\exists q \in \{1, \ldots, n\ell\} : (r_q \neq (0)_{m\ell})$ and $(\forall u \in \{1, \ldots, n\ell\} \setminus \{q\} :\ r_u = (0)_{m\ell})$.
It is easy to see that in such a case we have $b + (-1)_{n\ell} \cdot A = b - r_q \notin D_{m,i}$, contradicting (10).

Case 3.2: $\exists u \neq v \in \{1, \ldots, n\ell\} :\ \{r_u, r_v\} \subseteq \{1,2\}^{m\ell}$.
Then $r_u + r_v \in \{2, 3\ (mod\ p), 4\ (mod\ p)\}^{m\ell}$, i.e., $b + r_u + r_v \in \{1, 2, 3\ (mod\ p)\}^{m\ell}$. Clearly $b + r_u + r_v \notin D_{m,i}$ for $p \geq 5$ and so let $p = 3$. By definition of m we have $m > 2s_i$ and thus $m\ell > 2s_i\ell$. Combining this with (14) we get that the vector $b + r_u + r_v$ contains at least one symbol 1 and one symbol 2 ($= -1$) and so $b + r_u + r_v \notin D_{m,i}$. This completes the proof of Lemma 15. □

Lemma 16. *Let* $i \neq j$, $n := (p(j+1)-1)s_j + 1$, $m := (p(i+1)-1)s_i + 1$, $\{g_1, g_2, \ldots, g_m\} \subseteq (\mathrm{Str}\ (L))^{(n)}$ *and*

$$f(x_1, \ldots, x_n) := \alpha_i(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

Then either
$$\mathrm{dom}\ (\alpha_j) \not\subseteq \mathrm{dom}\ (f) \tag{15}$$
or
$$f_{|\mathrm{dom}\ (\alpha_j)} \in \mathrm{Str}\ (L). \tag{16}$$

Proof. We proceed by cases.

Case 1: $i < j$.

Since $g_1, \dots, g_m \in \mathrm{Str}\,(L)$, there are $h_1, \dots, h_m \in L$ such that $g_t \leq h_t$ for $t = 1, 2, \dots, m$.

Now as $h_t \in L$, in view of Proposition 11, there are for every $t = 1, \dots, m$, a vector $B_t \in \mathbf{p}^\ell$ and n matrices $A_{ut} \in \mathbf{p}^{\ell \times \ell}$, $u = 1, \dots, n$ such that

$$\forall X_1, \dots, X_n \in E : h_t(X_1, \dots, X_n) = B_t \oplus \textstyle\sum_{u=1}^n X_u \cdot A_{ut}.$$

Let

$$\begin{aligned}
&B_t := (b_{(t-1)\ell+1}, b_{(t-1)\ell+2}, \dots, b_{t\ell}),\\
&b := (b_1, \dots, b_\ell, b_{\ell+1}, \dots, b_{2\ell}, \dots, b_{(m-1)\ell+1}, \dots, b_{m\ell}),\\
&A_{ut} := \begin{pmatrix} a_{(u-1)\ell+1,(t-1)\ell+1} & a_{(u-1)\ell+1,(t-1)\ell+2} & \cdots & a_{(u-1)\ell+1,t\ell} \\ a_{(u-1)\ell+2,(t-1)\ell+1} & a_{(u-1)\ell+2,(t-1)\ell+2} & \cdots & a_{(u-1)\ell+2,t\ell} \\ \vdots & \vdots & & \vdots \\ a_{u\ell,(t-1)\ell+1} & a_{u\ell,(t-1)\ell+2} & \cdots & a_{u\ell,t\ell} \end{pmatrix},\\
&A := (a_{ij}) \text{ where } 1 \leq i \leq n\ell, 1 \leq j \leq m\ell,\\
&X_u := (x_{(u-1)\ell+1}, \dots, x_{u\ell}),\ u = 1, \dots, n,\\
&X := (X_1, \dots, X_n),\\
&x := (x_1, x_2 \dots, x_{n\ell}).
\end{aligned}$$

Then, for

$$b + x \cdot A\ (mod\ p) = (y_1, \dots, y_{m\ell}),$$

we have

$$\begin{aligned}
&(h_1(X), h_2(X), \dots, h_m(X))\\
&= ((y_1, \dots, y_\ell), (y_{\ell+1}, \dots, y_{2\ell}), \dots, (y_{(m-1)\ell+1}, \dots, y_{m\ell})).
\end{aligned}$$

If A is a zero matrix, then (16) holds by definition of α_i. So assume that A is not the zero matrix. We distinguish the two subcases $p = 2$ and p is an odd prime number.

Case 1.1: $p \geq 3$.

By Lemma 15 there is an $x \in D_{n,j}$ with $b + x \cdot A \notin D_{m,i}$, i.e., $x \notin \mathrm{dom}\,(f)$ and so the non-inclusion (15) holds.

Case 1.2: $p = 2$.

The map

$$\varphi : \{0,1\}^{n\ell} \longrightarrow \{0,1\}^{m\ell},\ x \mapsto b + x \cdot A$$

is an affine map and the set

$$W := \varphi(\{0,1\}^{n\ell}) := \{y \in \{0,1\}^{m\ell} \mid \exists\, x \in \{0,1\}^{n\ell} :\ y = b + x \cdot A\}$$

is an affine space with

$$\dim W = \text{rank } A \leq m\ell.$$

First we show, by contradiction, that

$$W \subseteq D_{m,i}.$$

Assume that there is a $\widehat{w} \in W$ with $w \notin \text{dom }(\alpha_i)$. Now clearly

$$\varphi^{-1}(w) := \{x \in \{0,1\}^{n\ell} \mid \varphi(x) = w\}$$

is an affine space with

$$\dim \varphi^{-1}(w) = n\ell - \dim W$$

and as $\dim W \leq m\ell$ and $s_j \geq m$ (see Lemma 15, Case 1.2.2) we have

$$n\ell - \dim W \geq n\ell - m\ell \geq n\ell - s_j\ell. \tag{17}$$

On the other hand we have

$$\varphi^{-1}(w) \subseteq \{0,1\}^{n\ell} \setminus D_{n,j} \subset \{0,1\}^{n\ell} \setminus R_{n\ell;s_j\ell}$$

and by Lemma 13 (b)

$$\dim \varphi^{-1}(w) \leq n\ell - s_j\ell - 1,$$

contradicting (17). This shows that $W \subseteq D_{m,i}$ and thus (16) follows from Lemma 14.

Case 2: $i > j$.
Let $\text{dom }(\alpha_j) \subseteq \text{dom }(f)$, we show that (16) holds. By definition of α_i we have $\alpha_i := \tau_{i+1,s_i}$, where $s_1 := 1$ and $s_t := (pt-1)s_{t-1} + 1$ for $t \geq 2$. Now by Lemma 12 $\alpha_i \in \text{pPol } \lambda_{p(i+1)}$ and $\text{Str }(L) \subseteq \text{pPol } \lambda_{p\cdot(i+1)} \subset \text{pPol } \lambda_{p(j+1)}$ (as $1 \leq j < i$), therefore

$$f \in \text{pPol } \lambda_{p(i+1)} \subset \text{pPol } \lambda_{p(j+1)} \subseteq \text{pPol } \lambda_{2p}. \tag{18}$$

For $1 \leq u \leq \ell$ let e_u denote the vector in $\{0,1\}^\ell$ consisting of a 1 on the position u and 0's elsewhere, i.e., $e_u := (0,\dots,0,1,0,\dots,0)$. Furthermore, let $e_0 := (\mathbf{0},\mathbf{0},\dots,\mathbf{0})$, $e_{q,u} := (\mathbf{0},\mathbf{0},\dots,\underbrace{e_u}_{q},\mathbf{0},\dots,\mathbf{0})$ be n-vectors in E^n and, for $q \in \{1,\dots,n\}$, let $A_q \in \{0,1\}^{\ell\times\ell}$ be the matrix whose columns are $(f(e_0) \ominus f(e_{q,v}))^T$, $1 \leq v \leq \ell$, i.e.,

$$A_q := \begin{pmatrix} f(e_0) \ominus f(e_{q,1}) \\ f(e_0) \ominus f(e_{q,2}) \\ \dots\dots\dots\dots \\ f(e_0) \ominus f(e_{q,\ell}) \end{pmatrix}^T.$$

Define the function f_1 by setting

$$f_1(x_1,\dots,x_n) := f(x_1,\dots,x_n) \ominus f(e_0) \oplus \sum_{q=1}^{n} x_q \otimes A_q.$$

Then f_1 has the following properties:

$$f_1(e_0) = f_1(e_{q,v}) = \mathbf{0} \text{ for all } q \in \{1,\dots,n\} \text{ and all } v \in \{1,\dots,\ell\}, \tag{19}$$

and

$$\mathrm{dom}\,(\alpha_j) \subseteq \mathrm{dom}\,(f_1) = \mathrm{dom}\,(f).$$

Combining this with Lemma 12 and (18) above we obtain:

$$f_1 \in \mathrm{pPol}\ \lambda_{p(i+1)} \subset \mathrm{pPol}\ \lambda_{p(j+1)} \subseteq \mathrm{pPol}\ \lambda_{2p}. \tag{20}$$

Furthermore it holds

$$f_{1|\mathrm{dom}\,(\alpha_j)} \in \mathrm{Str}\,(L) \iff f_{|\mathrm{dom}\,(\alpha_j)} \in \mathrm{Str}\,(L). \tag{21}$$

We now show that $f_{1|\mathrm{dom}\,(\alpha_j)}$ is a constant function. Assume that there is an $a \in E^n$ with $\widehat{a} := (a_1, \dots, a_{n\ell}) \in \{0,1\}^{n\ell}$, $\sum_{u=1}^{n\ell} a_u \le s_j\ell$ and $f_1(a) \ne \mathbf{0}$. Then we may choose a such that the number of 1's in the vector $\widehat{a}$ is minimal, let t be that number. Then $t \ge 2$ by (19) and w.l.o.g. let $\widehat{a} := (\underbrace{1, 1, \dots, 1}_{t}, 0, 0, \dots, 0)$.

By the minimality of t we have $f_1(a') = f(a'') = \mathbf{0}$, where $a', a'' \in E^n$, $\widehat{a'} := (0, \underbrace{1, \dots, 1}_{t-1}, 0, \dots, 0)$ and $\widehat{a''} := (1, 0, 0, \dots, 0)$. Here

$$a \oplus e_0 \oplus \underbrace{a' \oplus \cdots \oplus a'}_{p-1} \oplus \underbrace{a'' \oplus \cdots \oplus a''}_{p-1} = e_0$$

and thus the matrix in $E^{2p\times n}$ whose rows are

$$r_1 = a,\ r_2 = e_0,\ r_3 = \cdots = r_{p+1} = a' \text{ and } r_{p+2} = \cdots = r_{2p} = a''$$

has all its columns in λ_{2p} while

$$(f_1(a), f_1(e_0), \underbrace{f_1(a'), \dots, f_1(a')}_{p-1}, \underbrace{f_1(a'), \dots, f_1(a')}_{p-1}) \notin \lambda_{2p}$$

contradicting (20). This shows that

$$\forall\, b \in \mathrm{dom}\,(\alpha_j) \setminus \{(\underbrace{-\mathbf{1}, \dots, -\mathbf{1}}_{n})\} :\ f_1(b) = \mathbf{0}.$$

Finally we show that $f_1(-\mathbf{1}, -\mathbf{1}, \dots - \mathbf{1}) = \mathbf{0}$. Assume that $f_1(-\mathbf{1}, -\mathbf{1}, \dots - \mathbf{1}) \ne \mathbf{0}$ and consider the following matrix C with $p(i+1)$ rows $c_1, \dots, c_{p(i+1)}$ and $n = (p(j+1)-1)s_j + 1$ columns :

$$C := \begin{pmatrix}
\underbrace{\mathbf{1\,1} \dots \mathbf{1}}_{s_j} & \mathbf{0\,0} \dots \mathbf{0} & \dots\dots & \mathbf{0\,0} \dots \mathbf{0} & \mathbf{0} \\
\mathbf{0\,0} \dots \mathbf{0} & \underbrace{\mathbf{1\,1} \dots \mathbf{1}}_{s_j} & \dots\dots & \mathbf{0\,0} \dots \mathbf{0} & \mathbf{0} \\
\vdots & & & & \vdots \\
\mathbf{0\,0} \dots \mathbf{0} & \mathbf{0\,0} \dots \mathbf{0} & \dots\dots & \underbrace{\mathbf{1\,1} \dots \mathbf{1}}_{s_j} & \mathbf{0} \\
\mathbf{0\,0} \dots \mathbf{0} & \mathbf{0\,0} \dots \mathbf{0} & \dots\dots & \mathbf{0\,0} \dots \mathbf{0} & \mathbf{1} \\
\mathbf{0\,0} \dots \mathbf{0} & \mathbf{0\,0} \dots \mathbf{0} & \dots\dots & \mathbf{0\,0} \dots \mathbf{0} & \mathbf{0} \\
\vdots & & & & \vdots \\
\mathbf{0\,0} \dots \mathbf{0} & \mathbf{0\,0} \dots \mathbf{0} & \dots\dots & \mathbf{0\,0} \dots \mathbf{0} & \mathbf{0} \\
-\mathbf{1} \quad \dots & -\mathbf{1} \quad \dots & \dots\dots & -\mathbf{1} \dots -\mathbf{1} & -\mathbf{1}
\end{pmatrix}$$

Then the columns of C belong to $\lambda_{p(i+1)}$, but
$(f_1(c_1), f_1(c_2), \dots, f_1(c_{p(i+1)}) = (\mathbf{0}, \dots, \mathbf{0}, f_1(-\mathbf{1}, -\mathbf{1}, \dots - \mathbf{1})) \in E^{p(i+1)} \setminus \lambda_{p(i+1)}$,
contradicting (20).
Thus, we have shown that

$$\forall\, b \in \mathrm{dom}\,(\alpha_j): \;\; f_1(b) = \mathbf{0},$$

i.e., $f_{1|\mathrm{dom}\,(\alpha_j)}$ is a constant function with value $\mathbf{0}$, and so $f_{1|\mathrm{dom}\,(\alpha_j)} \in \mathrm{Str}\,(L)$. Then (16) follows from (21) and this completes the proof of the lemma. □

We need to recall the following statement shown in [3] (Lemma 2.10):

Lemma 17. ([3]) *Let $F \subset \mathrm{Par}(\mathbf{k})$ and $D_0 := F \cup J_{\mathbf{k}}$. Moreover for $\ell \geq 0$ set*

$$D_{\ell+1} := \{h[g_1, \dots, g_m] \;\mid\; h \in D_0^{(m)} \text{ and } g_1, \dots, g_m \in D_\ell \text{ for some } m \geq 1\}\;.$$

Then $\langle F \rangle = \bigcup\limits_{\ell \,\geq\, 0} D_\ell\;.$ □

We use Lemma 16 and Lemma 17 to show:

Theorem 18. *For every $j \geq 1$*

$$\alpha_j \notin \langle \{\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots\} \cup \mathrm{Str}\,(L)\rangle.$$

Proof. Let $F := \{\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots\} \cup \mathrm{Str}\,(L)$, $D_0 := F$ (notice that D_0 contains $J_{\mathbf{k}}$) and let $D_{\ell+1}$ be defined from D_ℓ as in Lemma 17. We show by induction on $\ell \geq 0$ that

$$\forall\, f \in D_\ell\; (\mathrm{dom}\,(\alpha_j) \subseteq \mathrm{dom}\,(f) \Longrightarrow f_{|\mathrm{dom}\,(\alpha_j)} \in \mathrm{Str}\,(L)). \tag{22}$$

The above statement clearly holds for $\ell = 0$ as $\mathrm{dom}\,(\alpha_j) \not\subseteq \mathrm{dom}\,(\alpha_i)$ for $i \neq j$. So assume that (22) holds for all $0 \leq t \leq \ell$ and consider $f \in D_{\ell+1} \setminus D_\ell$ with $\mathrm{dom}\,(\alpha_j) \subseteq \mathrm{dom}\,(f)$. Then there are $m \geq 1$, $h \in D_0^{(m)}$ and $g_1, \dots, g_m \in D_\ell^{(n)}$ such that $f = h[g_1, \dots, g_m]$, where $n := (p(j+1)-1)s_j+1$ and s_j is as in Lemma 15. As $\mathrm{dom}\,(\alpha_j) \subseteq \mathrm{dom}\,(f)$ we have $\mathrm{dom}\,(\alpha_j) \subseteq \mathrm{dom}\,(g_t)$ for all $t = 1, \dots, m$. Thus by the induction hypothesis the partial functions $\overline{g_t} := g_{t|\mathrm{dom}\,(\alpha_j)}$ satisfy $\overline{g_t} \in \mathrm{Str}\,(L)$ for all $t = 1, \dots, m$. Obviously, $f_{|\mathrm{dom}\,(\alpha_j)} = h[\overline{g_1}, \dots, \overline{g_m}]$. If $h \in \mathrm{Str}\,(L)$ then $f_{|\mathrm{dom}\,(\alpha_j)} \in \mathrm{Str}\,(L)$, since $\mathrm{Str}\,(L)$ is a partial clone. Thus we can assume that there is $i \in N^+ \setminus \{j\}$ with $h = \alpha_i$. As $\overline{g_t} \in \mathrm{Str}\,(L)$ for all $t = 1, \dots, m$ we have by Lemma 16 that $f_{|\mathrm{dom}\,(\alpha_j)} = \alpha_i[\overline{g_1}, \dots, \overline{g_m}] \in \mathrm{Str}\,(L)$, i.e. (22) holds.
Finally if $\alpha_j \in \langle \{\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots\} \cup \mathrm{Str}\,(L)\rangle$, then there is an $\ell \geq 0$ such that $\alpha_j \in D_\ell$ and by (22) $\alpha_j \in \mathrm{Str}\,(L)$, a contradiction. □

For $j = 1, 2, \dots$ let C_j denote the partial clone $\langle \{\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots\} \cup Str(L)\rangle$. By Theorem 18

$$\alpha_j \in C_i \Longleftrightarrow i \neq j$$

and thus the correspondence

$$\chi : \mathcal{P}(N^+) \longrightarrow [\text{Str } (L), \text{Par}(E)]$$

defined by

$$\chi(X) := \bigcap_{n \in N^+ \setminus X} C_n$$

is a one-to-one mapping. We have shown that

Theorem 19. *Let $E = \mathbf{p}^\ell$ where p is a prime number and $\ell \geq 1$ and let L be the maximal clone on E defined in Proposition* 11. *Then the interval of partial clones* $[\text{Str}(L),\ \text{Par}(E)]$ *is of continuum cardinality on E.* □

References

[1] Alekseev, V. B.; Voronenko, A. A.: *On some closed classes in partial two-valued logic.* (Russian, English) Discrete Math. Appl. **4**(5) (1994), 401–419; translation from Diskretn. Mat. **6**(4) (1994), 58–79. Zbl 0818.06013

[2] Börner, F.; Haddad L.; Pöschel R.: *Minimal partial clones.* Bull. Aust. Math. Soc. **44** (1991), 405–415. Zbl 0731.08005

[3] Börner, F.; Haddad, L.: *Maximal partial clones with no finite basis.* Algebra Univers. **40** (1998), 453–476. Zbl 0936.08004

[4] Frejvald, R. V.: *A completeness criterion for partial functions of logic and many-valued logic algebras.* (Russian, English) Sov. Phys. Dokl. **11** (1966), 288–289, translation from Dokl. Akad. Nauk. SSSR **167** (1966), 1249–1250. Zbl 0149.24405

[5] Haddad, L.; Rosenberg, I. G.: *Maximal partial clones determined by areflexive relations.* Discrete Appl. Math. **24** (1989), 133–143. Zbl 0695.08010

[6] Haddad, L.; Rosenberg, I. G.: *Partial Sheffer operations.* Eur. J. Comb. **12** (1991), 375–379. Zbl 0724.08005

[7] Haddad, L.; Rosenberg, I. G.: *Completeness theory for finite partial algebras.* Algebra Univers. **29** (1992), 378–401. Zbl 0771.08001

[8] Haddad, L.: *On the depth of the intersection of two maximal partial clones.* Mult.- Valued Log. **3** (1998), 259–270. Zbl 0939.08001

[9] Haddad, L.; Lau, D.: *Families of finitely generated maximal partial clones.* Mult. Valued Log. **5** (2000), 201–228. Zbl 0992.08004

[10] Haddad, L.; Lau, D.: *Pairwise intersections of Slupecki type maximal partial clones.* Beitr. Algebra Geom. **41**(2) (2000), 537–555. Zbl 0992.08003

[11] Haddad, L.; Lau, D.; Rosenberg, I. G.: *Intervals of partial clones containing maximal clones.* Journal of Automata, Languages and Combinatorics (submitted)

[12] Janov, Yu.; Mucnik, A. A.: *On the existence of k-valued closed classes having no finite basis.* (Russian) Dokl. Akad. Nauk SSSR **127** (1959), 44–46. Zbl 0100.01001

[13] Lau, D.: *Bestimmung der Ordnung maximaler Klassen von Funktionen der k-wertigen Logik.* Z. Math. Logik Grundlagen Math. **24** (1978), 79–96. Zbl 0401.03008

[14] Lau, D.: *Über partielle Funktionenalgebren.* Rostocker Math. Kolloq. **33** (1988), 23–48. Zbl 0659.08001

[15] Lau, D.: *Function algebras on finite sets. A basic course on many-valued logic and clone theory.* Springer Monographs in Mathematics. Springer-Verlag Berlin-Heidelberg 2006. Zbl pre05066368

[16] Pöschel, R.; Kalužnin, L. A.: *Funktionen- und Relationenalgebren.* Birkhäuser Verlag, Basel, Stuttgart 1979. Zbl 0421.03049

[17] Post, E.: *The two-valued iterative systems of mathematical logic.* Annals of Mathematics Studies **5**, Princeton University Press, Princeton, N.J. 1941. Zbl 0063.06326

[18] Romov, B. A.: *Maximal subalgebras of algebras of partial multivalued logic functions.* Kibernatika; English translation in: Cybernetics **16** (1980), 31-41. Zbl 0453.03068

[19] Romov, B. A.: *The algebras of partial functions and their invariants.* Kibernetika; English translation in: Cybernetics **17** (1981), 157–167. Zbl 0466.03026

[20] Romov, B. A.: *The completeness problem in the algebra of partial functions of finite-valued logic.* Kibernetika, English translation in: Cybernetics **26** (1990), 133–138. Zbl 0752.03010

[21] Rosenberg, I. G.: *Über die funktionale Vollständigkeit in den mehrwertigen Logiken.* Rozpr. Cesk. Akad. Ved, Rada Mat. Přír. Ved **80** (1970), 3–93. Zbl 0199.30201

[22] Rosenberg I. G.: *Composition of functions on finite sets, completeness and relations, a short survey.* In: D. Rine (ed.), Computer science and multiple-valued logic. 2nd edition, North-Holland, Amsterdam 1984, 150–192. Zbl 0546.94020

[23] Rosenberg, I. G.: *Partial algebras and clones via one-point extension.* Contrib. Gen. Algebra **6**, 227–242. Zbl 0695.08009

[24] Strauch, B.: *On partial classes containing all monotone and zero-preserving total Boolean functions.* Math. Log. Q. **43** (1997), 510–524. Zbl 0885.06006

[25] Strauch, B.: *The classes which contain all monotone and idempotent total Boolean functions.* Universität Rostock, preprint 1996.

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