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Addition and Subtraction of Homothety Classes of Convex Sets

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Abstract. Let S_H denote the homothety class generated by a convex set $S \subset \mathbb{R}^n$: $S_H = \{a + \lambda S \mid a \in \mathbb{R}^n, \lambda > 0\}$. We determine conditions for the Minkowski sum $B_H + C_H$ or the Minkowski difference $B_H \sim C_H$ of homothety classes B_H and C_H generated by closed convex sets $B, C \subset \mathbb{R}^n$ to lie in a homothety class generated by a closed convex set (more generally, in the union of countably many homothety classes generated by closed convex sets).

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1. Introduction and main results

In what follows, everything takes place in the Euclidean space $\mathbb{R}^n$. Let us recall that a set B is *homothetic* to a set A provided $B = a + \lambda A$ for a suitable point a and a scalar $\lambda > 0$. If A is a convex set, then the Minkowski sum of any two homothetic copies of A is again a homothetic copy of A. In other words, the homothety class

$$A_H = \{a + \lambda A \mid a \in \mathbb{R}^n, \lambda > 0\}$$

is closed with respect to the Minkowski addition. We will say that closed convex sets B and C form a pair of *H-summands* of a closed convex set A, or summands of A with respect to homotheties, provided the Minkowski sum of any homothetic copies of B and C is always homothetic to A. (See Schneider's monograph [7]

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for an extensive treatment of the Minkowski addition and subtraction of convex bodies.) In terms of homothety classes, B and C are *H-summands* of A if and only if $B_H + C_H \subset A_H$, where

$$B_H + C_H = \{B' + C' \mid B' \in B_H, C' \in C_H\}.$$

Our first result (see Theorem 1) describes the pairs of H-summands of a line-free closed convex set in terms of homothety classes. In what follows, $\operatorname{rec} S$ denotes the recession cone of a closed convex set S. In particular, $\operatorname{rec} S$ is a closed convex cone with apex 0 such that $S + \operatorname{rec} S = S$.

Theorem 1. *For a pair of line-free closed convex sets B and C, the following conditions* (1)–(3) *are equivalent.*

(1) *$B_H + C_H$ belongs to a unique homothety class generated by a line-free closed convex set.*
(2) *$B_H + C_H$ lies in the union of countably many homothety classes generated by line-free closed convex sets.*
(3) *There is a line-free closed convex set A such that:*
   (a) $\operatorname{rec} A = \operatorname{rec} B + \operatorname{rec} C$,
   (b) *each of the sets $B_0 = B + \operatorname{rec} A$ and $C_0 = C + \operatorname{rec} A$ is homothetic either to A or to $\operatorname{rec} A$,*
   (c) *if A is not a cone, then at least one of the sets B_0, C_0 is not a cone.*

As follows from the proof of Theorem 1, a line-free closed convex set A with properties (a)–(c) above satisfies the inclusion $B_H + C_H \subset A_H$.

Corollary 1. *For a pair of compact convex sets B and C, each of the conditions* (1)–(3) *from Theorem 1 holds if and only if B and C are homothetic.*

We note that Corollary 1 can be easily proved by using Rådström's cancellation law [5]. The proof of Theorem 1 is based on the properties of exposed points of the sum of two line-free closed convex sets formulated in Theorem 2. As usual, $\exp S$ and $\operatorname{ext} S$ stand, respectively, for the sets of exposed and extreme points of a convex set S.

Theorem 2. *Let a line-free closed convex set A be the Minkowski sum of closed convex sets B and C. Then both convex sets $B_0 = B + \operatorname{rec} A$ and $C_0 = C + \operatorname{rec} A$ are closed and satisfy the following conditions:*

(1) *for any point $a \in \exp A$ there are unique points $b \in \exp B_0$ and $c \in \exp C_0$ such that $a = b + c$,*
(2) *the sets*

$$\exp_C B = \{x \in \exp B \mid \exists\, y \in \exp C \text{ such that } x + y \in \exp A\},$$
$$\exp_B C = \{x \in \exp C \mid \exists\, y \in \exp B \text{ such that } x + y \in \exp A\}$$

are dense in $\exp B_0$ and $\exp C_0$, respectively.

Remark 1. Theorem 2 seems to be new even for the case of compact convex sets. Moreover, there are convex bodies B and C in $\mathbb{R}^2$ such that $\exp_C B \neq \exp B$ and $\exp_B C \neq \exp C$. Indeed, let $B = \{(x, y) \mid x^2 + y^2 \leq 1\}$ be the unit disk of the coordinate plane $\mathbb{R}^2$, and $C = \{(x, y) \mid 0 \leq x, y \leq 1\}$ be the unit square. Then $b = (0, 1)$ lies in $\exp B \setminus \exp_C B$.

Remark 2. Since $\exp A$ is dense in $\operatorname{ext} A$, Theorem 2 remains true if we substitute "ext" for "exp". Then $\operatorname{ext}_C B = \operatorname{ext} B$ and $\operatorname{ext}_B C = \operatorname{ext} C$ provided both B and C are compact (see [2]). One can easily construct unbounded closed convex sets B and C in $\mathbb{R}^2$ such that $\operatorname{ext}_C B \neq \operatorname{ext} B_0$ and $\operatorname{ext}_B C \neq \operatorname{ext} C_0$.

Let us recall that the Minkowski difference $X \sim Y$ of any sets X and Y in $\mathbb{R}^n$ is defined by $X \sim Y = \{x \in \mathbb{R}^n \mid x + Y \subset X\}$. If both X and Y are closed convex sets, then the equality $X \sim Y = \cap\{X - y \mid y \in Y\}$ implies that $X \sim Y$ is also closed and convex (possibly, empty). Given n-dimensional closed convex sets B and C, we put

$$B_H \underset{n}{\sim} C_H = \{B' \sim C' \mid B' \in B_H,\ C' \in C_H,\ \dim(B' \sim C') = n\}.$$

An important notion here is that of tangential set introduced by Schneider [7, p. 136]: a closed convex set D of dimension n is a *tangential set* of a convex body F provided $F \subset D$ and through each boundary point of D there is a support hyperplane to D that also supports F.

Theorem 3. *For a pair of convex bodies B and C, the following conditions* (1)–(4) *are equivalent:*

(1) $B_H \underset{n}{\sim} C_H \subset B_H$,

(2) $B_H \underset{n}{\sim} C_H$ *lies in a unique homothety class generated by a convex body,*

(3) $B_H \underset{n}{\sim} C_H$ *lies in the union of countably many homothety classes generated by convex bodies,*

(4) B *is homothetic to a tangential set of* C.

Remark 3. Theorem 3 cannot be directly generalized to the case of unbounded convex sets. Indeed, let B and C be convex sets in $\mathbb{R}^2$ given by

$$B = \{(x, y) \mid x \geq 0,\ xy \geq 1\}, \quad C = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

Then $B \sim \gamma C = B$ for any $\gamma > 0$, while B is not homothetic to a tangential set of C.

2. Proof of Theorem 2

We say that a closed halfspace P supports a closed convex set S provided the boundary hyperplane of P supports S and the interior of P is disjoint from S. If $P = \{x \in \mathbb{R}^n \mid \langle x, e\rangle \leq \alpha\}$ where e is a unit vector and α is a scalar, then e is called the *outward unit normal* to P.

Lemma 1. *Let S be a line-free closed convex set, and P be a closed halfspace such that $P \cap \operatorname{rec} S = \{0\}$. Then:*

(1) *there is a translate of P that supports S,*

(2) *no translate of P contains an asymptotic ray of S,*

(3) *if a translate Q of P is disjoint from S, then for any point $x \in \operatorname{bd} Q$ the tangent cone*

$$T_x(S) = \operatorname{cl}(\cup\{x + \lambda(S - x) \mid \lambda \geq 0\})$$

is line-free and satisfies the condition $Q \cap T_x(S) = \{x\}$.

Proof. First we claim that for any vector x the intersection $(x+P) \cap S$ is compact. Indeed, if $(x + P) \cap S$ were unbounded, then $\operatorname{rec}((x + P) \cap S)$ would contain a ray with apex 0. This and the equality $\operatorname{rec}((x + P) \cap S) = P \cap \operatorname{rec} S$ contradict the hypothesis.

(1) Let $x+P$ be a translate of P that intersects S. Because $(x+P) \cap S$ is compact, there is a translate $y+P$ that supports $(x+P) \cap S$. Obviously, $y+P$ also supports S.

(2) Assume for a moment that a translate $z + P$ of P contains an asymptotic ray l of S. If $x + P$ is a translate of P that intersects S, then $(x + P) \cap S$ should contain the ray $(x - z) + l$, contradicting (a).

(3) The cone $T_x(S)$ is line-free as a tangent cone of a line-free convex set S with $x \notin S$. Assume that $Q \cap T_x(S)$ contains a point $z \neq x$. Then the ray $[x, z)$ lies in $Q \cap T_x(S)$, which implies that $l = [x, z) - x$ lies in $P \cap \operatorname{rec} S$, a contradiction. □

Lemma 2. *Let S be a line-free closed convex set, P be a closed halfspace that supports S, and e be the outward unit normal to P. For any $\varepsilon > 0$ there is a closed halfspace P' such that $S \cap P'$ is an exposed point of S and the outward unit normal e' to P' satisfies the inequality $\|e - e'\| < \varepsilon$.*

Proof. Choose a point $a \in S \cap P$, and let $b = a - e$. Then the unit ball with center b lies in P and touches S at a. Let B_r be the ball with center b and radius $r \in]0, 1[$. We can choose r so close to 1 that for any closed halfspace Q that contains B_r and is disjoint from S, the outward unit normal q to Q satisfies the inequality $\|e - q\| < \varepsilon$.

As proved in [1], there is a pair of distinct parallel hyperplanes L and M both separating S and B_r such that the intersections $S \cap L$ and $B_r \cap M$ are exposed points of S and B_r, respectively. Let P' be the closed halfspace bounded by L and containing B_r. By the choice of r, the outward unit normal e' to P' satisfies the inequality $\|e - e'\| < \varepsilon$. □

I am indebted to Rolf Schneider for his comment that Lemma 2 can be proved by using a duality argument and the fact that the set of regular point of an n-dimensional closed convex set $S \subset \mathbb{R}^n$ is dense in the boundary of S.

Lemma 3. ([6, Corollary 9.1.2]) *Let B and C be line-free closed convex sets such that their sum $A = B + C$ is also line-free. Then A is closed and* $\operatorname{rec} A = \operatorname{rec} B + \operatorname{rec} C$. □

We continue with the proof of Theorem 2. Because A is line-free, both B_0 and C_0 are also line-free. Lemma 3 implies that B_0 and C_0 are closed sets and $\operatorname{rec} B_0 = \operatorname{rec} C_0 = \operatorname{rec} A$.

Let a be an exposed point of A, and let P be a closed halfspace supporting A such that $A \cap P = \{a\}$. If $a = b + c$, with $b \in B$ and $c \in C$, then, as is easily seen, the halfspace $Q = (b - a) + P$ supports B at b, and the halfspace $T = (c - a) + P$ supports C at c. Moreover, $B \cap Q = \{b\}$ and $C \cap T = \{c\}$ since otherwise A should intersect P along a set larger than $\{a\}$. Hence $b \in \exp B$ and $c \in \exp C$. Lemma 1 implies that $B_0 \cap Q = \{b\}$ and $C_0 \cap T = \{c\}$. Thus $b \in \exp B_0$ and $c \in \exp C_0$.

Regarding part (2) of the theorem, we will prove only that $\exp_C B$ is dense in $\exp B_0$, since the second inclusion holds by the symmetry argument. First we observe that $\exp_C B \subset \exp B_0$. Indeed, let $x \in \exp_C B$ and $y \in \exp_B C$ be such that $x + y \in \exp A$. Choose a closed halfspace P with $P \cap A = \{x + y\}$. As above, the halfspace $Q = P - y$ satisfies $Q \cap B_0 = \{x\}$. Hence $x \in \exp B_0$.

To prove the inclusion $\exp B_0 \subset \operatorname{cl} \exp_C B$, it suffices to show that

$$B_0 = \operatorname{conv}(\operatorname{cl} \exp_C B) + \operatorname{rec} A. \qquad (*)$$

Indeed, let $(*)$ be true. By [3, 4], we have $B_0 = \operatorname{conv}(\operatorname{ext} B_0) + \operatorname{rec} A$. Moreover, $\operatorname{ext} B_0 \subset X$ for any set $X \subset B_0$ with $B_0 = \operatorname{conv} X + \operatorname{rec} A$. Then $(*)$ implies that $\exp B_0 \subset \operatorname{ext} B_0 \subset \operatorname{cl} \exp_C B$.

Assume, for contradiction, that $B_0 \neq \operatorname{conv}(\operatorname{cl} \exp_C B) + \operatorname{rec} A$. Then there is a point $p \in \exp B_0$ that does not lie in the line-free closed convex set $B_1 = \operatorname{conv}(\operatorname{cl} \exp_C B) + \operatorname{rec} A$. Let Q be the closed halfspace such that $B_0 \cap Q = \{p\}$. Because $B_1 \subset B_0$ and $p \notin B_1$, we have $B_1 \cap Q = \varnothing$. Let e be the outward unit normal to Q.

Since $p \notin B_1$, the tangent cone $T_p(B_1)$ is line-free. Furthermore, $Q \cap T_p(B_1) = \{p\}$ (see Lemma 1). Hence there is a scalar $\varepsilon > 0$ such that any closed halfspace H with the properties $p \in \operatorname{bd} H$ and $\|e - h\| < \varepsilon$, where h is the outward unit normal for H, supports $T_p(B_1)$ at p only: $H \cap T_p(B_1) = \{p\}$.

Lemma 1 implies the existence of a translate of Q that supports A. By Lemma 2, there is a closed halfspace Q' whose outward unit normal e' satisfies $\|e - e'\| < \varepsilon$ and such that $A \cap Q'$ is an exposed point of A. Let $\{a\} = A \cap Q'$. As above, $a = b + c$ with $b \in \exp_C B$ and $c \in \exp_B C$. Moreover, the closed halfspace $P = (b - a) + Q'$ satisfies $B_0 \cap P = \{b\}$. By the choice of ε, the halfspace P should be disjoint from B_1. The last is in contradiction with $b \in \exp_C B \subset B_1$. Hence $B_0 = B_1$. □

3. Proof of Theorem 1

(3) $\Rightarrow$ (1) Given points b, c and scalars $\beta, \gamma > 0$, we have

$$\begin{aligned}(b+\beta B)+(c+\gamma C) &= b+\beta(B+\operatorname{rec} B)+c+\gamma(C+\operatorname{rec} C)\\ &= b+\beta(B+\operatorname{rec} B+\operatorname{rec} C)+c+\gamma(C+\operatorname{rec} B+\operatorname{rec} C)\\ &= b+\beta(B+\operatorname{rec} A)+c+\gamma(C+\operatorname{rec} A)\\ &= b+c+\beta B_0+\gamma C_0.\end{aligned}$$

If A is a cone then $A = \operatorname{rec} A$ and $\beta B_0 + \gamma C_0 = A$. Let A be distinct from a cone. By (3c), at least one of the sets B_0, C_0 is not a cone. Assume, for example, that B_0 is not a cone. In this case,

$$\beta B_0+\gamma C_0=\begin{cases}\beta x+\gamma z+(\beta\lambda+\gamma\mu)A, & \text{if } B_0=x+\lambda A,\ C_0=z+\mu A,\\ \beta x+\gamma z+\beta\lambda A, & \text{if } B_0=x+\lambda A,\ C_0=z+\operatorname{rec} A.\end{cases}$$

Summing up, $(b+\beta B)+(c+\gamma C)$ is homothetic to A. Hence $B_H + C_H \subset A_H$.
Since (1) $\Rightarrow$ (2) trivially holds, it remains to prove that (2) $\Rightarrow$ (3). We need some auxiliary lemmas.

Lemma 4. *Line-free closed convex sets S and T are homothetic if and only if $\operatorname{rec} S = \operatorname{rec} T$ and the sets $\operatorname{cl}\exp S$ and $\operatorname{cl}\exp T$ are homothetic.* □

Lemma 5. *If the sets B and C satisfy condition (2) of Theorem 1, then there are scalars $0 < \gamma_1 < \gamma_2$ such that $B + \gamma_1 C$ and $B + \gamma_2 C$ are homothetic.*

Proof. Indeed, consider the family $\mathcal{F} = \{B + \gamma C \mid \gamma > 0\}$. Since $\mathcal{F}$ lies in the union of countably many homothety classes, and since the elements of $\mathcal{F}$ depend on an uncountable parameter γ, there is a pair of scalars $0 < \gamma_1 < \gamma_2$ such that the sets $B + \gamma_1 C$ and $B + \gamma_2 C$ are homothetic. □

Continuing with (2) $\Rightarrow$ (3), we are going to show that the set $A = B + C$ satisfies condition (3). By Lemma 3, A is a closed convex set with $\operatorname{rec} A = \operatorname{rec} B + \operatorname{rec} C$. Furthermore, Theorem 2 obviously implies that A is a cone if and only if both B and C are cones, whence part (3c) also holds.
Hence it remains to prove (3b). If any of the sets B_0, C_0, say B_0, is a cone, then $B_0 = x + \operatorname{rec} B_0 = x + \operatorname{rec} A$ for a suitable point x, and

$$C_0 = C_0 + \operatorname{rec} A = C_0 + (B_0 - x) = A - x.$$

Thus we may assume that neither B_0 nor C_0 is a cone. In this case we will prove that both B_0 and C_0 are homothetic to A. Since $A = B_0 + C_0$, it is sufficient to show that B_0 and C_0 are homothetic. By Lemma 4, B_0 and C_0 are homothetic if and only if the sets $\operatorname{cl}\exp B_0$ and $\operatorname{cl}\exp C_0$ are homothetic, and Theorem 2 implies that the last are homothetic if and only if $\operatorname{cl}\exp_C B$ and $\operatorname{cl}\exp_B C$ are homothetic.

Choose any point $a_0 \in \exp A$. Then $a_0 = b_0 + c_0$ for suitable points $b_0 \in \exp_C B$ and $c_0 \in \exp_B C$. Translating B and C on vectors $-b_0$ and $-c_0$, respectively, we may consider that $a_0 = b_0 = c_0 = 0$. We divide our consideration into two steps.

1. *If points $a \in \exp A \setminus \{0\}$, $b \in \exp_C B$, and $c \in \exp_B C$ are such that $a = b + c$, then 0, b, and c are collinear.*

Indeed, assume the existence of a point $a \in \exp A \setminus \{0\}$ and of points $b \in \exp_C B$, $c \in \exp_B C$ such that $a = b + c$ but 0, b, and c are not collinear. Then no three of the points 0, $b + \gamma_1 c$, $b + \gamma_2 c$, with $0 < \gamma_1 < \gamma_2$, are collinear. Since $b + \gamma c$ is an exposed point of $B + \gamma C$, which has 0 as an exposed point, we conclude that no two elements of the family $\{B + \gamma C \mid \gamma > 0\}$ are homothetic, contradicting Lemma 5.

2. *There is a scalar $\mu > 0$ such that for any points $a \in \exp A \setminus \{0\}$, $b \in \exp_C B$, and $c \in \exp_B C$ with $a = b + c$, we have $c = \mu b$.*

Indeed, assume the existence of points $a_1, a_2 \in \exp A \setminus \{0\}$ and of corresponding points $b_1, b_2 \in \exp_C B$ and $c_1, c_2 \in \exp_B C$, with $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$, such that $c_1 = \mu_1 b_1$ and $c_2 = \mu_2 b_2$, where $\mu_1 \neq \mu_2$. In this case, both $b_1 + \gamma c_1 = (1 + \gamma\mu_1) b_1$ and $b_2 + \gamma c_2 = (1 + \gamma\mu_2) b_2$ are exposed points of $B + \gamma C$ for all $\gamma > 0$. Since 0 is an exposed point of $B + \gamma C$, $\gamma > 0$, and since the ratio

$$\frac{\|(1 + \gamma\mu_1) b_1 - 0\|}{\|(1 + \gamma\mu_2) b_2 - 0\|} = \frac{1 + \gamma\mu_1}{1 + \gamma\mu_2}$$

is a strictly monotone function of γ on $]0, \infty)$, we conclude that no two elements of the family $\{B + \gamma C \mid \gamma > 0\}$ are homothetic. The last is in contradiction with Lemma 5.

Summing up, we conclude the existence of a scalar $\mu > 0$ such that $\mathrm{cl} \exp_C B = \mu \, \mathrm{cl} \exp_B C$. By Lemma 4, B_0 and C_0 are homothetic. □

4. Proof of Theorem 3

The key role here plays the following lemma, which is a slight generalization of Lemma 3.1.10 from [7].

Lemma 6. *Given a closed convex set B of dimension n and a convex body C, the following conditions are equivalent:*

(1) *there is a scalar $\tau > 0$ such that B is a tangential set of τC,*

(2) *there is a scalar $\tau > 0$ such that $B \sim \gamma C = (1 - \gamma/\tau) B$ for all $\gamma \in]0, \tau[$,*

(3) *there is a scalar $\gamma > 0$ such that $B \sim \gamma C = \lambda B$ with $0 < \lambda < 1$.*

Proof. (1) ⇒ (2) If B is a tangential set of τC for some $\tau > 0$, then $\tau C \subset B$ and $\gamma C \subset \gamma/\tau B$ for any scalar $\gamma \in]0, \tau[$. In this case,

$$(1 - \gamma/\tau) B = B \sim \gamma/\tau B \subset B \sim \gamma C.$$

To prove the opposite inclusion, choose any point $x \in B \sim \gamma C$. Equivalently, $x + \gamma C \subset B$. We claim that $x + \gamma/\tau B \subset B$. Indeed, let $\mathcal{P} = \{P_\alpha\}$ be the family of closed halfspaces each containing B such that the boundary hyperplane H_α of every $P_\alpha \in \mathcal{P}$ supports B at a regular boundary point. Obviously, $B = \cap\{P_\alpha \mid$

$P_\alpha \in \mathcal{P}\}$. Since B is a tangential set of τC, each $P_\alpha \in \mathcal{P}$ contains τC and H_α supports τC. Hence each halfspace $\gamma/\tau P_\alpha$ contains γC and the hyperplane $\gamma/\tau H_\alpha$ supports γC. Then the inclusion $x + \gamma C \subset B$ implies that $x + \gamma/\tau P_\alpha \subset P_\alpha$ for all $P_\alpha \in \mathcal{P}$. Thus

$$x + \gamma/\tau B = \cap\{x + \gamma/\tau P_\alpha \mid P_\alpha \in \mathcal{P}\} \subset \{P_\alpha \mid P_\alpha \in \mathcal{P}\} = B,$$

implying that $x \in B \sim \gamma/\tau B = (1 - \gamma/\tau)B$. Finally, $B \sim \gamma C \subset (1 - \gamma/\tau)B$.
Since (2) trivially implies (3), it remains to show that (3) $\Rightarrow$ (1). Let $B \sim \gamma C = \lambda B$ with $0 < \lambda < 1$. Then

$$\begin{aligned}\lambda^2 B = \lambda(\lambda B) = \lambda(B \sim \gamma C) &= \lambda B \sim \lambda\gamma C \\ = (B \sim \gamma C) \sim \lambda\gamma C &= B \sim (1 + \lambda)\gamma C.\end{aligned}$$

By induction on $k = 1, 2, \ldots$ we get

$$\lambda^k B = B \sim (1 + \lambda + \cdots + \lambda^{k-1})\gamma C = B \sim \gamma \tfrac{1-\lambda^k}{1-\lambda} C.$$

As is easily seen, $\lambda^k B \to \operatorname{rec} B$ when $k \to \infty$. By the compactness argument, we have $B \sim \rho_k C \to B \sim \rho C$ when $\rho_k \to \rho$. Hence

$$\operatorname{rec} B = B \sim \tau C \quad \text{with} \quad \tau = \frac{\gamma}{1 - \lambda}.$$

It remains to prove that B is a tangential set of τC. Choose any point $x \in \operatorname{bd} B$. Then

$$\lambda x \in \lambda \operatorname{bd} B = \operatorname{bd}(\lambda B) = \operatorname{bd}(B \sim \gamma C).$$

In particular, $\lambda x \in B \sim \gamma C$, implying that $\lambda x + \gamma C \subset B$.

We claim that $\lambda x + \gamma C$ contains a boundary point of B. Indeed, assume for a moment that $\lambda x + \gamma C \subset \operatorname{int} B$. Since C is compact, there is an open ball U_ε of radius $\varepsilon > 0$ centered at 0 such that the ε-neighborhood $\lambda x + \gamma C + U_\varepsilon$ of $\lambda x + \gamma C$ lies in B. Hence $\lambda x + U_\varepsilon \subset B \sim \gamma C$, in contradiction to $\lambda x \in \operatorname{bd}(B \sim \gamma C)$.

Let y be a point of $\lambda x + \gamma C$ that belongs to $\operatorname{bd} B$. Then $y = \lambda x + \gamma c$ for a point $c \in C$ and

$$v = \frac{y - \lambda x}{1 - \lambda} = \frac{\gamma c}{1 - \lambda} = \tau c \in \tau C \subset B.$$

Since $y = (1 - \lambda)v + \lambda x$ with $x, y \in \operatorname{bd} B$ and $v \in B$ we conclude that the line segment $[x, v]$ lies in $\operatorname{bd} B$. Hence any support hyperplane of B through y contains x and v and thus supports B at x and τC at v. So B is a tangential set of τC.$\square$

Remark 3. From the proof of Lemma 6 we conclude that if the sets B and C satisfy condition (3) of the lemma, then B is a translate of a tangential set of $\gamma/(1 - \lambda)C$.

Let us recall (see [7, p. 136]) that the *inradius of a convex body B with respect to a convex body C* is defined by

$$r_C(B) = \max\{\lambda \geq 0 \mid x + \lambda C \subset B\}.$$

Lemma 7. *Given convex bodies B and C, we have $r_C(B)r_B(C) \leq 1$. The equality $r_C(B)r_B(C) = 1$ holds if and only if B and C are homothetic.*

Proof. Put $s = r_C(B)$ and $t = r_B(C)$. Then $x + sC \subset B$ and $z + tB \subset C$ for some vectors x, z. In this case, $tx + stC \subset tB \subset C - z$, implying that $st \leq 1$.

If $st = 1$ then from the inclusion above we deduce that $tB = C - z$, whence B is homothetic to C. Conversely, if $B = x + \gamma C$, $\gamma > 0$, then, as easy to see, $r_B(C) = \gamma$ and $r_C(B) = \gamma^{-1}$. □

Lemma 8. *Given convex bodies B and C and a scalar $\rho \in]0, r_C(B)[$, we have $r_C(B \sim \rho C) = r_C(B) - \rho$.*

Proof. Indeed,

$$\begin{aligned} r_C(B \sim \rho C) &= \max\{\lambda \geq 0 \mid x + \lambda C \subset B \sim \rho C,\ x \in \mathbb{R}^n\} \\ &= \max\{\lambda \geq 0 \mid x + \lambda C + \rho C \subset B,\ x \in \mathbb{R}^n\} \\ &= \max\{\lambda \geq 0 \mid x + (\lambda + \rho)C \subset B,\ x \in \mathbb{R}^n\} \\ &= r_C(B) - \rho. \end{aligned}$$

□

Lemma 9. *Let B and C be convex bodies such that $B \sim \rho C = z + \mu B$ for a vector z and scalars $\rho \in]0, r_C(B)[$ and $\mu > 0$. Then*

$$1 - \rho r_C^{-1}(B) \leq \mu \leq 1 - \rho r_B(C).$$

Proof. Let v be a vector such that $v + r_B(C)B \subset C$. According to Lemma 7, $\rho v + \rho r_B(C)B \subset \rho C$ with $\rho r_B(C) < r_C(B)r_B(C) \leq 1$. We have

$$\begin{aligned} B \sim \rho C &= \{x \in \mathbb{R}^n \mid x + \rho C \subset B\} \subset \{x \in \mathbb{R}^n \mid x + \rho v + \rho r_B(C)B \subset B\} \\ &= \{x \in \mathbb{R}^n \mid x + \rho r_B(C)B \subset B - \rho v\} = (B - \rho v) \sim \rho r_B(C)B \\ &= (B \sim \rho r_B(C)B) - \rho v = (1 - \rho r_B(C))B - \rho v. \end{aligned}$$

Hence

$$z + \mu B = B \sim \rho C \subset (1 - \rho r_B(C))B - \rho v,$$

which implies the inequality $\mu \leq 1 - \rho r_B(C)$.

On the other hand, there is a vector w such that $w + r_C(B)C \subset B$, which gives the inclusion $\rho C \subset \rho r_C^{-1}(B)(B - w)$. Thus

$$\begin{aligned} z + \mu B = B \sim \rho C &= \{x \in \mathbb{R}^n \mid x + \rho C \subset B\} \\ &\supset \{x \in \mathbb{R}^n \mid x + \rho r_C^{-1}(B)(B - w) \subset B + \rho r_C^{-1}(B)w\} \\ &= \{x \in \mathbb{R}^n \mid x + \rho r_C^{-1}(B)B \subset B + \rho r_C^{-1}(B)w\} \\ &= (B + \rho r_C^{-1}(B)w) \sim \rho r_C^{-1}(B)B \\ &= r_C^{-1}(B)w + (B \sim \rho r_C^{-1}(B)B) \\ &= r_C^{-1}(B)w + (1 - \rho r_C^{-1}(B))B, \end{aligned}$$

resulting in the inequality $1 - \rho r_C^{-1}(B) \le \mu$. $\square$

Lemma 10. *If $T_1, T_2, \dots$ is a convergent sequence of tangential convex bodies of a convex body C, then their limit is also a tangential body of C.*

Proof. Let $T = \lim_{k\to\infty} T_k$. Choose a boundary point x of T. Then there is a sequence of points $x_k \in \operatorname{bd} T_k$, $k = 1, 2, \dots$, such that $x = \lim_{k\to\infty} x_k$. For each point x_k there is a hyperplane H_k supporting T_k at x_k and also supporting C. The sequence $H_1, H_2, \dots$ contains a subsequence $H_1', H_2', \dots$ that converges to a hyperplane H. As is easily seen, H supports T at x and also supports C. Hence T is a tangential body of C. $\square$

Proof of Theorem 3. (4) $\Rightarrow$ (1) By Lemma 6, every n-dimensional set

$$(x + \lambda B) \sim (z + \gamma C) = (x - z) + \lambda(B \sim \gamma/\lambda C), \ \lambda, \gamma > 0,$$

is homothetic to B. Hence $B_H \sim C_H \subset B_H$.
Since the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial, it remains to show that (3) $\Rightarrow$ (4). Consider the intervals

$$I_k =]2^{-k} r_C(B), 2^{1-k} r_C(B)[, \quad k = 1, 2, \dots .$$

By the assumption, each family

$$\mathcal{D}_k = \{B \sim \lambda C \mid \lambda \in I_k, \dim(B \sim \lambda C) = n\}, \quad k = 1, 2, \dots ,$$

lies in the union of countably many homothety classes. Hence there are scalars $\delta_k, \gamma_k \in I_k$ and $\mu_k \in]0, 1[$ such that $\delta_k < \gamma_k$ and

$$B \sim \gamma_k C = x_k + \mu_k (B \sim \delta_k C), \quad x_k \in \mathbb{R}^n, \ k = 1, 2, \dots .$$

Since

$$B \sim \gamma_k C = B \sim (\delta_k C + (\gamma_k - \delta_k) C) = (B \sim \delta_k C) \sim (\gamma_k - \delta_k) C,$$

we have

$$(B \sim \delta_k C) \sim (\gamma_k - \delta_k) C = x_k + \mu_k (B \sim \delta_k C).$$

By Lemma 6 and Remark 3, $B \sim \delta_k C$ is a translate of a tangential set of $(\gamma_k - \delta_k)/(1 - \mu_k) C$, or, equivalently, the body

$$D_k = (1 - \mu_k)/(\gamma_k - \delta_k)(B \sim \delta_k C)$$

is a translate of a tangential set T_k of C. Lemma 9 implies that

$$\mu_k \ge 1 - (\gamma_k - \delta_k) r_C^{-1}(B - \delta_k C),$$

which gives

$$\frac{1 - \mu_k}{\gamma_k - \delta_k} \le \frac{1}{r_C(B - \delta_k C)}, \quad k = 1, 2, \dots .$$

By Lemma 8, $r_C(B \sim \delta_k C) = r_C(B) - \delta_k$. Since $\delta_1 > \delta_2 > \cdots > 0$, we have

$$\frac{1}{r_C(B - \delta_1 C)} > \frac{1}{r_C(B - \delta_2 C)} > \cdots > \frac{1}{r_C(B)}.$$

As a result, all of $D_1, D_2, \dots$ are contained in a neighborhood of $B \sim \delta_1 C$. Then we can select a subsequence $D_1', D_2', \dots$ of $D_1, D_2, \dots$ that converges to a convex body D. Since each D_k is a translate of the tangential body T_k that contains C, the respective subsequence $T_1', T_2', \dots$ converges to a convex body T. By Lemma 10, T is a tangential body of C.

Finally, $\lim_{k\to\infty}(B \sim \delta_k C) = B$ implies that B is homothetic to T. □

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