Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 48 (2007), No. 1, 209-215.

On Minimum Size Blocking Sets of External Lines to a Quadric in PG(3, q)

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Abstract. We present the characterization of the minimum size blocking sets with respect to the external lines to a quadric in $\mathrm{PG}(3, q)$, $q \geq 9$.

MSC 2000: 51E21
Keywords: blocking sets, quadrics

1. Introduction

A *blocking set* in a projective space $\mathbb{P} = \mathrm{PG}(d, q)$ is a subset of $\mathbb{P}$ which meets every line. Blocking sets have been investigated by several authors from many points of view. The reader is referred to [2, 5, 7, 8] and papers cited there for a survey on this topic.

Now, let $\mathcal{F}$ be a set of lines of $\mathbb{P}$. A point set B of $\mathbb{P}$ is a *blocking set with respect to* $\mathcal{F}$ (or an $\mathcal{F}$-*blocking set*) if every line in $\mathcal{F}$ is incident with at least one point of B. In [1] and [6], all minimum size blocking sets with respect to the set of the external lines to a conic in $\mathrm{PG}(2, q)$ have been determined for q odd and q even, respectively.

Let $\mathcal{Q}$ be a non-singular quadric or a cone in $\mathrm{PG}(3, q)$, and let $\mathcal{F}$ be the set of the external lines to $\mathcal{Q}$. If π is a plane, then $\pi \backslash \mathcal{Q}$ is an $\mathcal{F}$-blocking set. The minimum size of $\pi \backslash \mathcal{Q}$ is:

(a) $q^2 - q$, if $\mathcal{Q}$ is a hyperbolic quadric (π is a tangent plane);
(b) q^2, if $\mathcal{Q}$ is an elliptic quadric (π is a secant plane);
(c) $q^2 - q$, if $\mathcal{Q}$ is a cone (π is a plane sharing two distinct lines with $\mathcal{Q}$).

The following two questions immediately arise.

1. Are these the correct sizes for the smallest blocking sets with respect to the external lines to a non-singular quadric or a cone?
2. A blocking set of such a size is always of type $\pi \backslash \mathcal{Q}$, for some plane π?

In [4], an affirmative answer is given to both questions when $\mathcal{Q}$ is a hyperbolic quadric and q is even. In this paper, we examine the other cases and again the answer will be affirmative.

We note that if π is a plane and B is a blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $\pi \cap B$ is a blocking set in π with respect to the lines in π external to $\pi \cap \mathcal{Q}$. Moreover, if B is of minimum size, then $B \cap \mathcal{Q} = \emptyset$.

Now, we state the main results of [1] and [6] which will be useful for the sequel of this article.

Theorem 1.1. *Let Γ be an irreducible conic of $PG(2,q)$, $q \geq 9$ odd. If B is a blocking set with respect to the set of the external lines to Γ, then $|B| \geq q-1$ and $|B| = q-1$ if and only if $B = L \backslash \Gamma$, where L is a secant line to Γ.*

Theorem 1.2. *Let Γ be an irreducible conic of $PG(2,q)$, q even, and let n be its nucleus. If B is a blocking set with respect to the set of the external lines to Γ, then $|B| \geq q-1$ and $|B| = q-1$ if and only if one of the following cases occurs:*
(i) *$B = L \backslash \Gamma$, where L is a secant line to Γ;*
(ii) *$B = L \backslash (\Gamma \cup \{n\})$, where L is a tangent line to Γ;*
(iii) *q is a square and $B = \Pi \backslash (\Gamma \cup \{n\})$, where Π is a Baer subplane such that $|\Pi \cap \Gamma| = \sqrt{q} + 1$.*

Finally, if $\mathcal{Q}$ is a quadric in $\mathrm{PG}(3,q)$, by an *external* (*tangent* or *secant*) *line* we mean a line external (tangent or secant) to $\mathcal{Q}$. Similarly, a plane tangent or secant to $\mathcal{Q}$ will be sometimes referred to as a *tangent* or *secant plane.*

Acknowledgements. The research of the first two authors was supported by the Italian National Project "Strutture geometriche, Combinatoria e loro applicazioni" (COFIN 2005) and by the Department of Mathematics and Applications "R. Caccioppoli" of the University of Naples "Federico II".

The third author thanks the Fund for Scientific Research Flanders-Belgium for a research grant.

2. Hyperbolic quadric, *q* odd

Let $\mathcal{Q}$ be a hyperbolic quadric of $\mathrm{PG}(3,q)$, $q \geq 9$ odd, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. Throughout this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since, for any tangent plane π to $\mathcal{Q}$, $\pi\backslash\mathcal{Q}$ is an $\mathcal{F}$-blocking set of size q^2-q, necessarily

$$|B| \leq q^2 - q. \tag{2.1}$$

Proposition 2.1. *For an $\mathcal{F}$-blocking set B of minimum size, $|B| = q^2-q$. Moreover, $|L \cap B| = 1$ for any $L \in \mathcal{F}$.*

Proof. Counting in two ways the point-line pairs (x, L), $x \in B \cap L$ and $L \in \mathcal{F}$, yields

$$|B|\frac{q^2-q}{2} \geq \frac{q^2(q-1)^2}{2}. \tag{2.2}$$

From (2.1) and (2.2),

$$\frac{q^2(q-1)^2}{2} = (q^2-q)\frac{q^2-q}{2} \geq |B|\frac{q^2-q}{2} \geq \frac{q^2(q-1)^2}{2}. \tag{2.3}$$

From (2.3), it follows that the equality holds in both (2.1) and (2.2); so, the statement is proved. □

Proposition 2.2. *The following properties are valid:*

(i) *through any point in B, there exist q secant lines whose points not in $\mathcal{Q}$ are in B;*

(ii) *if π is a secant plane, then $|\pi \cap B| = q-1$ or q.*

Proof. (i) Let $p \in B$ and let L be an external line through p. By Proposition 2.1, $L \cap B = \{p\}$. Every plane π through L is a secant plane so, by Theorem 1.1, $|\pi \cap B| \geq q-1$. This implies, by Proposition 2.1, that there exist q planes $\pi_1, \pi_2, \ldots, \pi_q$ through L meeting B in $q-1$ points and just one plane through L intersecting B in q points. By Theorem 1.1, $\pi_i \cap B = L_i\backslash\mathcal{Q}$ $(i = 1, \ldots, q)$, where L_i is a secant line to $\mathcal{Q}$.

(ii) Consider a line L in π external to $\mathcal{Q}$. The arguments in (i) imply the statement. □

Proposition 2.3. *Through any point in B, there exists a tangent plane π such that $|\pi \cap B| \geq (q-1)^2$.*

Proof. Let $p \in B$. By Proposition 2.2 (i), p lies on q secant lines $L_1, L_2, \ldots, L_q$ such that $|L_i \cap B| = q-1$, $i = 1, 2, \ldots, q$. As $q \geq 9$, (ii) of Proposition 2.2 implies that the plane joining L_i and L_j, $i \neq j$, is tangent to $\mathcal{Q}$. Now, consider the two tangent planes π' and π'' through L_1. The lines $L_2, \ldots, L_q$ are in $\pi' \cup \pi''$; so, $q \geq 9$ implies that one of the two planes π' and π'', say π', contains at least two of such lines. Let $L_2, L_3 \subseteq \pi'$. If a line L_i, $i \geq 4$, exists not in π', then through L_i there pass three distinct tangent planes $\langle L_1, L_i\rangle, \langle L_2, L_i\rangle, \langle L_3, L_i\rangle$, a contradiction. Hence, the lines $L_1, L_2, \ldots, L_q$ all lie in π' and the statement is proved. □

Now we can prove the following classification result.

Theorem 2.4. *Let $\mathcal{Q}$ be a hyperbolic quadric in $PG(3, q)$, $q \geq 9$ odd. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^2-q$ and $B = \pi\backslash\mathcal{Q}$ for some plane π tangent to $\mathcal{Q}$.*

Proof. By Proposition 2.1, $|B| = q^2 - q$. Now, let $p \in B$. By Proposition 2.3, a tangent plane π through p exists sharing with B at least $(q-1)^2$ points. Assume that a point p' exists in $B \backslash \pi$. Again by Proposition 2.3, p' lies in a tangent plane π' such that $|\pi' \cap B| \geq (q-1)^2$. Since $|(\pi \cap \pi') \cap B| \leq q-1$, then $|(\pi \cup \pi') \cap B| \geq 2q^2 - 5q + 3$, a contradiction to $|B| = q^2 - q$ as $q \geq 9$. Hence, $B \subseteq \pi$; so, $|B| = q^2 - q$ implies that $B = \pi \backslash \mathcal{Q}$ and the statement is completely proved. $\square$

3. Elliptic quadric

Let $\mathcal{Q}$ be an elliptic quadric of $\mathrm{PG}(3, q)$, $q \geq 9$, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. In this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since, for any secant plane π, $\pi \backslash \mathcal{Q}$ is an $\mathcal{F}$-blocking set of size q^2, necessarily

$$|B| \leq q^2. \tag{3.1}$$

Proposition 3.1. *For any tangent plane π, $|\pi \cap B| \geq q$.*

If a tangent plane π meets B in exactly q points, then $B \cap \pi = L \backslash \mathcal{Q}$ for some line L in π tangent to $\mathcal{Q}$.

Proof. Counting in two ways the point-line pairs (x, L), $x \in \pi \cap B \cap L$ and L a line in π external to $\mathcal{Q}$, gives $|\pi \cap B| q \geq q^2$. The first statement follows.

For the proof of the second statement, let $\pi \cap \mathcal{Q} = \{p_0\}$. The set $(B \cap \pi) \cup \{p_0\}$ is a blocking set in π and $|(B \cap \pi) \cup \{p_0\}| = q + 1$; so, $(B \cap \pi) \cup \{p_0\}$ is a line. The statement follows. $\square$

Proposition 3.2. *There exists a tangent line skew to B.*

Proof. Assume that any tangent line shares at least one point with B. Counting in two ways the point-line pairs (x, L), $x \in B \cap L$ and L a tangent line, yields $|B|(q+1) \geq (q+1)(q^2+1)$, a contradiction to (3.1); so, the statement is proved. $\square$

Proposition 3.3. *If L is a tangent line skew to B, then the tangent plane through L meets B in exactly q points and $|B \cap \pi| = q - 1$ for any secant plane through L. Moreover, $|B| = q^2$.*

Proof. Counting points of B on planes through L, we obtain, by Proposition 3.1 and Theorems 1.1 and 1.2,

$$|B| \geq q + q(q-1) = q^2;$$

so, the statement follows from (3.1). $\square$

Proposition 3.4. *There exist $q+1$ lines $L_1, \ldots, L_{q+1}$ tangent to $\mathcal{Q}$ at distinct points such that $L_i \backslash \mathcal{Q} \subseteq B$, $i = 1, \ldots, q+1$, and such that the tangent plane through L_i only intersects B in $L_i \setminus \mathcal{Q}$.*

Proof. By Proposition 3.1, every tangent plane shares at least q points with B. Denote by t the number of tangent planes meeting B in exactly q points. Counting in two ways the point-plane pairs (x, π), $x \in B \cap \pi$ and π a tangent plane, gives

$$|B|(q+1) \geq tq + (q^2 + 1 - t)(q+1)$$

from which, by Proposition 3.3, $t \geq q+1$. The statement follows from Proposition 3.1. □

Now we can prove the following classification result.

Theorem 3.5. *Let $\mathcal{Q}$ be an elliptic quadric in $PG(3, q)$, $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{Q}$, then $|B| = q^2$ and $B = \pi \backslash \mathcal{Q}$ for some secant plane π to $\mathcal{Q}$.*

Proof. By Proposition 3.3, $|B| = q^2$.

Case 1 q odd. By Proposition 3.2, there exists a tangent line L skew to B. Set $\{p\} = L \cap \mathcal{Q}$. Moreover, denote by α_0 the tangent plane to the point p and by $\alpha_i, i = 1, \ldots, q$, the secant planes through L. By Propositions 3.3 and 3.1 and Theorem 1.1, $\alpha_0 \cap B = T \backslash \{p\}$, where T is a line in α_0 tangent to $\mathcal{Q}$, and $\alpha_i \cap B = L_i \backslash \mathcal{Q}$, where L_i is a secant line on p.

Let π_i be the plane through L_1 and L_i, $i = 2, \ldots, q$. Since π_i is a secant plane through p and $|\pi_i \cap B| \geq 2q - 2$, then Proposition 3.3 implies that $\pi_i \cap \alpha_0 = T$. Hence, the planes π_i, $i = 2, \ldots, q$, all coincide with the secant plane π joining L_1 and T. Since $|\pi \cap B| \geq q^2$, necessarily $B = \pi \backslash \mathcal{Q}$.

Case 2 q even. By Proposition 3.4, there exist $q + 1$ lines $L_1, \ldots, L_{q+1}$ tangent to $\mathcal{Q}$ at distinct points such that $L_i \backslash \mathcal{Q} \subseteq B$, and such that the tangent plane π_i through L_i meets B in $L_i \backslash \mathcal{Q}$. So, L_i meets π_j, $i \neq j$, at a point of L_j. Therefore, the lines L_i pairwise intersect in a point of B.

Consider the plane π through L_1 and L_2, and let $\{n\} = L_1 \cap L_2$. Obviously, n is the nucleus of the conic $\pi \cap \mathcal{Q}$. Since any line L_i, $i = 3, \ldots, q+1$, meets L_1 and L_2, then $L_i \subseteq \pi$ and $n \in L_i$ since L_i cannot intersect L_1 and L_2 in two distinct points. Otherwise, it would be skew or secant to $\pi \cap \mathcal{Q}$. It therefore passes through n. Since the only tangents to $\mathcal{Q}$ through n lie in π, the lines $L_3, \ldots, L_{q+1}$ all are contained in π. Hence, the lines $L_1, \ldots, L_{q+1}$ all are contained in the plane π; so, $|\pi \cap B| \geq q^2$. It follows that $B = \pi \backslash \mathcal{Q}$ and the statement is completely proved. □

4. Cone

Let $\mathcal{C}$ be a cone of PG$(3, q)$, $q \geq 9$, and let $\mathcal{F}$ be the set of all external lines to $\mathcal{Q}$. In this section, B denotes an $\mathcal{F}$-blocking set of minimum size. Since for any plane π sharing two distinct lines with $\mathcal{C}$, $\pi \backslash \mathcal{C}$ is an $\mathcal{F}$-blocking set of size $q^2 - q$, necessarily

$$|B| \leq q^2 - q. \tag{4.1}$$

Proposition 4.1. $|B| = q^2 - q$ *and, for any secant plane π, $|B \cap \pi| = q - 1$.*

Proof. By Theorems 1.1 and 1.2, $|\pi \cap B| \geq q-1$. Therefore, counting in two ways the point-plane pairs (x, π), $x \in B \cap \pi$ and π a secant plane, gives

$$|B|q^2 \geq q^3(q-1). \tag{4.2}$$

The statement follows from (4.1) and (4.2). □

Proposition 4.2. *For any secant plane π, $B \cap \pi = L \backslash \mathcal{C}$, where L is a line in π secant to $\mathcal{C}$.*

Proof. Let $B' = B \cap \pi$ and let $\Gamma = \mathcal{C} \cap \pi$. By Proposition 4.1, $|\pi \cap B| = q-1$.

If q is odd, then the statement follows from Theorem 1.1.

If q is even, then one of the three cases in Theorem 1.2 must occur. Assume that $B' = T \backslash (\Gamma \cup \{n\})$, where T is a line in π tangent to Γ and where n is the nucleus of Γ. If π' is a secant plane through T other than π, then $T \backslash (\Gamma \cup \{n\}) \subseteq B \cap \pi'$; so, by Proposition 4.1, $T \backslash (\Gamma \cup \{n\}) = B \cap \pi'$, a contradiction to Theorem 1.2 since the nucleus of the conic $\mathcal{C} \cap \pi'$ is a point of $T \backslash \Gamma$ distinct from n.

Now, assume that q is a square and $B' = \Pi \backslash (\Gamma \cup \{n\})$, with Π a Baer subplane of π sharing exactly $\sqrt{q}+1$ points with Γ. Consider a line T in π tangent to Γ at a point of Π and a secant plane π' through T, distinct from π, such that the nucleus n' of $\mathcal{C} \cap \pi'$ is in Π. By Proposition 4.1, $|B \cap \pi'| = q-1$; so, Theorem 1.2 implies that $n' \notin B \cap \pi'$, a contradiction.

Hence, only the case (i) of Theorem 1.2 can occur.

The statement is completely proved. □

Theorem 4.3. *Let $\mathcal{C}$ be a cone in $PG(3, q)$, $q \geq 9$. If B is a minimum size blocking set with respect to the set of the external lines to $\mathcal{C}$, then $|B| = q^2 - q$ and $B = \pi \backslash \mathcal{C}$ for some plane π sharing two distinct lines with $\mathcal{C}$.*

Proof. Let v be the vertex of $\mathcal{C}$ and let $p \in \mathcal{C} \backslash \{v\}$. By Proposition 4.1, $|B| = q^2 - q$; so, through p, a secant line L skew to B exists. Denote by r the point in $\mathcal{C} \cap L$ different from p, and by $\pi_1, \ldots, \pi_q$ the q secant planes through L.

By Proposition 4.2, $\pi_i \cap B = L_i \backslash \mathcal{C}$, L_i a secant for any $i = 1, \ldots, q$. Obviously, L_i passes through p or r, $i = 1, \ldots, q$. Assume that $p \in L_i$ and $r \in L_j$, $i \neq j$. Since $q \geq 9$, there exists a secant plane through L_i and a point in $L_j \cap B$; a contradiction to Proposition 4.2. Hence, the q lines L_i, $i = 1, \ldots, q$, all pass through one of the two points p and r, say p. By Proposition 4.2, the plane through L_1 and L_j, $j = 2, \ldots, q$, is not a secant plane, so it is the plane π joining L_1 and v. Therefore, $|\pi \cap B| \geq q(q-1)$; so $B = \pi \backslash \mathcal{C}$ and the statement is proved. □

References

[1] Aguglia, A.; Korchmáros, G.: *Blocking sets of external lines to a conic in $PG(2, q)$, q odd.* Combinatorica **26**(4) (2006), 379–394. Zbl pre05082371

[2] Ball, S.; Blokhuis, A.: *Polynomials in finite geometry.* Proceedings of the summer school, Braunschweig, Germany, In: Löwe, Stefan (ed.) et al.: Methods of discrete mathematics. Proceedings of the summer school, Braunschweig 1999. Germany. Quad. Mat. **5** (1999), 71–101. Zbl 0973.51010

[3] Biondi, P.; Lo Re, P. M.: *On blocking sets of external lines to a quadric in $PG(3,q)$, q prime.* To appear in Ars Combin.

[4] Biondi, P.; Lo Re, P. M.: *On blocking sets of external lines to a hyperbolic quadric in $PG(3,q)$, q even.* Submitted.

[5] Blokhuis, A.: *Blocking sets in Desarguesian planes.* In: Miklós, D. (ed.) et al, Combinatorics, Paul Erdös is eighty, Bolyai society Math. Studies **2** (1996), 133–155. Zbl 0849.51005

[6] Giulietti, M.: *Blocking sets of external lines to a conic in $PG(2,q)$, q even.* To appear in European J. Combin.

[7] Metsch, K.: *Blocking sets in projective spaces and polar spaces.* J. Geom. **76** (2003), 216–232. Zbl 1032.51010

[8] Szőnyi, T.: *Blocking sets in Desarguesian affine and projective planes.* Finite Fields Appl. **3** (1997), 187–202. Zbl 0912.51004

Received February 3, 2006