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# Groups in which the Bounded Nilpotency of Two-generator Subgroups is a Transitive Relation

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**Abstract.** In this paper we describe the structure of locally finite groups in which the bounded nilpotency of two-generator subgroups is a transitive relation. We also introduce the notion of (nilpotent of class $c$)-transitive kernel. Our results generalize several known results related to the groups in which commutativity is a transitive relation.

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## 1. Introduction

Let $c$ be a positive integer and let $\mathfrak{N}_c$ denote the class of all groups which are nilpotent of class $\leq c$. A group $G$ is said to be an $\mathfrak{N}_cT$*-group* if for all $x, y, z \in G\backslash\{1\}$ the relations $\langle x, y\rangle \in \mathfrak{N}_c$ and $\langle y, z\rangle \in \mathfrak{N}_c$ imply $\langle x, z\rangle \in \mathfrak{N}_c$. In the case $c = 1$ these groups are known as commutative-transitive groups (also $CT$-groups

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or $CA$-groups) and have been studied by several authors [2, 3, 4, 8, 11, 14, 15]. It is not difficult to see that $CT$-groups are precisely the groups in which centralizers of non-identity elements are abelian. The study of these groups was initiated by Weisner [14] in 1925, but there are some fallacies in his proofs. Nevertheless, it turns out that finite $CT$-groups are either soluble or simple. Finite nonabelian simple $CT$-groups have been classified by Suzuki [11]. He proved that every finite nonabelian simple $CT$-group is isomorphic to some $\mathrm{PSL}(2, 2^f)$, where $f > 1$. The complete description of finite soluble $CT$-groups has been given by Wu [15] (see also a paper of Lescot [8]), who has also obtained information on locally finite $CT$-groups and polycyclic $CT$-groups. At roughly the same time Fine et al. [4] introduced the notion of the commutative-transitive kernel of a group. This topic has been further explored by the first and the third author; see [2] and [3].

Passing to finite $\mathfrak{N}_c T$-groups with $c > 1$ we first note that in these groups centralizers of non-identity elements are nilpotent. The converse is not true, however, as the example of $\mathrm{PSL}(2, 9)$ shows (see Proposition 4.5). Compared to the $CT$-case, this may seem to be a certain disadvantage at first glance, but nevertheless we obtain satisfactory information on the structure of locally finite $\mathfrak{N}_c T$-groups. We show that soluble locally finite $\mathfrak{N}_c T$-groups are either Frobenius groups or belong to the class of groups in which every two-generator subgroup is nilpotent of class $\leq c$. Furthermore, we prove that finite $\mathfrak{N}_c T$-groups are either soluble or simple. This provides a generalization of results in [15]. Additionally, we show that the groups $\mathrm{PSL}(2, 2^f)$, where $f > 1$, and Suzuki groups $\mathrm{Sz}(q)$, with $q = 2^{2n+1} > 2$, are the only finite nonabelian simple $\mathfrak{N}_c T$-groups for $c > 1$. This result is probably the strongest evidence showing the gap between $CT$-groups and $\mathfrak{N}_c T$-groups with $c > 1$. We also show that locally finite $\mathfrak{N}_c T$-groups are either locally soluble or simple. In the latter case we give a classification of these groups.

Another notion closely related to $CT$-groups is the commutative-transitive kernel of a group. Given a group $G$, we can construct a characteristic subgroup $T(G)$ as the union of a chain $1 = T_0(G) \leq T_1(G) \leq \cdots$ in such way that $G/T(G)$ is a $CT$-group [4]. In [2] it is proved that if $G$ is locally finite, then $T(G) = T_1(G)$. Similar results have also been obtained in [3] for other classes of groups, such as supersoluble groups. In analogy with this we introduce the notion of the $\mathfrak{N}_c$-transitive kernel of a group and prove that it has similar properties like the commutative-transitive kernel.

In the final section we present some examples of $\mathfrak{N}_2 T$-groups. In particular, we present Frobenius $\mathfrak{N}_2 T$-groups with nonabelian kernel and Frobenius $\mathfrak{N}_2 T$-groups with noncyclic complement. We also show that some finite linear groups with nilpotent centralizers are in a certain sense far from being $\mathfrak{N}_c T$-groups.

## 2. $\mathfrak{N}_c T$-groups

In this section we investigate the structure of locally finite $\mathfrak{N}_c T$-groups. In the beginning we exhibit some basic properties of these groups. For positive integers $r > 1$ and $n$ denote by $\mathfrak{N}(r, n)$ the class of all groups in which every $r$-generator subgroup is nilpotent of class $\leq n$. Every finite $\mathfrak{N}(r, n)$-group is nilpotent by Zorn's theorem (see Theorem 12.3.4 in [10]). It is now clear that every locally nilpotent $\mathfrak{N}_cT$-group is also an $\mathfrak{N}(2,c)$-group. In fact, every $\mathfrak{N}_cT$-group with nontrivial center is an $\mathfrak{N}(2,c)$-group. On the other hand, the property $\mathfrak{N}_cT$ behaves badly under taking quotients and forming direct products. For, it is known that every free (soluble) group is a $CT$-group [15]. Moreover if $G$ and $H$ are $\mathfrak{N}_cT$-groups and there exist $x, y \in G$ such that $\langle x, y\rangle$ is not nilpotent, then it is easy to see that $G \times H$ is not an $\mathfrak{N}_dT$-group for any $d \in \mathbb{N}$.

Our first result shows that the classes of $\mathfrak{N}_cT$-groups form a chain.

**Proposition 2.1.** *Let $c$ and $d$ be integers, $c \geq d \geq 1$. Then every $\mathfrak{N}_dT$-group is also an $\mathfrak{N}_cT$-group.*

*Proof.* Let $G$ be an $\mathfrak{N}_dT$-group. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq c$. By the above remarks $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq d$. As $G$ is an $\mathfrak{N}_dT$-group, it follows that $\langle x, z\rangle$ is nilpotent of class $\leq d$, hence it is nilpotent of class $\leq c$. □

The following lemma is crucial for the description of soluble locally finite $\mathfrak{N}_cT$-groups.

**Lemma 2.2.** *Let $G$ be a locally finite $\mathfrak{N}_cT$-group with nontrivial Hirsch-Plotkin radical $H$. Then the factor group $G/H$ acts fixed-point-freely on $H$ by conjugation.*

*Proof.* As the Hirsch-Plotkin radical $H$ is a locally nilpotent $\mathfrak{N}_cT$-group, it is also an $\mathfrak{N}(2,c)$-group. Let $y$ be a nontrivial element in $H$. Suppose there exists $a \in C_G(y)\backslash H$. Since the group $\langle a, y\rangle$ is abelian and $H$ is an $\mathfrak{N}(2,c)$-group, we conclude that the group $\langle a, h\rangle$ is nilpotent of class $\leq c$ for every $h \in H$, since $G$ is an $\mathfrak{N}_cT$-group. By conjugation we get that $\langle a^g, h\rangle$ is also nilpotent of class $\leq c$ for all $g \in G$ and $h \in H$. As $G$ is an $\mathfrak{N}_cT$-group, this implies that the group $\langle a, a^g\rangle$ is nilpotent of class $\leq c$ for every $g \in G$. In particular, we have $1 = [a^g, {}_c a] = [a, g, {}_c a]$ for all $g \in G$, hence $a$ is a left $(c+1)$-Engel element of $G$. As $G$ is locally finite, this implies that $a \in H$ (see, for instance, Exercise 12.3.2 of [10]), which is a contradiction. □

**Theorem 2.3.** *Every locally finite soluble $\mathfrak{N}_cT$-group is either an $\mathfrak{N}(2,c)$-group or a Frobenius group whose kernel and complement are both $\mathfrak{N}(2,c)$-groups. Conversely, every locally finite Frobenius group in which kernel and complement are both $\mathfrak{N}(2,c)$-groups is an $\mathfrak{N}_cT$-group.*

*Proof.* Let $G$ be a locally finite soluble $\mathfrak{N}_cT$-group and suppose $G$ is not in $\mathfrak{N}(2,c)$. Let $N$ be its Hirsch-Plotkin radical. As $N$ is also an $\mathfrak{N}_cT$-group, it is an $\mathfrak{N}(2,c)$-group. By Lemma 2.2 $G/N$ acts fixed-point-freely on $N$, hence $G$ is a Frobenius group with the kernel $N$ and a complement $H$; see, for instance, Proposition 1.J.3 in [7]. Since $H$ has a nontrivial center [7, Theorem 1.J.2], we have that $H \in \mathfrak{N}(2,c)$. Besides, $N$ is nilpotent by the same result from [7].

Conversely, let $G$ be a locally finite Frobenius group with the kernel $N$ and a complement $H$ and suppose that both $N$ and $H$ are $\mathfrak{N}(2,c)$-groups. Let $x, y, z \in G\backslash\{1\}$ and let the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ be nilpotent of class $\leq c$. Suppose

$x \in N$ and $y \notin N$. Then the equation $[x, {}_c y] = 1$ implies $[x, {}_{c-1} y] = 1$, since $H$ acts fixed-point-freely on $N$. By the same argument we get $x = 1$, which is not possible. This shows that if $x \in N$ then $y \in N$ and similarly also $z \in N$. But in this case $\langle x, z \rangle$ is clearly nilpotent of class $\leq c$, since $N$ is an $\mathfrak{N}(2, c)$-group. Thus we may assume that $x, y, z \notin N$. Let $x \in H^g$ and $y \in H^k$ for some $g, k \in G$ and suppose $H^g \neq H^k$. We clearly have $C_G(x) \leq H^g$ and $C_G(y) \leq H^k$. Let $\alpha$ be any simple commutator of weight $c$ with entries in $\{x, y\}$. As $\langle x, y \rangle$ is nilpotent of class $\leq c$, we have $\alpha \in C_G(x) \cap C_G(y) = 1$. This implies that $\langle x, y \rangle$ is nilpotent of class $\leq c - 1$. Continuing with this process, we end at $x = y = 1$ which is impossible. Hence we conclude that $\langle x, y \rangle \leq H^g$ and similarly also $\langle y, z \rangle \leq H^g$. Therefore we have $\langle x, z \rangle \leq H^g$. But $H^g$ is an $\mathfrak{N}(2, c)$-group, hence the group $\langle x, z \rangle$ is nilpotent of class $\leq c$. This concludes the proof. □

Theorem 2.3 can be further refined when we restrict ourselves to finite groups.

**Theorem 2.4.** *Let $G$ be a finite group. Then $G$ is a soluble $\mathfrak{N}_c T$-group if and only if it is either an $\mathfrak{N}(2, c)$-group or a Frobenius group with the kernel which is an $\mathfrak{N}(2, c)$-group and a complement which is nilpotent of class $\leq c$.*

*Proof.* By Theorem 2.3 we only need to show that if $G$ is a finite soluble $\mathfrak{N}_c T$-group which is not an $\mathfrak{N}(2, c)$-group, then every complement $H$ of the Frobenius kernel $N$ of $G$ is nilpotent of class $\leq c$. Suppose $N$ is not abelian. Then the order of $H$ is odd, hence all Sylow subgroups of $H$ are cyclic. This implies that $H$ is cyclic. Assume now that $N$ is abelian. Then all the Sylow $p$-subgroups of $H$ are cyclic for $p \neq 2$, whereas the Sylow 2-subgroup is either cyclic or a generalized quaternion group $Q_{2^n}$ [5]. Moreover, since $H \in \mathfrak{N}(2, c)$, we obtain $n \leq c+1$. As $H$ is nilpotent and all its Sylow subgroups are nilpotent of class $\leq c$, the nilpotency class of $H$ does not exceed $c$. □

Let $G$ be a finite $\mathfrak{N}_c T$-group and suppose $G \notin \mathfrak{N}(2, c)$. If the Fitting subgroup of $G$ is nontrivial, then Lemma 2.2 together with Theorem 2.4 shows that $G$ is soluble and so its structure is completely determined by Theorem 2.4. The complete classification of finite insoluble $\mathfrak{N}_c T$-groups is described in our next result. Note that it has been shown in [11] that the groups $\mathrm{PSL}(2, 2^f)$, where $f > 1$, are the only finite insoluble $\mathfrak{N}_1 T$-groups. Passing to finite $\mathfrak{N}_c T$-groups with $c > 1$, we obtain an additional family of simple groups.

**Theorem 2.5.** *Let $G$ be a finite $\mathfrak{N}_c T$-group with $c > 1$. Then $G$ is either soluble or simple. Moreover, $G$ is a nonabelian simple $\mathfrak{N}_c T$-group if and only if it is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$.*

*Proof.* It is easy to see that in every finite $\mathfrak{N}_c T$-group $G$ the centralizers of nontrivial elements are nilpotent, i.e., $G$ is a $CN$-group. Suppose that $G$ is not soluble. By a result of Suzuki [12, Part I, Theorem 4], $G$ is a $CIT$-group, i.e., the centralizer of any involution in $G$ is a 2-group. Let $P$ and $Q$ be any Sylow $p$-subgroups of $G$ and suppose that $P \cap Q \neq 1$. Since $P$ and $Q$ are $\mathfrak{N}(2, c)$-groups and $G$ is an $\mathfrak{N}_c T$-group, we conclude that $\langle P, Q\rangle$ is an $\mathfrak{N}(2,c)$-group, hence it is nilpotent. This shows that $\langle P, Q\rangle$ is a $p$-group, which implies $P = Q$. Therefore Sylow subgroups of $G$ are independent. Combining Theorem 1 in Part I and Theorem 3 in Part II of [12], we conclude that $G$ has to be simple. Additionally, we also obtain that $G$ is a $ZT$-group, that is, $G$ is faithfully represented as a doubly transitive permutation group of odd degree in which the identity is the only element fixing three distinct letters. The structure of these groups is described in [13]. It turns out that $G$ is isomorphic either to $\mathrm{PSL}(2, 2^f)$, where $f > 1$, or to $\mathrm{Sz}(q)$ with $q = 2^{2n+1} > 2$.

It remains to prove that $\mathrm{PSL}(2, 2^f)$ and $\mathrm{Sz}(q)$ are $\mathfrak{N}_c T$-groups. For projective special linear groups this has been done in [11]. Now, let $G = \mathrm{Sz}(q)$ where $q = 2^{2n+1} > 2$. By Theorem 3.10 c) in [6] $G$ has a nontrivial partition $(G_i)_{i\in I}$, where for every $i \in I$ the group $G_i$ is either cyclic or nilpotent of class $\leq 2$. Moreover, the proof of result 3.11 in [6] implies that for all $g \in G\backslash\{1\}$ the relation $g \in G_i$ implies that $C_G(g) \leq G_i$. Let $x, y, z \in G\backslash\{1\}$ and suppose that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq 2$. Let $a$ and $b$ be nontrivial elements in $Z(\langle x, y\rangle)$ and $Z(\langle y, z\rangle)$, respectively, and suppose that $a \in G_i$ and $b \in G_j$ for some $i, j \in I$. Then $y \in C_G(a) \cap C_G(b) \leq G_i \cap G_j$, hence $i = j$. But now we get $x, z \in G_i$ and since $G_i$ is nilpotent of class $\leq 2$, the same is true for the group $\langle x, z\rangle$. Hence $G$ is an $\mathfrak{N}_2 T$-group. By Proposition 2.1 $G$ is an $\mathfrak{N}_c T$-group for every $c > 1$. □

It is proved in [15] that every locally finite insoluble $CT$-group is isomorphic to $\mathrm{PSL}(2, F)$ for some locally finite field $F$. For $\mathfrak{N}_c T$-groups, where $c > 1$, we have the following result.

**Theorem 2.6.** *Let $c > 1$ and let $G$ be a locally finite $\mathfrak{N}_c T$-group which is not locally soluble. Then there exists a locally finite field $F$ such that $G$ is isomorphic either to $\mathrm{PSL}(2, F)$ or to $\mathrm{Sz}(F)$.*

*Proof.* Let $G$ be a locally finite $\mathfrak{N}_c T$-group and suppose that $G$ is not locally soluble. Then $G$ contains a finite insoluble subgroup, hence every finite subgroup of $G$ is contained in some finite insoluble subgroup of $G$. Using Theorem 2.5, we conclude that every finitely generated subgroup of $G$ has a faithful representation of degree 4 over some field of even characteristic. By Mal'cev's representation theorem [7, Theorem 1.L.6], $G$ has a faithful representation of the same degree over a field which is an ultraproduct of some finite fields. Hence $G$ is a linear periodic group. It is not difficult to see that $G$ has to be simple. Namely, the set of all finite nonabelian simple subgroups of G is a local system of $G$. By a theorem of Winter [7] the group $G$ is countable. Thus we obtain a chain $(G_i)_{i\in\mathbb{N}}$ of nonabelian finite simple subgroups in $G$ such that $G$ is the union of this chain. By Theorem 2.5 we have either $G_i \cong \mathrm{PSL}(2, F_i)$ or $G_i \cong \mathrm{Sz}(F_i)$ for suitable finite fields $F_i$, $i \in \mathbb{N}$. On the other hand, $PSL(2, F)$ does not contain any Suzuki group as a subgroup and vice versa (this follows from [13] and Dickson's theorem in [5]). Therefore we either have $G_i \cong \mathrm{PSL}(2, F_i)$ for all $i \in \mathbb{N}$ or $G_i \cong \mathrm{Sz}(F_i)$ for all $i \in \mathbb{N}$. By a theorem of Kegel [7, Theorem 4.18] there exists a locally finite field $F$ such that either $G \cong \mathrm{PSL}(2, F)$ or $G \cong \mathrm{Sz}(F)$. □

Let the group $G$ be locally finite and locally soluble. If $G$ is an $\mathfrak{N}_2T$-group, then Theorem 2.5 implies that every finitely generated subgroup of $G$ is either a 2-Engel group or a Frobenius group with the kernel which is a 2-Engel group and a complement which is nilpotent of class $\leq 2$. As every 2-Engel group is nilpotent of class $\leq 3$ (see [9, p. 45]), the derived length of finitely generated subgroups of $G$ is bounded, so $G$ is actually soluble. Therefore we have:

**Corollary 2.7.** *Let $G$ be a locally finite $\mathfrak{N}_2T$-group. Then $G$ is either soluble or simple.*

The structure of locally finite $\mathfrak{N}_cT$-groups, where $c > 2$, is more complicated. Namely, Bachmuth and Mochizuki [1] constructed an insoluble $\mathfrak{N}(2,3)$-group of exponent 5. This is a locally finite $\mathfrak{N}_3T$-group in which all finite subgroups are nilpotent. Therefore the result of Corollary 2.7 is no longer true for $\mathfrak{N}_cT$-groups with $c > 2$.

### 3. $\mathfrak{N}_c$-transitive kernel

Let $G$ be a group and let $c$ be a positive integer. Put $T_0^{(c)}(G) = 1$ and let $T_1^{(c)}(G)$ be the group generated by all commutators $[x_1, x_2, \ldots, x_{c+1}]$ for $x_i \in \{a, b\}$, where $a$ and $b$ are nontrivial elements of $G$ such that there exist $t \in \mathbb{N}_0$ and $y_1, \ldots, y_t \in G\backslash\{1\}$ with $\langle a, y_1\rangle \in \mathfrak{N}_c, \langle y_1, y_2\rangle \in \mathfrak{N}_c, \ldots, \langle y_t, b\rangle \in \mathfrak{N}_c$. It is clear that $T_1^{(c)}(G)$ is a characteristic subgroup of $G$. For $n > 1$ we define $T_n^{(c)}(G)$ inductively by $T_n^{(c)}(G)/T_{n-1}^{(c)}(G) = T_1^{(c)}(G/T_{n-1}^{(c)}(G))$. So we get a chain $1 = T_0^{(c)}(G) \leq T_1^{(c)}(G) \leq \cdots \leq T_n^{(c)}(G) \leq \cdots$ of characteristic subgroups of the group $G$. We define

$$T^{(c)}(G) = \bigcup_{n\in\mathbb{N}_0} T_n^{(c)}(G)$$

to be the *(nilpotent of class c)-transitive kernel* or, shorter, *$\mathfrak{N}_c$-transitive kernel* of the group $G$. In the case $c = 1$ this definition coincides with the usual definition of the commutative-transitive kernel given in [4]. From the definition it also follows that $T^{(c)}(G)$ is a characteristic subgroup of $G$ and that $T^{(c)}(G) = 1$ if and only if $G$ is an $\mathfrak{N}_cT$-group. Moreover, $G/T^{(c)}(G)$ is an $\mathfrak{N}_cT$-group for every group $G$. Additionally, notice that $T^{(c)}(G) = T_n^{(c)}(G)$ for some $n \in \mathbb{N}_0$ if and only if $G/T_n^{(c)}(G)$ is an $\mathfrak{N}_cT$-group. We use the notation $\Gamma_t(G) = \langle \gamma_t(\langle a, b\rangle) \mid a, b \in G\rangle$. It is easy to see that $T^{(c)}(G) \leq \Gamma_{c+1}(G)$.

In [2] it is proved that if $G$ is a locally finite group, then $T^{(1)}(G) = T_1^{(1)}(G)$. In this section we shall show that we have an analogous result for the $\mathfrak{N}_c$-transitive kernel.

**Proposition 3.1.** *Let $G$ be a group and $H$ a subgroup of $G$. Let $c$ be a positive integer and suppose that the set $\mathcal{S} = \{h \in H \mid \langle h, k\rangle \in \mathfrak{N}_c$ for all $k \in H\}$ contains a nontrivial element. Then the group $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group.*

*Proof.* Let $z \in \mathcal{S}\backslash\{1\}$. For all $a, b \in H\backslash\{1\}$ we have $\gamma_{c+1}(\langle a, b\rangle) \leq T_1^{(c)}(H)$, since the groups $\langle a, z\rangle$ and $\langle z, b\rangle$ are nilpotent of class $\leq c$. This implies that $\Gamma_{c+1}(H) = T_1^{(c)}(H) \leq T_1^{(c)}(G)$, so $HT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group. □

Note that Proposition 3.1 implies that if $G$ is a finite group, then every Sylow subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group. In particular, if $G$ is finite then the Fitting subgroup of $G/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group.

**Proposition 3.2.** *The class of finite $\mathfrak{N}_cT$-groups is closed under taking quotients.*

*Proof.* By Theorem 2.5 it suffices to consider finite soluble $\mathfrak{N}_cT$-groups. So suppose that $G$ is a finite soluble $\mathfrak{N}_cT$-group. If $G \in \mathfrak{N}(2, c)$, then we are done. Otherwise, $G$ is a Frobenius group with the kernel $F = \mathrm{Fitt}(G)$ which is an $\mathfrak{N}(2, c)$-group and a complement $H$ which is nilpotent of class $\leq c$ by Theorem 2.4. If $N$ is a normal subgroup of $G$, then we have either $N \leq F$ or $F \leq N$. If $F \leq N$, then $G/N$ is nilpotent of class $\leq c$, hence it is an $\mathfrak{N}_cT$-group. Assume now that $N$ is a proper subgroup of $F$. Then $G/N = F/N \rtimes H$, where the action of $H$ on $F/N$ is induced by the conjugation on $F$ with elements of $H$. Since the subgroup $N$ is invariant under the action of $H$, we conclude that $H$ acts fixed-point-freely on $F/N$ by Satz 8.10 in [5]. Therefore $G/N$ is an $\mathfrak{N}_cT$-group by Theorem 2.4. □

The following result is a generalization of Theorem 3 in [2]:

**Theorem 3.3.** *Let $G$ be a finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer $c$.*

*Proof.* If $T_1^{(c)}(G) = 1$ or $T_1^{(c)}(G) = \Gamma_{c+1}(G)$, then we have nothing to prove. So we may assume that $1 \neq T_1^{(c)}(G) < \Gamma_{c+1}(G)$. Additionally, we may suppose that $T^{(c)}(H) = T_1^{(c)}(H)$ for every proper subgroup $H$ of $G$. Let $\mathcal{F} = \{1 \neq H \triangleleft G \mid \Gamma_{c+1}(H) \leq T_1^{(c)}(G)\}$. Then this set is not empty since $T_1^{(c)}(G) \in \mathcal{F}$. So $\mathcal{F}$ has a maximal element $N$. First of all, it is clear that $N \neq G$, since $T_1^{(c)}(G) \neq \Gamma_{c+1}(G)$. Furthermore, since $NT_1^{(c)}(G)/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, the group $NT_1^{(c)}(G)$ also belongs to $\mathcal{F}$, so we have $T_1^{(c)}(G) \leq N$ by the maximality of $N$. Let $F/T_1^{(c)}(G)$ be the Fitting subgroup of $G/T_1^{(c)}(G)$. Since $N/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, it is nilpotent, hence $N/T_1^{(c)}(G) \leq F/T_1^{(c)}(G)$. On the other hand, since $F/T_1^{(c)}(G)$ is an $\mathfrak{N}(2, c)$-group, we have that $\Gamma_{c+1}(F) \leq T_1^{(c)}(G)$. Thus $F \in \mathcal{F}$, hence $F = N$ by the maximality of $N$ in $\mathcal{F}$. Consider now the set $\mathcal{S} = \{h \in N \mid \langle h, k\rangle \in \mathfrak{N}_c \text{ for all } k \in N\}$. Here we have to consider the following two cases.

*Case* 1. Suppose that $\mathcal{S} \neq \{1\}$ and let $h$ be a nontrivial element of $\mathcal{S}$. Let $y \in N\backslash\{1\}$ and let $a \in C_G(y)$. For every $b \in N$ we have $\gamma_{c+1}(\langle a, b\rangle) \leq T_1^{(c)}(G)$, since $\langle a, y\rangle$, $\langle y, h\rangle$ and $\langle h, b\rangle$ are in $\mathfrak{N}_c$. Additionally we have that $\langle a^g, y^g\rangle$, $\langle y^g, h\rangle$, $\langle h, y^k\rangle$ and $\langle y^k, a^k\rangle$ are in $\mathfrak{N}_c$ for all $g, k \in G$. Hence $\gamma_{c+1}(\langle a^g, a^k\rangle) \leq T_1^{(c)}(G)$ for all $g, k \in G$. In particular, this implies that $aT_1^{(c)}(G)$ is a left $(c + 1)$-Engel element of the group $G/T_1^{(c)}(G)$, hence it is contained in the Fitting subgroup of $G/T_1^{(c)}(G)$ by Theorem 12.3.7 in [10]. This gives that $a \in N$. By Satz 8.5 in [5] $G$ is a Frobenius group and $N$ is its kernel. Let $A$ be a complement of $N$ in $G$. Since $T_1^{(c)}(A) \leq A \cap T_1^{(c)}(G) \leq A \cap N = 1$, it follows that $A$ is an $\mathfrak{N}_cT$-group. Moreover the center of $A$ is nontrivial by [5, Satz 8.18], so $A$ is an $\mathfrak{N}(2,c)$-group. Therefore $G$ is soluble. If the nilpotency class of $N$ does not exceed $c$, then $G$ is an $\mathfrak{N}_cT$-group by Theorem 2.3 and $T_1^{(c)}(G) = 1$, which is a contradiction. Hence we may suppose that the nilpotency class of $N$ is greater than $c$. Consider the group $G/T_1^{(c)}(G) = N/T_1^{(c)}(G) \rtimes AT_1^{(c)}(G)/T_1^{(c)}(G)$. This is a Frobenius group with the kernel $N/T_1^{(c)}(G) \in \mathfrak{N}(2,c)$ and complement $AT_1^{(c)}(G)/T_1^{(c)}(G)$ which is also an $\mathfrak{N}(2,c)$-group. By Theorem 2.3 the group $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, hence $T^{(c)}(G) = T_1^{(c)}(G)$ in this case.

*Case* 2. Suppose now that $\mathcal{S} = \{1\}$. Let $\Phi(G)$ be the Frattini subgroup of $G$. If $T_1^{(c)}(G) \leq \Phi(G)$, then the nilpotency of the group $N/T_1^{(c)}(G)$ implies that $N$ is nilpotent, which is a contradiction. Hence $T_1^{(c)}(G) \not\leq \Phi(G)$, so there exists a maximal subgroup $M$ of $G$ such that $T_1^{(c)}(G) \not\leq M$. Then $G = MT_1^{(c)}(G)$ and $T_1^{(c)}(M) = T^{(c)}(M)$ since $M < G$. From $T_1^{(c)}(M) \leq T_1^{(c)}(G) \cap M$ we now obtain that $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group, since it is a homomorphic image of the $\mathfrak{N}_cT$-group $M/T_1^{(c)}(M)$. So $T^{(c)}(G) = T_1^{(c)}(G)$, as required. □

**Corollary 3.4.** *Let $G$ be a locally finite group. Then $T^{(c)}(G) = T_1^{(c)}(G)$ for every positive integer $c$.*

*Proof.* It suffices to show that if $G$ is locally finite, then $G/T_1^{(c)}(G)$ is an $\mathfrak{N}_cT$-group. Let $x, y, z \in G \backslash T_1^{(c)}(G)$ and suppose that the groups $\langle x, y\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ and $\langle y, z\rangle T_1^{(c)}(G)/T_1^{(c)}(G)$ are nilpotent of class $\leq c$. This means that $\gamma_{c+1}(\langle x, y\rangle) \leq T_1^{(c)}(G)$ and $\gamma_{c+1}(\langle y, z\rangle) \leq T_1^{(c)}(G)$. Let $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\overline{\alpha}_1, \ldots, \overline{\alpha}_{r'}\}$ be the sets of all simple commutators of weight $c+1$ with entries from $\{x, y\}$ and $\{y, z\}$, respectively. For every $i = 1, \ldots, r$ we have

$$\alpha_i = \prod_{t=1}^{n_i} [x_{i,t,1}, \ldots, x_{i,t,c+1}]^{\epsilon_{i,t}},$$

where $\epsilon_{i,t} = \pm 1$, $x_{i,t,j} \in \{a_{i,t}, b_{i,t}\}$ for some $a_{i,t}, b_{i,t} \in G$ for which there exist $y_{i,t,1}, \ldots, y_{i,t,s_{i,t}}$ in $G$ such that $\langle a_{i,t}, y_{i,t,1}\rangle, \langle y_{i,t,1}, y_{i,t,2}\rangle, \ldots, \langle y_{i,t,s_{i,t}}, b_{i,t}\rangle$ are nilpotent of class $\leq c$, for all $i = 1, \ldots, r$, $j = 1, \ldots, c+1$ and $t = 1, \ldots, n_i$. Similarly,

$$\overline{\alpha}_{i'} = \prod_{t'=1}^{m_{i'}} [\overline{x}_{i',t',1}, \ldots, \overline{x}_{i',t',c+1}]^{\overline{\epsilon}_{i',t'}},$$

where $\overline{\epsilon}_{i',t'} = \pm 1$, $\overline{x}_{i',t',j} \in \{\overline{a}_{i',t'}, \overline{b}_{i',t'}\}$ for some $\overline{a}_{i',t'}, \overline{b}_{i',t'} \in G$ for which there exist $\overline{y}_{i',t',1}, \ldots, \overline{y}_{i',t',s'_{i',t'}}$ in $G$ such that $\langle \overline{a}_{i',t'}, \overline{y}_{i',t',1}\rangle, \langle \overline{y}_{i',t',1}, \overline{y}_{i',t',2}\rangle, \ldots, \langle \overline{y}_{i',t',s'_{i',t'}}, \overline{b}_{i',t'}\rangle$ are nilpotent of class $\leq c$, for all $i' = 1, \ldots, r'$, $j = 1, \ldots, c+1$ and $t' = 1, \ldots, m_{i'}$. Let $H$ be the subgroup of $G$ generated by all

$$x,\ y,\ z,\ x_{i,t,j},\ \overline{x}_{i',t',j},\ a_{i,t},\ \overline{a}_{i',t'}, y_{i,t,k},\ \overline{y}_{i',t',k'},$$

where $i = 1, \dots, r$, $i' = 1, \dots, r'$, $t = 1, \dots, n_i$, $t' = 1, \dots, m_{i'}$, $j = 1, \dots, c+1$, $k = 1, \dots, s_{i,t}$ and $k' = 1, \dots, s'_{i',t'}$. Then $\gamma_{c+1}(\langle x, y\rangle) \le T_1^{(c)}(H)$ and $\gamma_{c+1}(\langle y, z\rangle) \le T_1^{(c)}(H)$. Since $H/T_1^{(c)}(H)$ is an $\mathfrak{N}_c T$-group by Theorem 3.3, we have $\gamma_{c+1}(\langle y, z\rangle) \le T_1^{(c)}(H) \le T_1^{(c)}(G)$. This concludes the proof. □

**Remark 3.5.** Let $G$ be a locally nilpotent group, and let $c \ge 1$ be any positive integer. It easily follows from Proposition 3.1 that $T_1^{(c)}(G) = T^{(c)}(G) = \Gamma_{c+1}(G)$.

**Remark 3.6.** Let $G$ be a supersoluble group. It is proved in [3] that $T^{(1)}(G) = T_1^{(1)}(G)$. It is to be expected that the same holds true for $\mathfrak{N}_c$-transitive kernel where $c > 1$, and that the proofs require only suitable modifications of those in [3].

## 4. Examples and non-examples

Theorem 2.4 completely describes the structure of finite soluble $\mathfrak{N}_c T$-groups. At least in the case $c \le 2$ we are able to obtain more detailed information about these groups, using the descriptions of fixed-point-free actions on finite abelian groups obtained by Zassenhaus [16].

**Example 4.1.** Let $G$ be a finite soluble $\mathfrak{N}_1 T$-group (or $CT$-group) which is not abelian. Then $G = F \rtimes \langle x\rangle$ where $F$ is abelian and $\langle x\rangle$ acts fixed-point-freely on $F$ (see Theorem 2.4 or Theorem 10 of [15]). Suppose $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Let $k$ be the order of $\langle x\rangle$. Then it follows from [16] that $x = (x_1, \dots, x_m)$ where $\langle x_i\rangle$ is a fixed-point-free automorphism group of order $k$ on $G_i$ for all $i = 1, \dots, m$. Conversely, for every $x$ with this property the group $\langle x\rangle$ acts fixed-point-freely on $F$. Note also that a necessary and sufficient condition for the existence of a fixed-point-free automorphism on $F$ is given in Theorem 2 of [15].

As the class of $\mathfrak{N}(2,2)$-groups coincides with the variety of 2-Engel groups, Theorem 2.4 implies that a finite soluble $\mathfrak{N}_2 T$-group is either 2-Engel or it is a Frobenius group with the kernel $F$ which is 2-Engel and a complement $H$ which is nilpotent of class $\le 2$. Thus it follows from Levi's theorem (see [9, p. 45]) that $F$ is nilpotent of class $\le 3$. Moreover, if $|H|$ is even, then $F$ is abelian. In this case, $H$ is either a cyclic group or the quaternion group $Q_8$ of order 8 or $C_m \times Q_8$ where $m$ is odd. Our next example shows that there is essentially only one possibility of having a Frobenius $\mathfrak{N}_2 T$-group with the prescribed kernel and a complement isomorphic to $Q_8$.

**Example 4.2.** Let $F$ be a finite abelian group and $F = \bigoplus_{i=1}^m F_i$ where $F_i \cong \mathbb{Z}_{p_i^{e_i}}^{n_i}$ and $e_i \neq e_j$ if $p_i = p_j$. Then it follows from [16] that $F$ admits a quaternion fixed-point-free automorphism group $H$ of order 8 if and only if $2 \nmid p_i$ and $2 | n_i$

for all $i = 1, \ldots, m$. In this case, $H$ is conjugated to the group $\langle x, y\rangle$ where the restrictions of $x$ and $y$ on $F_i$ can be presented by matrices

$$A_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_i = \bigoplus_{j=1}^{n_i/2} \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i & -\alpha_i \end{pmatrix},$$

where $i = 1, \ldots, m$ and $\alpha_i^2 + \beta_i^2 \equiv -1 \mod p_i^{e_i}$ for all $i = 1, \ldots, m$.

In the following example we present a Frobenius group $G$ with abelian kernel $F$ and a complement $H$ which is isomorphic to $C_p \times Q_8$, where $p$ is an arbitrary odd prime. Of course, in this case $G$ is an $\mathfrak{N}_2T$-group.

**Example 4.3.** Let $q$ be a prime such that $p|(q-1)$ and let $F = C_q^2$. Let $a, b \in \mathbb{Z}_q$ be such that $a^2+b^2+1 \equiv 0 \mod q$. Consider the automorphisms of $C_q^2$ represented by the following matrices over $\mathbb{Z}_q$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \ B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \ , \ X = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}.$$

Here $\zeta$ is a primitive $p$-th root modulo $q$. Then we have $\langle A, B, X\rangle \cong C_p \times Q_8$ and it can be verified that $H = \langle A, B, X\rangle$ acts fixed-point-freely on $F$. The corresponding Frobenius group $F \rtimes H$ is an $\mathfrak{N}_2T$-group, but it is not an $\mathfrak{N}_1T$-group.

On the other hand, if the order of $H$ is odd, then $H$ is cyclic and the group $F$ may be nonabelian. In the next example we show that this is indeed so.

**Example 4.4.** Let $D = \langle x_1\rangle \times \langle x_2\rangle \times \langle x_3\rangle \times \langle x_4\rangle$ be an elementary group of order 16. Put $D_1 = D \rtimes \langle a\rangle$, where $a$ is an element of order 2 acting on $D$ in the following way: $[x_1, a] = x_3x_4$, $[x_2, a] = x_4$, $[x_3, a] = [x_4, a] = 1$. We make another split extension $F = D_1 \rtimes \langle b\rangle$, where $b$ induces an automorphism of order 2 on $D_1$ in the following way: $[x_1, b] = x_3$, $[x_2, b] = x_3x_4$ and $[x_3, b] = [x_4, b] = [a, b] = 1$. The group $F$ is nilpotent of class 2 and $|F| = 64$. Consider the following map on $F$:

$$x_1^\alpha = x_2 \ , \ x_2^\alpha = x_1x_2 \ , \ x_3^\alpha = x_4 \ , \ x_4^\alpha = x_3x_4 \ , \ a^\alpha = ab \ , \ b^\alpha = a.$$

It can be verified that $\alpha$ is an automorphism of order 3 on $F$. Moreover, $\alpha$ acts fixed-point-freely on $F$. The corresponding split extension $G = F \rtimes \langle\alpha\rangle$ is an $\mathfrak{N}_2T$-group of order 192 with the kernel $F$. One can verify that this is the smallest example of a non-nilpotent soluble $\mathfrak{N}_2T$-group having the nonabelian Frobenius kernel.

Finite simple groups with nilpotent centralizers are classified in [12] and [13]. It turns out that every finite nonabelian simple $CN$-group is of one of the following types:

(i) $\mathrm{PSL}(2, 2^f)$, where $f > 1$;

(ii) $\mathrm{Sz}(q)$, the Suzuki group with parameter $q = 2^{2n+1} > 2$;

(iii) $\mathrm{PSL}(2,p)$, where $p$ is either a Fermat prime or a Mersenne prime;
(iv) $\mathrm{PSL}(2,9)$;
(v) $\mathrm{PSL}(3,4)$.

By Theorem 2.5 only groups listed under (i) and (ii) are $\mathfrak{N}_cT$-groups for $c > 1$. Our aim is to show that in groups (iii)-(v) we can always find such nontrivial elements $x$, $y$ and $z$ that the groups $\langle x, y\rangle$ and $\langle y, z\rangle$ are nilpotent of class $\leq 2$, yet the group $\langle x, z\rangle$ is not even nilpotent. We call such a triple of elements a *bad triple.*

**Proposition 4.5.** *In the groups* $\mathrm{PSL}(2,9)$ *and* $\mathrm{PSL}(3,4)$ *there exist bad triples of elements.*

*Proof.* First we want to show that our proposition holds true for $\mathrm{PSL}(3,4)$. To this end, consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

over the Galois field $\mathrm{GF}(4)$. It is easy to see that $A$, $B$ and $C$ belong to $\mathrm{SL}(3,4)$. Besides, these matrices are not in the center of $\mathrm{SL}(3,4)$ and a straightforward calculation shows that $[A,B] = [B,C,C] = [C,B,B] = 1$. Let $\overline{A}$, $\overline{B}$ and $\overline{C}$ be the homomorphic images of $A$, $B$ and $C$, respectively, under the canonical homomorphism $\mathrm{SL}(3,4) \to \mathrm{PSL}(3,4)$. Then the group $\langle\overline{A},\overline{B}\rangle$ is abelian and $\langle\overline{B},\overline{C}\rangle$ is nilpotent of class 2. On the other hand, $\langle\overline{A},\overline{C}\rangle$ is not nilpotent, since $[A,C],[A,C,C] \notin Z(\mathrm{SL}(3,4))$ and $[A,C,C,C] = [A,C,C]$.

A similar argument also works for the group $\mathrm{PSL}(2,9)$. In this case, we have to consider the following matrices in $\mathrm{SL}(2,9)$:

$$A = \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^5 \end{pmatrix} \quad , \quad B = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^6 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & \zeta^4 \\ \zeta^4 & \zeta^4 \end{pmatrix}.$$

Here $\zeta$ is a generator of the multiplicative group of $\mathrm{GF}(9)$. If $\overline{A}$, $\overline{B}$ and $\overline{C}$ are the corresponding elements of $\mathrm{PSL}(2,9)$, then it is a routine to verify that the group $\langle\overline{A},\overline{B}\rangle$ is abelian and $\langle\overline{B},\overline{C}\rangle$ is nilpotent of class 2, but $\langle\overline{A},\overline{C}\rangle$ is not nilpotent. $\square$

Finally we consider the groups $\mathrm{PSL}(2,p)$ where $p$ is a Fermat prime or a Mersenne prime. If $p = 5$, then $\mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4)$ is an $\mathfrak{N}_1T$-group by [11]. For $p > 5$ the situation is completely different.

**Proposition 4.6.** *If $p$ is a Fermat prime or a Mersenne prime and $p \neq 5$, then* $\mathrm{PSL}(2,p)$ *contains a bad triple of elements.*

*Proof.* First we cover the case of Fermat primes. For this we need the following number-theoretical result:

*Claim* 1. If $p$ is a Fermat prime, then there exists $x \in \mathbb{Z}_p$ such that $2x^2 \equiv -1 \mod p$.

*Proof of Claim* 1. Let $p = 2^{2^n} + 1$ for some $n > 1$. It is enough to show that $2^{2^n-1}$ is a quadratic residue modulo $p$. Let $P$ be the set of all integers $a \in \{0, \ldots, p-1\}$ which are primitive roots modulo $p$ and let $Q$ be the set of all $a \in \{0, \ldots, p-1\}$ which are not quadratic residues modulo $p$. We shall show that $P = Q$. First, if $a \notin Q$, then there exists an integer $t$ such that $t^2 \equiv a \mod p$. By Euler's theorem, $a^{\phi(p)/2} \equiv t^{\phi(p)} \equiv 1 \mod p$, hence $a$ is not a primitive root modulo $p$ (here $\phi$ is the Euler function). This shows that $P \subseteq Q$. To prove the converse inclusion, note that $p$ has exactly $\phi(\phi(p))$ incongruent primitive roots and exactly $(p-1)/2$ quadratic non-residues. Hence

$$|P| = \phi(\phi(p)) = \phi(p-1) = \phi(2^{2^n}) = 2^{2^n-1} = \frac{p-1}{2} = |Q|$$

and therefore $P = Q$. Since $2^{2^n-1} \notin P = Q$, we have that $2^{2^n-1} \equiv x^2 \mod p$ for some $x \in \mathbb{Z}_p$, hence $2x^2 \equiv -1 \mod p$, as desired.

Now we are ready to finish the proof. Let $c, x \in \mathbb{Z}_p$ be such that $c^2 \equiv -1 \mod p$, $c \not\equiv -c \mod p$ and $2x^2 \equiv -1 \mod p$ (such $x$ exists by Claim 1). Let

$$A = \begin{pmatrix} 2x & 0 \\ 0 & -x \end{pmatrix} \quad , \quad B = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} x & x \\ x & -x \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p) \backslash Z(\mathrm{SL}(2,p))$. It is clear that $A$ and $B$ commute, and a short calculation shows that $[B, C, C]$ and $[C, B, B]$ belong to $Z(\mathrm{SL}(2,p))$. To prove that $\mathrm{PSL}(2,p)$ is not an $\mathfrak{N}_c T$-group for any $c > 1$ it suffices to show that $[C, {}_n A] \notin Z(\mathrm{SL}(2,p))$ for any $n \in \mathbb{N}$. More precisely, we shall prove that

$$[C, {}_n A] = x^{3 \cdot 2^n - 2} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

where $a_n, b_n, c_n, d_n \in \mathbb{Z}_p$ are such that at least one of $b_n$, $c_n$ and at least one of $a_n$, $d_n$ are not zero. First note that this is true for $n = 1$, hence we may assume that $n > 1$. Then

$$[C, {}_{n+1} A] = x^{3 \cdot 2^{n+1} - 2} \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix},$$

where $a_{n+1} = -2a_n d_n - 4b_n c_n$, $b_{n+1} = 3b_n d_n$, $c_{n+1} = 2a_n c_n$ and $d_{n+1} = b_n c_n - 2a_n d_n$. If both $b_{n+1}$ and $c_{n+1}$ are zero, then $a_n = d_n = 0$ which is not possible by the induction assumption. Similarly, if $a_{n+1} = d_{n+1} = 0$, then $a_n d_n = -2b_n c_n$ and $b_n c_n = 2a_n d_n$, hence $5b_n c_n = 0$, a contradiction since $p > 5$. This concludes the proof for Fermat primes.

Assume now that $p$ is a Mersenne prime. In this case we need the following auxiliary result:

*Claim* 2. If $p$ is a Mersenne prime, then there exist $x, y \in \mathbb{Z}_p$ such that $x^2 - x + 1 \equiv 0 \mod p$ and $xy^4 \equiv 2y^2 + 1 \mod p$.

*Proof of Claim* 2. First note that since $p$ is a Mersenne prime, $p-1$ is divisible by 6. The congruence equation $x^3 \equiv -1 \mod p$ is clearly solvable, hence it has $\gcd(3, p-1) = 3$ incongruent solutions. This shows that the equation $x^2 - x + 1 = 0$ is solvable in $\mathbb{Z}_p$. Let $x_1$ and $x_2$ be its solutions. Then $x_2 = x_1^{-1} = 1 - x_1$. We claim that at least one of $1+x_1$, $1+x_2$ is a quadratic residue modulo $p$. For this note that since $(p-1)/2$ is odd, Euler's criterion implies that for every $a \in \mathbb{Z}_p\backslash\{0\}$ we have that precisely one of $a$ and $-a$ is a quadratic residue modulo $p$. Furthermore, since $\gcd(2^k, p-1) = \gcd(2, p-1)$, every quadratic residue modulo $p$ is also a $2^k$-power residue modulo $p$. Suppose $1+x_1$ is not a square residue modulo $p$. Then $-1-x_1$ is a quadratic residue modulo $p$ and $1+x_2 = 2-x_1 = 1-x_1^2 = x_1^2(-1-x_1)$ is a square residue modulo $p$. So from now on we assume $x$ is such that $1-x+x^2 \equiv 0 \mod p$ and $1+x$ is a square residue modulo $p$. Then the equation $xt^2-2t-1=0$ has two solutions in $\mathbb{Z}_p$, namely $t_{1,2} = x^{-1}(1\pm c) = x^2(-1\mp c)$, where $c^2 = 1+x$ in $\mathbb{Z}_p$. In order to ensure the existence of $y$ it suffices to prove that $-1\mp c$ are square residues modulo $p$. Since $(-1+c)(-1-c) = -x = x^4$, we have that $-1+c$ and $-1-c$ are either both squares or both non-squares in $\mathbb{Z}_p$. Assume that they are not squares. Then $1+c$ and $1-c$ are squares in $\mathbb{Z}_p$. For every square $q$ in $\mathbb{Z}_p$ denote by $\sqrt{q}$ the square in $\mathbb{Z}_p$ for which $(\sqrt{q})^2 = q$. Let $u = \sqrt{1-c}$ and $v = \sqrt{1+c}$. Then $(u+v)^2 = u^2+v^2+2uv = 2(1+\sqrt{1-c^2}) = 2(1+\sqrt{-x}) = 2(1+x^2)$. Since $p \equiv -1 \mod 8$, 2 is a square residue modulo $p$, hence $1+x^2$ is a square in $\mathbb{Z}_p$. On the other hand, $-1-x^2 = -x = x^4$ is also a square in $\mathbb{Z}_p$. This leads to a contradiction, hence our claim is proved.

Let $x$ and $y$ be as above and let

$$A = \begin{pmatrix} 0 & x^2 \\ x & 0 \end{pmatrix}, \quad B = \begin{pmatrix} x & x \\ -1 & -x \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}$$

be matrices in $\mathrm{SL}(2,p)\backslash Z(\mathrm{SL}(2,p))$. It is not difficult to check that $[A,B] = -1$, hence $[A,B] \in Z(\mathrm{SL}(2,p))$. Beside that, we have $[B,C,C] = (a_{ij})_{i,j}$, and a straightforward calculation shows that $a_{11} - a_{22} = x - x^2y^4 - 2x^4y^2 = 0$ by Claim 2. Similarly, we obtain $a_{21} = a_{12} = 0$, hence $[B,C,C]$ belongs to $Z(\mathrm{SL}(2,p))$. Furthermore, it can be checked that the same holds true for $[C,B,B]$. On the other hand, an induction argument shows that

$$[A, {}_nC] = \begin{pmatrix} y^{(-2)^n} x^{2^{n+1}} & 0 \\ 0 & y^{-(-2)^n} x^{2^n} \end{pmatrix}$$

for every $n \in \mathbb{N}$. If $[A, {}_nC] \in Z(\mathrm{SL}(2,p))$ for some $n \in \mathbb{N}$, then $y^{(-2)^m} x^{2^{m+1}} = y^{-(-2)^m} x^{2^m} = 1$ in $\mathbb{Z}_p$ for every $m > n$. Besides we have that $x^{2^k}$ is either $x-1$ or $-x$, depending on whether $k$ is odd or even, respectively. Suppose $m > n$ and let $m$ be even. Then $[A, {}_mC] = 1$ implies $y^{2^m}(x-1) = 1$ and $y^{-2^m}x = -1$. Similarly, from $[A, {}_{m+1}C] = 1$ we obtain $y^{-2^{m+1}}x = -1$ and $y^{2^{m+1}}(x-1) = 1$. This implies $y^{2^m} = 1$ and hence $x = -1$, which contradicts the choice of $x$. □

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