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Petrie-Coxeter Maps Revisited

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Abstract. This paper presents a technique for constructing new chiral or regular polyhedra (or maps) from self-dual abstract chiral polytopes of rank 4. From improperly self-dual chiral polytopes we derive "Petrie-Coxeter-type" polyhedra (abstract chiral analogues of the classical Petrie-Coxeter polyhedra) and investigate their groups of automorphisms.

1. Introduction

The modern theory of regular polytopes and their geometric realizations in Euclidean spaces has been greatly influenced by the discovery of the Petrie-Coxeter polyhedra in $\mathbb{E}^3$ and Coxeter's regular skew polyhedra in $\mathbb{E}^4$ (see Coxeter [3], Dress [7], Grünbaum [11], McMullen [14, 15], and [8, 17]). Two of the Petrie-Coxeter polyhedra and all of Coxeter's skew polyhedra inherit their symmetry properties directly from self-dual regular figures, namely the cubical tessellation in $\mathbb{E}^3$, and the self-dual regular convex polytopes (the 4-simplex $\{3,3,3\}$ and 24-cell $\{3,4,3\}$) or the double p-gonal prisms (or $\{p,2,p\}$) in $\mathbb{E}^4$, respectively (see Section 4). However, the symmetries of these figures provide only one half of the symmetries of the polyhedra; the other half corresponds to the dualities (see [16, 20]).

Coxeter credited Petrie with the original idea of admitting infinite regular polyhedra in $\mathbb{E}^3$ with skew vertex-figures. In particular, Petrie discovered two

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of the Petrie-Coxeter polyhedra, and then Coxeter found the third and carried out the corresponding enumeration in $\mathbb{E}^4$. In [10] and [13], this enumeration is extended to hyperbolic 3-space $\mathbb{H}^3$, where there are thirty-one regular skew polyhedra.

The connection between self-dual polytopes and skew polyhedra was further investigated in [16] in the context of abstract regular polytopes, where a construction of regular "Petrie-Coxeter-type" polyhedra from self-dual regular 4-polytopes is described. In this paper we extend these ideas to abstract chiral polytopes. We exploit the two different kinds of self-duality (proper and improper self-duality) for chiral polytopes and present constructions of polyhedra, which are chiral or regular, depending on the kind of self-duality of the original chiral 4-polytope. From improperly self-dual chiral 4-polytopes we obtain chiral "Petrie-Coxeter-type" polyhedra (or maps), that is, abstract chiral analogues of the ordinary Petrie-Coxeter polyhedra.

2. Regular and chiral polytopes

In this section we briefly review some definitions and basic results from the theory of abstract polytopes. For details we refer to [17].

A *polytope of rank* n, or an n*-polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function having range $\{-1, 0, \dots, n\}$. The elements of $\mathcal{P}$ with rank j are the j*-faces* of $\mathcal{P}$. The faces of ranks 0, 1 or $n-1$ are called *vertices*, *edges* or *facets*, respectively. There is a smallest face F_{-1} (of rank -1) and a greatest face F_n (of rank n), and each *flag* (maximal chain) contains exactly $n+2$ faces. Also, $\mathcal{P}$ is *strongly flag-connected*, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ such that each Φ_{i-1} and Φ_i are *adjacent* (they differ by just one face), and $\Phi \cap \Psi \subseteq \Phi_i$ for each i. Furthermore, $\mathcal{P}$ has the following homogeneity property: whenever $F \leq G$, with $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, then there are exactly two j-faces H with $F \leq H \leq G$. These conditions essentially say that $\mathcal{P}$ shares many combinatorial properties with classical polytopes.

If Φ is a flag, then by Φ^i we denote the unique flag adjacent to Φ and differing from Φ in the face of rank i. Note that $(\Phi^i)^i = \Phi$. We extend our notation and write $\Phi^{i_1,\dots,i_{k-1},i_k} := (\Phi^{i_1,\dots,i_{k-1}})^{i_k}$. For any two faces F and G with $F \leq G$, we call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$; if F is a vertex and $G = F_n$, then this is the *vertex-figure* of $\mathcal{P}$ at F. Note that G/F is an abstract polytope of rank equal to $\mathrm{rank}(G) - \mathrm{rank}(F) - 1$.

The group of automorphisms (order preserving bijections) of $\mathcal{P}$ is denoted by $\Gamma(\mathcal{P})$. If $\mathcal{P}$ admits an order reversing bijection, then $\mathcal{P}$ is said to be *self-dual* and any such bijection is called a *duality* of $\mathcal{P}$. In this case the group of all automorphisms and dualities is called the *extended group* of $\mathcal{P}$ and is denoted by $\overline{\Gamma}(\mathcal{P})$. A *polarity* is an involutory duality. Note that, if Φ is any flag and δ any duality, then $\Phi^j\delta = (\Phi\delta)^{n-j-1}$ for $j = 0, \dots, n-1$.

A polytope $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive on the flags of $\mathcal{P}$. The group of a regular polytope $\mathcal{P}$ is generated by involutions ρ_i, $i = 0, \dots, n-1$, mapping a fixed, or *base*, flag Φ to the adjacent flags Φ^i. These generators satisfy (at least) the relations

$$(\rho_i\rho_j)^{p_{ij}} = \epsilon \quad \text{for } i,j = 0,\dots,n-1, \tag{1}$$

where $p_{ii} = 1$, $p_{ji} = p_{ij} =: p_{i+1}$ if $j = i+1$, and $p_{ij} = 2$ otherwise. We then say that $\mathcal{P}$ is of *type* $\{p_1,\dots,p_{n-1}\}$. Furthermore, these generators satisfy the *intersection condition*,

$$\langle \rho_i \mid i \in I\rangle \cap \langle \rho_i \mid i \in J\rangle = \langle \rho_i \mid i \in I \cap J\rangle \quad (I,J \subseteq \{0,1,\dots,n-1\}). \tag{2}$$

Conversely, if $\Gamma = \langle \rho_0,\dots,\rho_{n-1}\rangle$ is a group whose generators satisfy relations (1) and condition (2) (such groups are called *C-groups*), then there exists a regular polytope with $\Gamma(\mathcal{P}) \cong \Gamma$.

We note that every self-dual regular polytope possesses a polarity mapping the base flag to itself (but reversing the order of its elements).

A polytope $\mathcal{P}$ (of rank $n \geq 3$) is *chiral* if $\mathcal{P}$ is not regular, but if for some base flag $\Phi = \{F_{-1}, F_0, \dots, F_n\}$ there exist automorphisms $\sigma_1,\dots,\sigma_{n-1}$ of $\mathcal{P}$ such that σ_i fixes all faces in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces of $\mathcal{P}$ in the section $F_{i+1}/F_{i-2} = \{G \mid F_{i-2} \leq G \leq F_{i+1}\}$ of rank 2. We may choose σ_i in such a way that

$$\Phi\sigma_i = \Phi^{i,i-1}, \quad \text{for all } i \in \{1,\dots,n-1\},$$

and hence

$$\Phi\sigma_i^{-1} = \Phi^{i-1,i}, \text{ for all } i \in \{1,\dots,n-1\}.$$

The automorphisms $\sigma_1,\dots,\sigma_{n-1}$ generate $\Gamma(\mathcal{P})$ and satisfy (at least) the relations

$$\begin{aligned} \sigma_i^{p_i} &= \epsilon \quad \text{for } 1 \leq i \leq n-1, \\ (\sigma_i\sigma_{i+1}\dots\sigma_j)^2 &= \epsilon \quad \text{for } 1 \leq i < j \leq n-1, \end{aligned} \tag{3}$$

where again $\{p_1,\dots,p_{n-1}\}$ is the type of $\mathcal{P}$. For general background on chiral polytopes see [18, 19].

Alternatively, chiral polytopes can be characterized as those polytopes whose automorphism groups have two flag orbits, with adjacent flags in distinct orbits (see [18, 19]). Consequently, if a chiral polytope $\mathcal{P}$ is self-dual, it can be so in one of two ways. If a duality of $\mathcal{P}$ maps a flag to a flag (in reverse order) in the same orbit under $\Gamma(\mathcal{P})$, then each duality must do so with each flag; in this case $\mathcal{P}$ is said to be *properly self-dual*. However, if a duality maps a flag to a flag in a different orbit under $\Gamma(\mathcal{P})$, then again each duality must do so with each flag, and in this case $\mathcal{P}$ is said to be *improperly self-dual*.

For a regular n-polytope $\mathcal{P}$ with group $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \dots, \rho_{n-1}\rangle$, define $\sigma_i := \rho_{i-1}\rho_i$ for $i = 1,\dots,n-1$. Then $\sigma_1,\dots,\sigma_{n-1}$ generate the (*combinatorial*) *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$ (of index at most 2) and have properties similar to those of the distinguished generators for chiral polytopes.

From now on we restrict our considerations to polytopes of rank $n \leq 4$. The generators σ_i of a chiral polytope $\mathcal{P}$ also satisfy an *intersection condition*, which for $n = 4$ is

$$\begin{gathered} \langle \sigma_1, \sigma_2\rangle \cap \langle \sigma_2, \sigma_3\rangle = \langle \sigma_2\rangle, \\ \langle \sigma_1\rangle \cap \langle \sigma_2\rangle = \{\epsilon\} = \langle \sigma_2\rangle \cap \langle \sigma_3\rangle, \end{gathered} \tag{4}$$

while for $n = 3$ we only have

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{\epsilon\}. \tag{5}$$

Conversely, if $\Gamma = \langle \sigma_1, \ldots, \sigma_n \rangle$ is a group whose generators satisfy the relations (3) and this intersection condition, namely (4) for $n = 4$ or (5) for $n = 3$, then there exists a chiral polytope $\mathcal{P}$ with $\Gamma(\mathcal{P}) \cong \Gamma$ or a regular polytope $\mathcal{P}$ with $\Gamma^+(\mathcal{P}) \cong \Gamma$.

3. Holes, zigzags, and Petrie polygons

Each 3-polytope $\mathcal{P}$ gives rise to a map on a surface, which in the case of a chiral polytope must be orientable. A *j-hole* of $\mathcal{P}$ is an edge-path on the surface which leaves a vertex by the j-th edge from which it entered, always in the same sense (in some local orientation), keeping to the left (say). Similarly, a *j-zigzag* of $\mathcal{P}$ is an edge-path on the surface which leaves a vertex by the j-th edge from which it entered, alternating the sense. For example, the 1-holes are simply the 2-faces of $\mathcal{P}$, and the 1-zigzags are precisely the *Petrie polygons* of $\mathcal{P}$ (see [5]).

For a regular 3-polytope $\mathcal{P}$, the length of a j-hole or j-zigzag is given by the period of $\rho_0\rho_1(\rho_2\rho_1)^{j-1}$ or $\rho_0(\rho_1\rho_2)^j$ in $\Gamma(\mathcal{P})$, respectively (see [17, pp. 196–197]); note here that $\rho_0\rho_1(\rho_2\rho_1)^{j-1} = \sigma_1\sigma_2^{1-j} \in \Gamma^+(\mathcal{P})$ (but $\rho_0(\rho_1\rho_2)^j \in \Gamma^+(\mathcal{P})$ only if $\Gamma^+(\mathcal{P}) = \Gamma(\mathcal{P})$). Similarly, for a chiral 3-polytope $\mathcal{P}$, the length of a j-hole is the period of $\sigma_1\sigma_2^{1-j}$ in $\Gamma(\mathcal{P})$.

If a regular 3-polytope of type $\{p, q\}$ is completely determined by the lengths h_j of its j-holes, for $2 \leq j \leq k := \lfloor q/2 \rfloor$, and lengths r_j of its j-zigzags, for $1 \leq j \leq k := \lfloor q/2 \rfloor$, then we denote it by

$$\mathcal{P} = \{p, q \,|\, h_2, \ldots, h_k\}_{r_1,\ldots,r_k},$$

with the convention that any unnecessary h_j or r_j (that is, one not needed for the specification) is replaced by a $\cdot$, with those at the end of the sequence omitted. More generally, we say that a regular 3-polytope is of *type* $\{p, q \,|\, h_2, \ldots, h_k\}_{r_1,\ldots,r_k}$ if its j-holes and j-zigzags have lengths h_j or r_j, respectively, with j as above; we also use this terminology for chiral 3-polytopes when applicable (that is, when only the h_j's are specified).

For chiral or regular 4-polytopes $\mathcal{P}$ we define two particular automorphisms

$$\pi_L := \sigma_1\sigma_3 \quad \text{and} \quad \pi_R := \sigma_1\sigma_3^{-1}.$$

Let F_0 and F_1 be the base vertex and base edge (of the base flag Φ), respectively. Then the orbits $F_0\langle \pi_L \rangle$ and $F_1\langle \pi_L \rangle$, respectively, are the vertex-set and edge-set of a polygon; its images under $\Gamma(\mathcal{P})$ are called the *left-handed Petrie polygons*, or for short *left-Petrie polygons*, of $\mathcal{P}$. Similarly, we obtain *right-Petrie polygons* from π_R. Generally, for chiral polytopes, π_L and π_R have different orders, so the left- and right-Petrie polygons have different lengths. Left- and right-Petrie polygons were employed by Coxeter in [4] to construct chiral (or, in his terminology, *twisted*) 4-polytopes. For regular 4-polytopes, the elements π_L and π_R are conjugate in $\Gamma(\mathcal{P})$, so left- and right-Petrie polygons necessarily have the same lengths.

4. Chiral maps from improperly self-dual chiral polytopes

From [16] we know that with any self-dual regular 4-polytope $\mathcal{P}$ of type $\{p,q,p\}$ is associated a certain regular map of type $\{4,2q\,|\,p\}$ (with group $\overline{\Gamma}(\mathcal{P})$), called the *Petrie-Coxeter polyhedron of* $\mathcal{P}$; its dual is of type $\{2q,4\,|\,p\}$. When $\mathcal{P}$ is the universal polytope $\{p,q,p\}$ (see [17]), we obtain (the universal) $\{4,2q\,|\,p\}$ (as well as its dual $\{2q,4\,|\,p\}$). For example, from the 4-simplex $\{3,3,3\}$ and 24-cell $\{3,4,3\}$ in Euclidean 4-space $\mathbb{E}^4$ we derive Coxeter's finite skew polyhedra $\{4,6\,|\,3\}$ and $\{4,8\,|\,3\}$ in $\mathbb{E}^4$ (see [3], and for figures of projections into $\mathbb{E}^3$ see [20]), while the regular tessellations $\{4,3,4\}$ and $\{5,3,5\}$ of Euclidean or hyperbolic 3-space $\mathbb{E}^3$ or $\mathbb{H}^3$, respectively, yield the (classical) Petrie-Coxeter polyhedron $\{4,6\,|\,4\}$ in $\mathbb{E}^3$ and the hyperbolic skew polyhedron $\{4,6\,|\,5\}$ in $\mathbb{H}^3$ (see [3, 10, 13]). The toroidal regular skew polyhedra $\{4,4\,|\,p\}$ in $\mathbb{E}^4$ are related in a similar fashion to the double p-gonal prism $\{p\}\times\{p\}$ (or $\{p,2,p\}$ in $\mathbb{S}^3$).

The construction of [16] employs the self-duality of $\mathcal{P}$. Since $\mathcal{P}$ is regular, it has a (unique) polarity ω that fixes the base flag and thus permutes (by conjugation) the generators of $\Gamma(\mathcal{P})=\langle\rho_0,\ldots,\rho_3\rangle$ according to $\omega\rho_i\omega=\rho_{3-i}$, with $i=0,1,2,3$. For the universal polytope $\{p,q,p\}$, this polarity corresponds to the symmetry of the underlying (string) Coxeter diagram. For an arbitrary (self-dual and regular) polytope $\mathcal{P}$, the generators τ_0,τ_1,τ_2 of the automorphism group $\overline{\Gamma}(\mathcal{P})$ of its Petrie-Coxeter polyhedron are obtained from a twisting operation (in the sense of [17, Chapter 8]) on $\Gamma(\mathcal{P})$,

$$(\rho_0,\rho_1,\rho_2,\rho_3,\omega)\mapsto(\rho_0,\omega,\rho_2)=:(\tau_0,\tau_1,\tau_2). \tag{6}$$

Alternatively we may view (6) as a mixing operation (in the sense of [17, Chapter 7]) on $\overline{\Gamma}(\mathcal{P})$.

We now make use of the fact that the automorphism groups of chiral polytopes of rank greater than 3 are generated by involutions, and apply similar techniques to construct new "Petrie-Coxeter type" maps from self-dual chiral polytopes. In this section we discuss improper self-duality.

Henceforth, we assume that $\mathcal{P}$ is a chiral 4-polytope of type $\{p,q,p\}$, Φ its base flag, and $\sigma_1,\sigma_2,\sigma_3$ the distinguished generators of $\Gamma(\mathcal{P})$ (with respect to Φ). Then these generators satisfy (at least) the following relations

$$\sigma_1^p=\sigma_2^q=\sigma_3^p=(\sigma_1\sigma_2)^2=(\sigma_1\sigma_2\sigma_3)^2=(\sigma_2\sigma_3)^2=\epsilon. \tag{7}$$

Note that $\sigma_1\sigma_2,\sigma_1\sigma_2\sigma_3,\sigma_2\sigma_3$ is a set of involutory generators of $\Gamma(\mathcal{P})$.

Let $\mathcal{P}$ be improperly self-dual. Then there exists a duality δ which exchanges the two flag orbits. In fact, by multiplying δ with an automorphism of $\mathcal{P}$ (if need be), we may choose δ so that $\Phi\delta=\Phi^0$. Then,

$$\Phi^0\delta=\Phi^{0,3},\;\;\Phi^{0,3}\delta=\Phi^3,\;\;\Phi^3\delta=\Phi.$$

Bear in mind here that δ, being a duality, maps a j-face in a flag to the $(3-j)$-face in the image flag. It follows that the *distinguished duality* δ has period 4 and satisfies the relations

$$\begin{array}{rcl} \delta^2 &=& \sigma_1\sigma_2\sigma_3, \\ \delta^{-1}\sigma_1\delta &=& \sigma_3^{-1}, \\ \delta^{-1}\sigma_2\delta &=& \sigma_1\sigma_2\sigma_1^{-1}, \\ \delta^{-1}\sigma_3\delta &=& \sigma_1, \end{array} \tag{8}$$

in $\overline{\Gamma}(\mathcal{P})$ (see [12, Theorem 3.5]). Moreover, it is easy to see that δ fixes the basic involution $\sigma_1\sigma_2\sigma_3$, that is, $\delta^{-1}\sigma_1\sigma_2\sigma_3\delta = \sigma_1\sigma_2\sigma_3$, while

$$\begin{array}{rcl} \delta^{-1}\sigma_1\sigma_2\delta &=& \sigma_1\sigma_2\sigma_3\sigma_1^{-1}, \\ \delta^{-1}\sigma_1\sigma_2\sigma_3\sigma_1^{-1}\delta &=& \sigma_3^{-1}\sigma_1\sigma_2\sigma_3, \\ \delta^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_3\delta &=& \sigma_2\sigma_3, \\ \delta^{-1}\sigma_2\sigma_3\delta &=& \sigma_1\sigma_2. \end{array} \tag{9}$$

Note that the action of δ on the five involutory generators

$$\sigma_1\sigma_2,\ \sigma_1\sigma_2\sigma_3\sigma_1^{-1},\ \sigma_3^{-1}\sigma_1\sigma_2\sigma_3,\ \sigma_2\sigma_3,\ \sigma_1\sigma_2\sigma_3$$

of $\Gamma(\mathcal{P})$ can be illustrated by a Coxeter-type diagram on five nodes such that δ corresponds to a symmetry of the diagram (see Figure 3 below for an example of a diagram on three nodes). In particular, if the nodes are given by the vertices and the center of a square diagram, with the central node representing $\sigma_1\sigma_2\sigma_3$ and the vertices representing the other four generators (in the order in which they are listed), then the action of δ on these generators corresponds to the 4-fold rotational symmetry of the diagram.

We now consider the mixing operation

$$(\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3, \sigma_2\sigma_3, \delta) \mapsto (\delta, \sigma_1\sigma_2\delta^{-1}) =: (\kappa_1, \kappa_2) \tag{10}$$

on $\overline{\Gamma}(\mathcal{P})$, bearing in mind that the first three involutions on the left side generate $\Gamma(\mathcal{P})$. Set $\Lambda := \langle\kappa_1, \kappa_2\rangle$. Then $\Lambda = \overline{\Gamma}(\mathcal{P})$; in fact, we have

$$\begin{array}{l} \sigma_1 = \delta^2\sigma_2\sigma_3 = \delta^{-2}\sigma_2\sigma_3\delta\delta^{-1} = \delta^{-1}\sigma_1\sigma_2\delta^{-1} = \kappa_1^{-1}\kappa_2, \\ \sigma_2 = \sigma_2(\sigma_3\sigma_1\sigma_2)^2 = \delta\delta^{-1}\sigma_2\sigma_3\delta^2\sigma_1\sigma_2 = (\delta\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2\delta^{-1})^{-2} = \kappa_2^{-2}, \\ \sigma_3 = \sigma_1\sigma_2\delta^2 = \sigma_1\sigma_2\delta^{-2} = \kappa_2\kappa_1^{-1}. \end{array} \tag{11}$$

Clearly, κ_1 is of order 4, $\kappa_1\kappa_2 = \delta\sigma_1\sigma_2\delta^{-1}$ is an involution, and $\kappa_1\kappa_2^{-1} = \sigma_3^{-1}$ is of order p. Furthermore,

$$\kappa_2^2 = \sigma_1\sigma_2\delta^{-1}\sigma_1\sigma_2\delta\delta^2 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_1^{-1}\delta^2 = \sigma_3\sigma_1^{-1}\sigma_1\sigma_2\sigma_3 = \sigma_2^{-1}, \tag{12}$$

and hence $\kappa_2^{2q} = \epsilon$. Moreover, κ_2 has order $2q$; in fact, κ_2, being a duality, has even order, and $\sigma_2^{-1} = \kappa_2^2$ has order q. Hence, κ_1, κ_2 satisfy (at least) the following relations

$$\kappa_1^4 = \kappa_2^{2q} = (\kappa_1\kappa_2)^2 = (\kappa_1\kappa_2^{-1})^p = \epsilon. \tag{13}$$

These observations suggest that $\langle\kappa_1, \kappa_2\rangle$ could be the automorphism group of a chiral map. Indeed, this is true.

Theorem 1. *Let $\mathcal{P}$ be an improperly self-dual chiral 4-polytope of type $\{p, q, p\}$, let $\overline{\Gamma}(\mathcal{P})$ be its extended group, and let $\Lambda := \langle \kappa_1, \kappa_2 \rangle$, with κ_1, κ_2 as in (10). Then Λ is the automorphism group of a chiral 3-polytope $\mathcal{M}$ of type $\{4, 2q\,|\,p\}$, and $\Lambda = \overline{\Gamma}(\mathcal{P})$.*

Proof. We need to verify the intersection condition $\langle \kappa_1 \rangle \cap \langle \kappa_2 \rangle = \{\epsilon\}$ for Λ. Suppose that $\gamma \in \langle \kappa_1 \rangle \cap \langle \kappa_2 \rangle$. Then $\gamma = \kappa_1^j = \delta^j$ for some $j \in \{0, 1, 2, 3\}$, and $\gamma = \kappa_2^i = (\sigma_1 \sigma_2 \delta^{-1})^i$ for some i with $0 \leq i < 2q$. It suffices to rule out the possibility that $j = 2$; then, by replacing γ by γ^2, we have also ruled out the cases $j = 1, 3$, and hence must have $j = 0$.

Now suppose that $j = 2$. Then $\delta^2 = \gamma = \kappa_2^i$ is an automorphism of $\mathcal{P}$, and hence i is even, $i = 2k$ (say). But then (12) implies that

$$\sigma_1 \sigma_2 \sigma_3 = \delta^2 = (\kappa_2^2)^k = \sigma_2^{-k},$$

and hence $\sigma_1 \in \langle \sigma_2, \sigma_3 \rangle$, a contradiction to the intersection condition of $\Gamma(\mathcal{P})$. Hence we cannot have $j = 2$.

We now know that Λ determines a chiral or regular map $\mathcal{M}$. It remains to show that $\mathcal{M}$ cannot be regular. Suppose that $\mathcal{M}$ is regular. Then $\Lambda = \Gamma^+(\mathcal{M})$. Let ρ denote the 0^{th} distinguished involutory generator of $\Gamma(\mathcal{M})$. Then conjugation in $\Gamma(\mathcal{M})$ by ρ gives

$$\rho \kappa_1 \rho = \kappa_1^{-1}, \quad \rho \kappa_2 \rho = \kappa_1^2 \kappa_2.$$

Hence $\rho \delta \rho = \delta^{-1}$, and by (9) and (11) we have

$$\rho \sigma_1 \rho = \rho \kappa_1^{-1} \kappa_2 \rho = \kappa_1 \kappa_1^2 \kappa_2 = \kappa_1^3 \kappa_2 = \kappa_1^{-1} \kappa_2 = \sigma_1,$$

$$\begin{aligned} \rho \sigma_2 \rho &= \rho \kappa_2^{-2} \rho = (\kappa_2^{-1} \kappa_1^{-2})^2 = \kappa_1 \kappa_2^2 \kappa_1^{-1} = \delta (\sigma_1 \sigma_2 \delta^{-1})^2 \delta^{-1} = \delta \sigma_1 \sigma_2 \delta^{-1} \sigma_1 \sigma_2 \delta^2 \\ &= \delta^2 (\delta^{-1} \sigma_1 \sigma_2 \delta) \delta^2 \sigma_1 \sigma_2 \delta^2 = \delta^2 (\sigma_1 \sigma_2 \sigma_3 \sigma_1^{-1}) \sigma_1 \sigma_2 \sigma_3^2 = \sigma_2 \sigma_3^2, \end{aligned}$$

$$\rho \sigma_3 \rho = \rho \kappa_2 \kappa_1^{-1} \rho = \kappa_1^2 \kappa_2 \kappa_1 = \delta^2 \sigma_1 \sigma_2 \delta^{-1} \delta = \sigma_3^{-1}.$$

Thus ρ determines an involutory group automorphism of $\Gamma(\mathcal{P})$ that acts on the generators $\sigma_1, \sigma_2, \sigma_3$ as indicated. Now it follows from [18, Theorem 1] that $\mathcal{P}$ must be regular, contrary to our assumptions; in fact, ρ corresponds to conjugation by the 3^{rd} distinguished involutory generator of $\Gamma(\mathcal{P})$. Hence $\mathcal{M}$ must be chiral. This completes the proof. □

It is interesting to investigate the above construction in the context of regular polytopes. Let $\mathcal{P}$ be a self-dual regular 4-polytope, let $\rho_0, \rho_1, \rho_2, \rho_3$ be the distinguished generators of $\Gamma(\mathcal{P})$ with respect to some base flag Φ, and let $\sigma_1, \sigma_2, \sigma_3$ be the corresponding distinguished generators of the rotation subgroup $\Gamma^+(\mathcal{P})$, that is, $\sigma_i := \rho_{i-1} \rho_i$ for $i = 1, 2, 3$. As before, let δ be the duality (of period 4) defined by $\Phi \delta = \Phi^0$, and let κ_1, κ_2 be as in (10). Then the equations (8), (9) and (11) carry over. Clearly, $\overline{\Gamma}(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2, \rho_3, \delta \rangle$. Since $\mathcal{P}$ is regular, we also have a polarity ω that fixes Φ. Then $\delta = \omega \rho_0$ $(= \rho_3 \omega)$. Once again, bear in mind here that dualities map a j-face in a flag to the $(3-j)$-face in the image flag. Moreover,

$$\delta^{-1} \rho_j \delta = \rho_0 \omega \rho_j \omega \rho_0 = \rho_0 \rho_{3-j} \rho_0,$$

so δ acts on the generators of $\Gamma(\mathcal{P})$ as follows:

$$\delta^{-1}\rho_0\delta = \rho_3, \ \ \delta^{-1}\rho_1\delta = \rho_2, \ \ \delta^{-1}\rho_2\delta = \rho_0\rho_1\rho_0, \ \ \delta^{-1}\rho_3\delta = \rho_0.$$

Define $\rho := \rho_3$. (Recall that the supposed group automorphism of $\Gamma(\mathcal{P})$ in the proof of Theorem 1 was determined by conjugation with ρ_3.) Then, using $\delta = \omega\rho_0$, the generators of $\Gamma(\mathcal{P})$ can be expressed in terms of κ_1, κ_2, ρ as follows:

$$\rho_0 = \rho\kappa_1^2\,(= \rho\delta^2), \ \ \rho_1 = \rho\kappa_1\kappa_2, \ \ \rho_2 = \rho\kappa_1\kappa_2^{-1}, \ \ \rho_3 = \rho.$$

Now the mixing operation (10) can be rewritten in terms of the generators ρ_i as follows:

$$(\rho_0, \rho_1, \rho_2, \rho_3, \delta) \mapsto (\rho_3, \rho_3\delta, \rho_1) = (\rho, \rho\kappa_1, \rho\kappa_1\kappa_2). \tag{14}$$

(Recall here that, in the proof of Theorem 1, the element ρ acted on $\mathcal{M}$ like the 0^{th} distinguished generator.) Note that $(\rho\kappa_1)^2 = \epsilon$, so the generators of the corresponding rotation subgroup are indeed κ_1, κ_2. Moreover, $\rho_3\delta = \omega$, so the operation (14) is simply the dual version of (6) (obtained by conjugating the new generators with ω).

We summarize our observations in

Remark 1. Let $\mathcal{P}$ be a self-dual regular 4-polytope of type $\{p, q, p\}$, let $\overline{\Gamma}(\mathcal{P})$ be its extended group, and let $\Lambda := \langle\kappa_1, \kappa_2\rangle$, with κ_1, κ_2 as in (10). Then Λ is the rotation subgroup of a regular 3-polytope $\mathcal{M}$ of type $\{4, 2q\,|\,p\}$, whose full automorphism group is $\overline{\Gamma}(\mathcal{P})$. In particular, $\mathcal{M}$ is the regular map described in [16] and defined by (6).

We now give some examples of chiral maps that are associated with improperly self-dual locally toroidal chiral 4-polytopes. For notation we refer to [19, §2].

Example 1. The universal chiral 4-polytope $\mathcal{P} = \{\{4,4\}_{(1,3)}, \{4,4\}_{(1,3)}\}$ is improperly self-dual and has an automorphism group of order 2000 (see [12, p. 132]). The induced chiral map $\mathcal{M}$ of Theorem 1, with an automorphism group of order 4000, is of type $\{4, 8\,|\,4\}$. Eight quadrangular faces meet at each vertex of the map. The local structure is seen in Figure 1, where the 4-gonal holes of the map are highlighted (they are not faces of the map). In a sense, the set of quadrangular faces splits into blocks of 4, each associated with a hole and corresponding to the block of rectangular faces (in the mantle) of a 4-gonal prism; in the figure, the central vertex is common to the four prisms that are associated with the four highlighted square-shaped holes. It should be noted that the valency of each vertex is 8, but some edges in the figure are hidden. In addition we observe that the polytope $\mathcal{P}$ has no polarity and hence the group of the map $\mathcal{M}$ cannot be generated by involutions.

Example 2. The universal chiral 4-polytope $\mathcal{P} = \{\{6,3\}_{(1,2)}, \{3,6\}_{(2,1)}\}$ is improperly self-dual and has an automorphism group of order 20160. The polytope

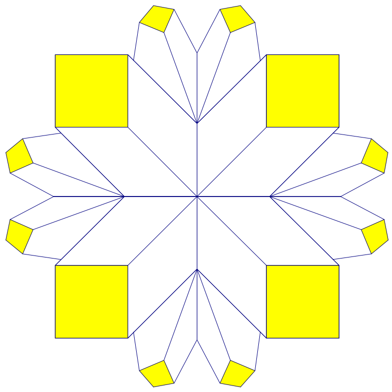

Figure 1. A chiral map of type $\{4, 8\,|\,4\}$

has (at least) two quotients, with groups of orders 10080 and 5040 (in the latter case, isomorphic to the symmetric group S_7), which are also improperly self-dual chiral polytopes of type $\{6, 3, 6\}$ with toroidal facets $\{6, 3\}_{(1,2)}$ and vertex-figures $\{3, 6\}_{(2,1)}$; both can be derived from $\mathcal{P}$ by identifying vertices separated by a certain number of steps along Petrie polygons. The polytope $\mathcal{P}$ and its two quotients have Petrie polygons of length 28, 14 and 7, respectively. Now all three polytopes possess a polarity (see [12, Theorem 3.4]), so their extended groups are generated by involutions. The automorphism groups of the three induced maps $\mathcal{M}$, of type $\{4, 6\,|\,6\}$, are thus generated by involutions. The local structure of the maps, with hexagonal holes highlighted, can be seen in Figure 2; now the set of quadrangular faces splits into blocks of 6, each associated with a hole and corresponding to a 6-gonal prism.

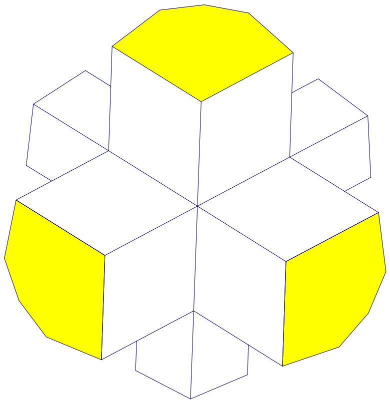

Figure 2. A chiral map of type $\{4, 6\,|\,6\}$

It should be noted that, for improperly self-dual chiral 4-polytopes, the elements $\pi_L = \sigma_1\sigma_3$ and $\pi_R = \sigma_1\sigma_3^{-1} = \delta^{-1}\sigma_3\sigma_1\delta$ are conjugate in $\overline{\Gamma}(\mathcal{P})$ (since $\sigma_3\sigma_1$ and

$\sigma_1\sigma_3$ are conjugate in $\Gamma(\mathcal{P})$) and the left- and right-handed Petrie polygons are thus of the same length.

5. Regular maps from properly self-dual chiral polytopes

As before, let $\mathcal{P}$ be a chiral 4-polytope of type $\{p, q, p\}$, and let $\sigma_1, \sigma_2, \sigma_3$ be the distinguished generators of $\Gamma(\mathcal{P})$ defined with respect to a base flag Φ. We now assume that $\mathcal{P}$ is properly self-dual. Then $\mathcal{P}$ admits a (unique) polarity ω which fixes Φ, and

$$\omega\sigma_1\omega = \sigma_3^{-1}, \quad \omega\sigma_2\omega = \sigma_2^{-1} \tag{15}$$

in $\overline{\Gamma}(\mathcal{P})$ (see [12, Theorem 3.1]). It follows that

$$\omega\sigma_1\sigma_2\omega = \sigma_2\sigma_3, \quad \omega\sigma_1\sigma_2\sigma_3\omega = \sigma_1\sigma_2\sigma_3, \tag{16}$$

so that (under conjugation) ω fixes $\sigma_1\sigma_2\sigma_3$ and interchanges $\sigma_1\sigma_2$ with $\sigma_2\sigma_3$. Hence ω induces the twisting operation

$$(\sigma_1\sigma_2\sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_3, \omega) \mapsto (\sigma_1\sigma_2\sigma_3, \sigma_1\sigma_2, \omega) =: (\tau_0, \tau_1, \tau_2) \tag{17}$$

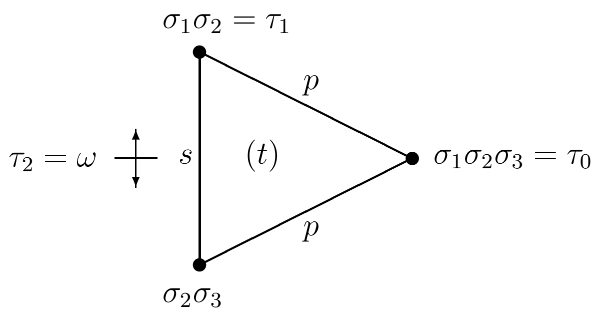

Figure 3. The polarity ω acting on $\Gamma(\mathcal{P})$.

on $\Gamma(\mathcal{P})$ shown in Figure 3; the notation used in the figure is explained in more detail below. Note that each τ_i is an involution, and that τ_0 and τ_2 commute. Furthermore,

$$\begin{aligned}
\sigma_1 &= \sigma_1(\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\omega\sigma_1\sigma_2\omega = \tau_0\tau_2\tau_1\tau_2 = \tau_2\tau_0\tau_1\tau_2,\\
\sigma_2 &= \sigma_1^{-1}\tau_1 = \tau_2\tau_1\tau_2\tau_0\tau_1,\\
\sigma_3 &= \sigma_1\sigma_2\tau_0 = \tau_1\tau_0.
\end{aligned}$$

Hence $\langle\tau_0, \tau_1, \tau_2\rangle = \overline{\Gamma}(\mathcal{P})$, and $\tau_0\tau_1 = \sigma_3^{-1}$ is of order p.

Theorem 2. *Let $\mathcal{P}$ be a properly self-dual chiral 4-polytope of type $\{p, q, p\}$, let $\overline{\Gamma}(\mathcal{P})$ be its extended group, and let $\Lambda := \langle\tau_0, \tau_1, \tau_2\rangle$, with τ_0, τ_1, τ_2 as in (17). Then Λ is the automorphism group of a regular 3-polytope $\mathcal{M}$ of type $\{p, 2s\}$, where s is the length of the left-Petrie polygons of $\mathcal{P}$. Moreover, $\Lambda = \overline{\Gamma}(\mathcal{P})$.*

Proof. First observe that

$$(\tau_1\tau_2)^2 = (\sigma_1\sigma_2\omega)^2 = \sigma_1\sigma_2^2\sigma_3 = \sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1} = \sigma_2^{-1}\sigma_2\sigma_3\sigma_1 = \sigma_3\sigma_1,$$

which is conjugate to $\pi_L := \sigma_1\sigma_3$ and hence is of order s. Then, since the order of a duality is even, $\tau_1\tau_2$ must have order $2s$.

It remains to verify the intersection condition for Λ. Suppose that $\gamma \in \langle\tau_0, \tau_1\rangle \cap \langle\tau_1, \tau_2\rangle$. By multiplying γ by τ_1 if need be, we may assume that $\gamma = (\tau_1\tau_0)^j = \sigma_3^j$ for some j, so that γ fixes the base vertex F_0 of Φ. We claim that $\gamma = \epsilon$. Now, since $\gamma \in \langle\tau_1, \tau_2\rangle = \langle\sigma_1\sigma_2, \sigma_2\sigma_3\rangle \ltimes \langle\omega\rangle$ and $\gamma \in \Gamma(\mathcal{P})$, we also have

$$\gamma = (\sigma_1\sigma_2^2\sigma_3)^k(\sigma_2\sigma_3)^l = (\sigma_3\sigma_1)^k(\sigma_2\sigma_3)^l$$

for some integer k and $l = 0, 1$. If $l = 0$, then γ must shift the base vertex F_0 k steps along a left-Petrie polygon (the image under σ_3^{-1} of the left-Petrie polygon associated with π_L), so that necessarily $k = 0$; it follows that $\gamma = \epsilon$, as required. On the other hand, the case $l = 1$ cannot occur. In fact, when $l = 1$, the element $(\sigma_3\sigma_1)^k = \gamma(\sigma_2\sigma_3)^{-1}$ also fixes F_0, so that again $k = 0$ and now $\gamma = \sigma_2\sigma_3$; but from $\gamma = \sigma_3^j$ we then obtain $\sigma_2 = \sigma_3^{j-1}$, a contradiction.

It follows that Λ is the group of a regular 3-polytope, which then must be of type $\{p, 2s\}$. □

We can also determine the lengths of the Petrie polygons and 2-zigzags of $\mathcal{M}$.

Remark 2. Let $\mathcal{P}$ and $\mathcal{M}$ be as in Theorem 2, and let t be the length of the right-Petrie polygons of $\mathcal{P}$. Then $\mathcal{M}$ is of type $\{p, 2s\}_{2t,q}$; that is, the length of the Petrie polygons of $\mathcal{M}$ is $2t$, and the length of the 2-zigzags of $\mathcal{M}$ is q.

The proof is straightforward. In fact,

$$(\tau_0\tau_1\tau_2)^2 = (\sigma_3^{-1}\omega)^2 = \sigma_3^{-1}\sigma_1$$

is conjugate to $\pi_R = \sigma_1\sigma_3^{-1}$ and hence has order t. On the other hand, since dualities have even order, the duality $\tau_0\tau_1\tau_2$ then must have order $2t$. Finally, $\tau_0(\tau_1\tau_2)^2$ is conjugate to σ_2.

The diagram shown in Figure 3 exhibits the group $\Gamma(\mathcal{P})$ as a quotient of the abstract group $\Gamma^3(s, t; p, p)$ described in [17, Sect. 9D]. As usual, the mark on a branch of the diagram is the period of the product of the two generators at its ends. The interior mark, t, represents the extra relation

$$(\sigma_1\sigma_2 \cdot \sigma_1\sigma_2\sigma_3 \cdot \sigma_2\sigma_3 \cdot \sigma_1\sigma_2\sigma_3)^t = \epsilon,$$

which holds in $\Gamma(\mathcal{P})$ because of

$$\sigma_1\sigma_2 \cdot \sigma_1\sigma_2\sigma_3 \cdot \sigma_2\sigma_3 \cdot \sigma_1\sigma_2\sigma_3 \;=\; (\sigma_1\sigma_2)^2\sigma_3\sigma_1^{-1}(\sigma_1\sigma_2\sigma_3)^2 \;=\; \sigma_3\sigma_1^{-1} \;=\; \pi_R^{-1}.$$

Note that the mark t is placed in parentheses to indicate the fact that the other analogous products of four generators, with one repeated, will not in general have the same order t (see [17, p. 320] for more details).

Example 3. The universal chiral 4-polytope $\mathcal{P} = \{\{3,6\}_{(1,2)}, \{6,3\}_{(1,2)}\}$ is properly self-dual. Its automorphism group, of order 672, is the group $\mathrm{L}_2^{\langle\pm1\rangle}(\mathbb{Z}_7)$ of all 2×2 matrices of determinant ±1 over $\mathbb{Z}_7$ (see [19]). The induced regular map $\mathcal{M}$ is of type $\{3,16\}$ (more exactly, of type $\{3,16\}_{28,6}$) and has 42 vertices and 224 faces; its automorphism group is $L_2^{\langle\pm1\rangle}(\mathbb{Z}_7) \ltimes C_2$ of order 1344. The quotient $\mathcal{P}'$ of $\mathcal{P}$ by the central subgroup $\langle\pm I\rangle$ (with I the identity matrix) of $\mathrm{L}_2^{\langle\pm1\rangle}(\mathbb{Z}_7)$ again has facets $\{3,6\}_{(1,2)}$ and vertex-figures $\{6,3\}_{(1,2)}$, and yields a regular map of type $\{3,8\}$ with 42 vertices, 112 faces, and automorphism group $\mathrm{PGL}_2(7) \ltimes \mathrm{C}_2$ of order 672. The latter has Petrie polygons of length 14, and doubly covers $\{3,8\}_7$, the dual of the Petrial of the Klein map $\{7,3\}_8$ (see [5]).

6. Involutory generators for chiral maps

In the construction of polytopes from those of higher rank we often encounter the question whether or not the automorphism group or a certain subgroup is generated by involutions. We know that the group of a regular polytope of any rank n is a C-group and comes with a distinguished set of generators consisting of involutions. The group of a chiral polytope $\mathcal{P}$ of rank $n \geq 4$ is also generated by involutions (see [18, p. 496]), although its distinguished generators $\sigma_1,\dots,\sigma_{n-1}$ are not involutory; for example, when $n = 4$, the elements $\sigma_1\sigma_2\sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_3$ are involutory generators. However, this is no longer true for rank 3, as shown by the chiral map described in Example 1 (as well as the toroidal maps below). On the other hand, many chiral 3-polytopes (see Example 2) do have the desired property. In this section we briefly discuss obstructions for the existence of involutory generators for rank 3.

Let $\mathcal{P}$ be a chiral 3-polytope of type $\{p,q\}$ with group $\Gamma(\mathcal{P}) = \langle\sigma_1,\sigma_2\rangle$. Let $\tau := \sigma_1\sigma_2$, and let

$$N(\tau) := \langle \varphi^{-1}\tau\varphi \mid \varphi \in \Gamma(\mathcal{P})\rangle \tag{18}$$

(that is, $N(\tau)$ is the normal closure of τ in $\Gamma(\mathcal{P})$). Then τ is the "half-turn" about the base edge (in Φ), and since $\Gamma(\mathcal{P})$ is edge-transitive, $N(\tau)$ is the group generated by all the half-turns about edges of $\mathcal{P}$. Clearly, since τ is an involution, $N(\tau)$ is generated by involutions (although their number may not be finite if $\mathcal{P}$ is infinite). Now, since $\Gamma(\mathcal{P}) = \langle\tau,\sigma_2\rangle = \langle\sigma_1,\tau\rangle$, we have

$$\Gamma(\mathcal{P}) = N(\tau)\cdot\langle\sigma_2\rangle = N(\tau)\cdot\langle\sigma_1\rangle$$

(a product of subgroups), where the second factors are cyclic of orders q or p, respectively. In particular, $\Gamma(\mathcal{P})/N(\tau)$ is a cyclic group, whose order divides p and q if the latter are finite.

Lemma 1. *Let Γ be any group, and let N be a normal subgroup of Γ such that Γ/N is cyclic. If Γ is generated by involutions, then N has index 1 or 2 in Γ.*

Proof. Suppose that $N \neq \Gamma$. If Γ is generated by involutions, then so is its quotient Γ/N. A non-trivial cyclic group contains an involution only if it has finite even order, and it is generated by involutions only if its order is 2. This proves the lemma. □

The following proposition follows immediately from Lemma 1, applied with $\Gamma = \Gamma(\mathcal{P})$ and $N = N(\tau)$.

Proposition 1. *Let $\mathcal{P}$ be a chiral 3-polytope of type $\{p, q\}$ with group $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$, and let $N(\tau)$ be as in* (18).
(a) *If $\Gamma(\mathcal{P})$ is generated by involutions, then $N(\tau)$ has index 1 or 2 in $\Gamma(\mathcal{P})$.*
(b) *If $\Gamma(\mathcal{P}) = N(\tau)$ or $N(\tau) \ltimes C_2$, then $\Gamma(\mathcal{P})$ is generated by involutions.*

The proposition basically says that, if $\mathcal{P}$ is a chiral 3-polytope with involutory generators, then in fact the half-turns about edges provide a "nice" generating set of involutions for a subgroup of very small index, at most 2. The second part of Proposition 1 is a partial converse of the first. Note that, if $N(\tau)$ has index 2 and $p \equiv 2 \pmod 4$ or $q \equiv 2 \pmod 4$, respectively, then $\Gamma(\mathcal{P}) = N(\tau) \ltimes \langle \sigma_1^{p/2} \rangle$ or $\Gamma(\mathcal{P}) = N(\tau) \ltimes \langle \sigma_2^{q/2} \rangle$; in any case, $\Gamma(\mathcal{P}) = N(\tau) \ltimes C_2$ and hence $\Gamma(\mathcal{P})$ is generated by involutions.

We remark that there is an exact analogue to Proposition 1 for the rotation subgroup $\Gamma^+(\mathcal{P})$ of a regular 3-polytope. The rotation subgroups of regular polytopes of rank at least 4 are always generated by involutions.

We conclude this section with a brief discussion of the chiral maps on the torus. None of these maps $\{4, 4\}_{(b,c)}$, $\{3, 6\}_{(b,c)}$ and $\{6, 3\}_{(b,c)}$ has a group generated by involutions.

Consider, for example, $\mathcal{P} := \{4, 4\}_{(b,c)}$. Then $\mathcal{P}$ is a quotient of the regular plane tessellation $\{4, 4\}$ with vertex-set $\mathbb{Z}^2$. Let $[4, 4]^+$ denote the rotation subgroup of $\{4, 4\}$, with distinguished generators σ_1, σ_2, let T be its translation subgroup generated by the translations τ_1, τ_2 by the vectors $(1, 0)$ and $(0, 1)$, respectively, and let $T_{(b,c)}$ denote the subgroup of T defined by

$$T_{(b,c)} = \langle \tau_1^b \tau_2^c, \tau_1^{-c} \tau_2^b \rangle.$$

Then $[4, 4]^+ = T \ltimes \langle \sigma_2 \rangle \cong \mathbb{Z}^2 \ltimes C_4$ and

$$[4, 4]_{(b,c)} := \Gamma(\{4, 4\}_{(b,c)}) \cong [4, 4]^+ / T_{(b,c)} \cong (T/T_{(b,c)}) \ltimes C_4$$

(see [5, §8.3]); note here that $\langle \sigma_2 \rangle \cap T_{(b,c)}$ is trivial, so the semi-direct product structure is preserved under taking the quotient. But now we can simply appeal to Lemma 1, applied with $\Gamma = [4, 4]_{(b,c)}$ and $N = T/T_{(b,c)}$. It follows that $[4, 4]_{(b,c)}$ is not generated by involutions, since the index of N is 4.

Similar arguments also apply to the chiral maps $\{3, 6\}_{(b,c)}$. Now we must take

$$T_{(b,c)} := \langle \tau_1^{b+c} \tau_2^c, \tau_1^{-c} \tau_2^b \rangle$$

(see [5, §8.4]). The dual map $\{6, 3\}_{(b,c)}$ has the same group as $\{3, 6\}_{(b,c)}$, and hence is also not generated by involutions.

Moreover, the same arguments establish that the rotation subgroups for the regular torus maps $\{4, 4\}_{(b,c)}$, $\{3, 6\}_{(b,c)}$ and $\{6, 3\}_{(b,c)}$ (with $c = 0$ or $b = c$) cannot be generated by involutions.

7. Concluding remarks

In his 1948 paper [2], Coxeter enumerated the chiral maps on the torus and showed that they fall into the three infinite families of maps of type $\{4,4\}$, $\{3,6\}$ or $\{6,3\}$. Garbe [9] proved that there are no chiral maps on surfaces of genus $2,3,4,5$ or 6. While there are many regular maps known to exist, the occurrence of chiral maps is rather sporadic. Several attempts have been made to construct chiral maps, notably by Sherk [21] and Wilson [6, 22]. More recently, in [1], Conder and Dobcsányi have enumerated all regular and chiral maps on surfaces up to genus 15. However, of the sixteen chiral maps listed, only thirteen are in fact abstract polyhedra, that is, their automorphism groups satisfy the intersection condition. (Every abstract polyhedron yields a map on a surface, but not vice versa. A map may be "too small" to be polytopal. An example is the torus map $\{4,4\}_{(1,0)}$, which has a single vertex, two edges and a single face.) The construction described in this paper can be used to provide new examples of chiral maps. They all are related to improperly self-dual chiral 4-polytopes. Unfortunately, at present, while many examples (including infinite families) of properly self-dual chiral 4-polytopes are known (see, for example, [19]), not many examples of improperly self-dual chiral 4-polytopes can be found in the literature.

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