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Circumscribed Simplices of Minimal Mean Width*

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Abstract. It is proved that the minimal mean width of all simplices circumscribed about a convex body of given mean width attains its maximum precisely if the body is a ball. An analogous result holds for circumscribed parallelepipeds, with balls replaced by bodies of constant width.

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1. Introduction and main result

For a convex body K (a compact convex set with interior points) in Euclidean space $\mathbb{R}^n$ ($n \geq 2$), we denote by $M(K)$ its mean width and by T_K a simplex of minimal mean width circumscribed about K. Let T^n be a regular simplex circumscribed about the unit ball B^n of $\mathbb{R}^n$. In this note, we prove the following result.

Theorem 1. *For any convex body* $K \subset \mathbb{R}^n$,

$$M(T_K) \leq \frac{1}{2} M(K) M(T^n). \tag{1}$$

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Equality holds if and only if K is a ball.

Every simplex of minimal mean width circumscribed about a given ball is regular.

Remark 1. For $n = 2$ (where the mean width is the perimeter divided by π), a more general result is known (see [5]). If $L_m(K)$ denotes the minimum of the perimeters of all convex m-gons circumscribed about the planar convex body K and $L(K)$ is the perimeter of K, then

$$L_m(K) \le L(K)\frac{m}{\pi}\tan\frac{\pi}{m}$$

for $m = 3, 4, \ldots$, with equality if and only if K is circular; every m-gon of minimal perimeter circumscribed about a given circle is regular.

Remark 2. The value of $M(T^n)$ for $n = 2$ is given by $6\sqrt{3}/\pi = 3.30797$ and for $n = 3$ by $(3\sqrt{6}/\pi)\arccos(-1/3) = 4.4691$. From these values, one can obtain the value for $n = 4$ by using formula (3) in [7]. Further,

$$M(T^n) \sim 2\sqrt{2n\ln n}$$

as n tends to infinity, according to a result obtained in [1].

Remark 3. The 'dual' analogue of the last part of Theorem 1 is (for $n \ge 3$) a long-standing open problem: among all simplices contained in a given ball, do the regular ones have maximal mean width? We refer to the discussion in Gritzmann and Klee [2, Section 9.10.2].

2. Proof of Theorem 1

We fix some notation. We denote the scalar product of $\mathbb{R}^n$ by $\langle\cdot,\cdot\rangle$ and the induced norm by $\|\cdot\|$. The set $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere, σ denotes spherical Lebesgue measure on S^{n-1}, and $\omega_n := \sigma(S^{n-1})$ is the total area of the unit sphere. The support function of a convex body K is defined by $h_K(u) := \max_{x\in K}\langle x, u\rangle$ for $u \in S^{n-1}$. The mean width of K is given by

$$M(K) = \frac{1}{\omega_n}\int_{S^{n-1}}[h_K(u) + h_K(-u)]\,\sigma(\mathrm{d}u) = \frac{2}{\omega_n}\int_{S^{n-1}} h_K\,\mathrm{d}\sigma.$$

First we deal with the last assertion of Theorem 1. This follows by the same argument as Hadwiger [3] used it for the isoperimetric quotient. We assume that T is a simplex circumscribed to the unit ball B^n and with minimal mean width. Suppose T is not regular, then it has vertices v_0, v_1, v_2 such that $\|v_2 - v_0\| \ne \|v_2 - v_1\|$. Let T' arise from Steiner symmetrization of T with respect to the hyperplane which is the perpendicular bisector of the segment with endpoints v_0 and v_1. Then T' is a simplex with $M(T') < M(T)$ ([3], p. 261, Zusatz IV), and T' clearly contains a unit ball. This is a contradiction, hence T is congruent to T^n.

To prove the first part of Theorem 1, we need the following lemma.

Lemma. *In $\mathbb{R}^n$, let K be a convex body and suppose that T is a regular simplex circumscribed about K with $M(T) = M(T_K)$. Then each facet F of T touches K in the centroid of F.*

Proof. Let T satisfy the assumption, and let F be a facet of T. We suppose that the centroid c of F is not contained in K, and seek a contradiction. We may assume that the origin o is the vertex of T opposite to F, and that the vectors $v_1, \dots, v_n$ giving the vertices of F are unit vectors. Later we will determine numbers $s_1, \dots, s_n \in \mathbb{R}$ with the following properties:

(i) $s_i < 1$ for $i = 1, \dots, n$, and $\sum_{i=1}^n s_i < 0$;

(ii) for all sufficiently small $\varrho > 0$, the simplex $T(\varrho)$ with vertices o and $v_i(\varrho) := (1 + \varrho s_i)v_i$, $i = 1, \dots, n$, contains K.

Let us assume that (i) and (ii) are satisfied and that $\varrho > 0$ is so small that $1 + \varrho s_i > 0$ for $i = 1, \dots, n$. Let $U_i(\varrho)$ denote the spherical image of the vertex $v_i(\varrho)$ of $T(\varrho)$, that is, the intersection of the unit sphere with the normal cone of $T(\varrho)$ at $v_i(\varrho)$. It can be represented by

$$U_i(\varrho) = \{x \in S^{n-1} : \langle x, v_i(\varrho)\rangle \geq 0 \ \text{ and } \ \langle x, v_i(\varrho) - v_j(\varrho)\rangle \geq 0,\ j = 1, \dots, n\}.$$

Write $U_i := U_i(0)$ for $i = 1, \dots, n$. If $x \in U_i \cap U_j(\varrho)$, then

$$\begin{aligned} h_{T(\varrho)}(x) - h_T(x) &= \langle x, (1 + \varrho s_j)v_j\rangle - \langle x, v_i\rangle \\ &\leq (1 + \varrho s_j)\langle x, v_i\rangle - \langle x, v_i\rangle = \varrho s_j \langle x, v_i\rangle. \end{aligned}$$

Below, the implied constants in $O(\cdot)$ depend on n and $s_1, \dots, s_n$. Since $\sigma(U_i \cap U_j(\varrho))$ is a continuously differentiable function of ϱ, we have $\sigma(U_i \cap U_j(\varrho)) = O(\varrho)$ for $i \neq j$ and hence $\sigma(U_i \Delta U_i(\varrho)) = O(\varrho)$, where Δ denotes the symmetric difference. Observing that

$$M(T(\varrho)) = \frac{2}{\omega_n} \sum_{j=1}^n \int_{U_j(\varrho)} h_{T(\varrho)}\, \mathrm{d}\sigma = \frac{2}{\omega_n} \sum_{j=1}^n \sum_{i=1}^n \int_{U_i \cap U_j(\varrho)} h_{T(\varrho)}\, \mathrm{d}\sigma,$$

since $h_{T(\rho)} = 0$ on the spherical image of the vertex o (and this spherical image is independent of ϱ), and that $\int_{U_i} \langle x, v_i\rangle\, \sigma(\mathrm{d}x)$ is independent of i, we obtain

$$M(T(\varrho)) - M(T) \leq \frac{2}{\omega_n} \int_{U_1} \langle x, v_1\rangle\, \sigma(\mathrm{d}x) \left(\sum_{i=1}^n s_i\right) \varrho + O(\varrho^2).$$

Therefore, if $\varrho > 0$ is sufficiently small, then $M(T(\varrho)) < M(T)$, which is a contradiction.

To finish the proof of the lemma, we have to find $s_1, \dots, s_n$ satisfying (i) and (ii). Since $c \notin K$, we can choose an $(n-2)$-dimensional linear subspace L of $\mathbb{R}^n$ such that $c + L \subset \operatorname{aff} F$ and $(c + L) \cap K = \emptyset$. Let $u_1, \dots, u_n \in \mathbb{R}^n$ denote the dual basis of $v_1, \dots, v_n$; namely, $\langle u_i, v_j\rangle = 1$ if $i = j$, and $\langle u_i, v_j\rangle = 0$ if $i \neq j$. Then $\sum_{i=1}^n u_i$ is orthogonal to $\operatorname{aff} F$. We can determine $\tau_1, \dots, \tau_n \in \mathbb{R}$, not all zero, so that $\sum_{i=1}^n \tau_i u_i$ is orthogonal to L and orthogonal to $\sum_{i=1}^n v_i = nc$. In particular, $\sum_{i=1}^n \tau_i = 0$. Without loss of generality, we suppose that $\tau_i < 1$ for all i. We may assume that v_1 and v_2 are strictly separated by $c + L$ in $\operatorname{aff} F$, that v_1 and $F \cap K$ lie on the same side of $c + L$, and that $\tau_1 > 0$ (hence $\tau_2 < 0$).

Let S be any $(n-2)$-dimensional affine subspace of $\operatorname{aff} F$ that does not meet K. It divides $\operatorname{aff} F$ into two halfspaces; let w be the unit vector parallel to F and normal to S that points into the halfspace not containing $F \cap K$. The vector $z := c/\|c\|$ is the unit normal vector of F pointing away from o. For $\varphi \in [0, \pi]$, let $H(\varphi)$ denote the hyperplane with normal vector $(\cos\varphi)z + (\sin\varphi)w$ and containing S. There is a largest number $\varphi(S) \in (0, \pi)$ such that $H(\varphi) \cap K = \emptyset$ for $0 < \varphi < \varphi(S)$.

For $\delta > 0$, we define the δ-neighborhood N_δ of $c+L$ as the set of all subspaces S as above which have distance less than δ from c and for which there exists a vector $\sum_{i=1}^n \alpha_i u_i$ orthogonal to S and with $|\alpha_i - \tau_i| < \delta$ for $i = 1, \dots, n$. Elementary continuity and compactness arguments yield the existence of numbers $\delta_0 > 0$ and $\varphi_0 > 0$ such that $\varphi(S) \geq \varphi_0$ for all $S \in N_{\delta_0}$.

Let $0 < \varepsilon < \tau_1$. For small $\varrho > 0$, let $t_{\varrho 1} := \varrho(\tau_1 - \varepsilon)$ and $t_{\varrho i} := \varrho\tau_i$, $i = 2, \dots, n$. Let H_ϱ be the hyperplane passing through $(1 + t_{\varrho i})v_i$ for $i = 1, \dots, n$, and let $S_\varrho := H_\varrho \cap \operatorname{aff} F$. We put $\beta := \varepsilon/(\tau_1 - \tau_2 - \varepsilon)$ and suppose that ε is so small that $\beta < 1$. Let

$$c_\varrho := \frac{1+\beta}{n}(1 + t_{\varrho 1})v_1 + \frac{1-\beta}{n}(1 + t_{\varrho 2})v_2 + \frac{1}{n}\sum_{i=3}^n (1 + t_{\varrho i})v_i.$$

Then c_ϱ is a convex combination of $(1 + t_{\varrho i})v_i$, $i = 1, \dots, n$, hence $c_\varrho \in H_\varrho$. Moreover, $c_\varrho - c$ is orthogonal to $\sum_{i=1}^n u_i$. It follows that $c_\varrho \in S_\varrho$.

The vector $\sum_{i=1}^n (1 + t_{\varrho i})^{-1} u_i$ is orthogonal to H_ϱ, hence the difference of u and this vector, namely $\sum_{i=1}^n \alpha_i u_i := \sum_{i=1}^n t_{\varrho i}(1 + t_{\varrho i})^{-1} u_i$, is orthogonal to S_ϱ. We can now choose $\tilde\varrho > 0$ so small that for all $\varrho \in [0, \tilde\varrho)$ we have $|\alpha_i - \tau_i| < \delta_0$ for $i = 1, \dots, n$. Next, we can decrease $\tilde\varrho > 0$, if necessary, and choose the number $\varepsilon > 0$ so small that $\|c_\varrho - c\| < \delta_0$ for all $\varrho \in [0, \tilde\varrho)$. With these choices, for $\varrho \in [0, \tilde\varrho)$ we have $S_\varrho \in N_{\delta_0}$ and, therefore, $\varphi(S_\varrho) \geq \varphi_0$. Again decreasing $\tilde\varrho > 0$, if necessary, we can achieve that for all $\varrho \in [0, \tilde\varrho)$ the hyperplane H_ϱ makes an angle with the hyperplane $H_0 = \operatorname{aff} F$ which is smaller than φ_0. This implies that $H_\varrho \cap K = \emptyset$ and, hence, that K is contained in the convex hull of o and $(1 + t_{\varrho i})v_i$, $i = 1, \dots, n$. Now we see that the numbers defined by $s_1 := \tau_1 - \varepsilon$ and $s_i := \tau_i$ for $i = 2, \dots, n$ satisfy (i) and (ii). This concludes the proof of the lemma.

Now we finish the proof of Theorem 1. Let T be a regular simplex, and let $u_1, \dots, u_{n+1}$ be the exterior unit normal vectors of its facets. Then

$$M(T) = \frac{M(T^n)}{n+1}\sum_{i=1}^{n+1} h_T(u_i),$$

since this holds if T is circumscribed about B^n, and both sides of the equation are invariant under translations (since $\sum_{i=1}^{n+1} u_i = o$) and homogeneous of degree one under positive dilatations.

Let SO_n be the rotation group of $\mathbb{R}^n$, and denote by ν its normalized Haar measure; then

$$\int_{SO_n} f(\vartheta u_0)\,\nu(\mathrm{d}\vartheta) = \frac{1}{\omega_n}\int_{S^{n-1}} f\,\mathrm{d}\sigma$$

for any integrable function f on S^{n-1} and arbitrary $u_0 \in S^{n-1}$.

For $\vartheta \in SO_n$, let T_ϑ be the regular simplex circumscribed about K with exterior normal vectors $\vartheta u_1, \dots, \vartheta u_{n+1}$. Then

$$\int_{SO_n} M(T_\vartheta)\,\nu(\mathrm{d}\vartheta) = \int_{SO_n} \frac{M(T^n)}{n+1}\sum_{i=1}^{n+1} h_K(\vartheta u_i)\,\nu(\mathrm{d}\vartheta) = \frac{1}{2}M(K)M(T^n).$$

We conclude that $M(T_K) \le \frac{1}{2}M(K)M(T^n)$, and equality implies that $M(T_\vartheta) = M(T_K)$ for each $\vartheta \in SO_n$. In particular, we have proved the inequality (1), and the last assertion of the theorem shows that equality holds for balls.

To prove uniqueness, let us assume that K is a convex body satisfying $M(K) = M(B^n) = 2$ (without loss of generality) and $M(T_K) = M(T^n)$, then $M(T_\vartheta) = M(T^n)$ for any $\vartheta \in SO_n$. For $u \in S^{n-1}$, we denote by H_u the supporting hyperplane of K with outer unit normal vector u, and we define $\widetilde{H}_u := H_u - \frac{n+1}{n}u$. For given u, let T be a regular simplex circumscribed about K for which u is an exterior normal vector, and F be the corresponding facet of T. We assume, without loss of generality, that B^n is the inball of T. Then u is the centroid of F. By the lemma, $u \in K$. We assume first that $H_u \cap K = \{u\}$. The centroids of the other facets of T are also points of K, by the lemma, as well as of B^n, and they are contained in the hyperplane $\widetilde{H}_u$. Let ϑ be a rotation fixing u. Let $u, u_2, \dots, u_{n+1}$ be the unit normal vectors of T. The simplex T_ϑ circumscribed to K with normal vectors $u, \vartheta u_2, \dots, \vartheta u_{n+1}$ has the same mean width as T, and its facet with normal u has centroid u, hence $T_\vartheta = \vartheta T$. It follows that $\widetilde{H}_u \cap K = \widetilde{H}_u \cap B^n$. This property extends by continuity to all $u \in S^{n-1}$, since the set of the vectors u for which $H_u \cap K$ contains only one point is dense in S^{n-1}. Thus,

$(*)$ $\widetilde{H}_u \cap K$ is an $(n-1)$-ball of radius $\sqrt{1 - 1/n^2}$, for all $u \in S^{n-1}$.

Let again u and T (with inball B^n) be as above, and assume that $H_u \cap K = \{u\}$. If $v \in \widetilde{H}_u \cap S^{n-1}$, then v is an exterior normal to K at $v \in \partial K$, hence $\widetilde{H}_v \cap K$ contains u and the point of $\widetilde{H}_v \cap S^{n-1}$ opposite to u. The distance of these two points is $2\sqrt{1 - 1/n^2}$, therefore $(*)$ yields $\widetilde{H}_v \cap K = \widetilde{H}_v \cap B^n$. So far we have proved that the part of K in the half space bounded by $\widetilde{H}_u$ and containing u coincides with the corresponding part of B^n. Now choosing u suitably in this part and using $(*)$ we see that K is a ball, and this completes the proof of Theorem 1.

3. The scope of the averaging method

As mentioned in Remark 1, the planar version of Theorem 1 has an extension from circumscribed triangles to circumscribed m-gons. The proof uses a similar averaging argument, which is not obstructed by the fact that some edges of a circumscribed polygon with given normal vectors may have length zero. In higher dimensions, the averaging argument works only for exceptional polytopes. Let

$Q \subset \mathbb{R}^n$ be a convex polytope, and let $u_1, \dots, u_m$ be the exterior unit normal vectors of its facets. For a convex body $K \subset \mathbb{R}^n$ and for $\vartheta \in SO_n$, let

$$Q_\vartheta(K) := \bigcap_{i=1}^{m} H^-(K, \vartheta u_i),$$

where $H^-(K, v)$ denotes the supporting halfspace of K with exterior normal vector v. Thus, $Q_\vartheta(K)$ is a polytope circumscribed to K with normal vectors in $\{\vartheta u_1, \dots, \vartheta u_m\}$, but $Q_\vartheta(K)$ may have fewer than m facets if K has singularities. The mean width of $Q_\vartheta(K)$ can be represented by

$$M(Q_\vartheta(K)) = \sum_{i=1}^{m} \alpha_i h_K(\vartheta u_i), \tag{2}$$

with coefficients α_i that depend only on the strong isomorphism type of the polytope $Q_\vartheta(K)$ (see [6], p. 100, for the notion of strongly isomorphic polytopes). The existence of the representation (2) follows from the fact that the mean width of a polytope P can be written in the form

$$M(P) = c_n \sum_{E \in \mathcal{F}_1(P)} \gamma(E, P) V_1(E) \tag{3}$$

(by [6], (4.2.17) and (5.3.12)), where $\mathcal{F}_1(P)$ is the set of edges of P, $V_1(E)$ is the length of the edge E, $\gamma(E, P)$ is the external angle of P at its edge E, and c_n is a constant depending only on the dimension. Let us now assume that the polytope Q has the following property: ($\star$) every polytope with the same system of normal vectors (of facets) as Q is strongly isomorphic to Q. Then the coefficients α_i in (2) are independent of ϑ, and we can conclude, as in the proof of Theorem 1, that

$$\int_{SO_n} M(Q_\vartheta(K))\, \nu(\mathrm{d}\vartheta) = \frac{1}{2} M(K) M(Q_{\mathrm{id}}(B^n)).$$

This part of the argument, however, breaks down if Q does not satisfy ($\star$).

The polytopes satisfying ($\star$) have been called *monotypic polytopes*; they were investigated in [4]. For $n > 2$, a complete classification has only been achieved for $n = 3$ or under the assumption of central symmetry. We consider here only the case of parallelepipeds, where it is easy to obtain a counterpart to Theorem 1.

For a convex body $K \subset \mathbb{R}^n$, we denote by P_K a parallelepiped of minimal mean width circumscribed about K. Let C^n be a cube circumscribed about B^n.

Theorem 2. *For any convex body $K \subset \mathbb{R}^n$,*

$$M(P_K) \le \frac{1}{2} M(K) M(C^n). \tag{4}$$

Equality holds if and only if K is a body of constant width.

Every parallelepiped of minimal mean width circumscribed about a given body of constant width is a cube.

To verify the second assertion of the theorem, we note that an n-dimensional parallelepiped P is a Minkowski sum of n segments, hence its mean width is the sum of the mean widths of the segments and is thus a constant multiple of the sum of the edge lengths of P. If the parallelepiped P is circumscribed about a convex body of constant width b, then the length of a given edge E is not smaller than the distance between the pair of parallel facets of P through the endpoints of E, which is equal to b, and equality holds if and only if the edge is orthogonal to the facets. Now the assertion is clear.

To prove the first assertion of Theorem 2, we argue precisely as in the proof of Theorem 1, replacing the set of normal vectors of a regular simplex by the set of normal vectors of the cube C^n. This yields equality (4), and equality holds if K is a body of constant width. To prove uniqueness, let K be a convex body satisfying $M(K) = M(B^n) = 2$ and $M(P_K) = M(C^n)$, then $M(P_\vartheta) = M(C^n)$ for any $\vartheta \in SO_n$; here P_ϑ is the rectangular parallelepiped circumscribed about K whose normal vectors arise from the normal vectors of C^n by applying the rotation ϑ. Let L be a two-dimensional linear subspace of $\mathbb{R}^n$ and $L^\perp$ its orthogonal complement. Let $u_1, \ldots, u_n$ be unit vectors such that (u_1, u_2) is a basis of L (not necessarily orthogonal) and $(u_3, \ldots, u_n)$ is an orthonormal basis of $L^\perp$. Let P be the parallelepiped circumscribed about K with normal vectors $\pm u_1, \ldots, \pm u_n$. Let $\cdot|L$ denote the orthogonal projection to L. Then P is the direct orthogonal sum of the parallelogram $P|L$ in L, which is circumscribed about $K|L$, and a certain $(n-2)$-dimensional cube C. The mean width of P is obtained from $M(P) = M(P|L) + M(C)$. Since the minimal mean width $M(P_K)$ is realized by all rectangular parallelepipeds circumscribed about K, the minimal perimeter of all parallelograms circumscribed about $K|L$ is realized by each circumscribed rectangle. This property is shared by the centrally symmetric body $S := (K|L - K|L)/2$, since parallelograms with the same normals circumscribed about $K|L$ and S, respectively, are translates of each other. By the argument used in [5], p. 381, modified for centrally symmetric convex sets and circumscribed parallelograms, this implies that $K|L$ is a circular disc. Since L was an arbitrary two-dimensional linear subspace, it follows that $(K - K)/2$ is a ball, hence K is a body of constant width.

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